MS&E 334 Lecture 2

Consider n goods, m agents, and agent i with utility function $u_i : \mathbf{R}_+^n \to \mathbf{R}$, which is concave and monotone. Three popular utility functions are

- Linear: $u_i(x) = \sum_j a_{ij} x_{ij}$.
- Cobb-Douglas: $u_i(x) = \prod_j (x_{ij})^{a_{ij}}$ with $\sum_j a_{ij} = 1$.
- Leontief: $u_i(x) = \min_j a_{ij} x_{ij}$.

A CES function is defined as

$$u_i(x) = \left(\sum_j \alpha_{ij} x_{ij}^{\rho} \right)^{1/\rho}$$

where $\rho = 1$ corresponds to linear function, $\rho = 0$ corresponds to Cobb-Douglas function, and $\rho \to -\infty$ corresponds to Leontief function.

Now suppose agent i has money m_i and has linear utility function. The prices $p_1, \cdots, p_n$ are called *market clearing prices* if, after each agent is assigned an optimal amount of goods relative to these prices, there is no surplus or deficiency of any of the goods. These prices can be solved from the Eisenberg-Gale convex program under two assumptions:

- If the utilities of any agent are scaled by a constant, the optimal allocation remain unchanged.
- If the money of agent i is split among two new agents with same utility function as agent i, then the sum of the optimal allocations of the new agents should be an optimal allocation for agent i.

The Eisenberg-Gale convex program is as follows:

$$\begin{aligned} \text{maximize} \quad & \sum_i m_i \log u_i \\ \text{subject to} \quad & u_i = \sum_j u_{ij} x_{ij}, \forall i \\ & \sum_i x_{ij} \le 1, \forall j \\ & x_{ij} \ge 0, \forall i, j \end{aligned}$$

and the optimal dual of the second set of constraints gives the market clearing prices.

For initial endowment $w_i = (w_{i1}, \cdots, w_{in})$, a price $\pi = (\pi_1, \cdots, \pi_n)$ is considered an *equilibrium price* if agent i sells her endowment and ask for goods $x_i = (x_{i1}, \cdots, x_{in})$ that

- x_i maximizes $u_i(x_i)$ subject to the budget constraint $\pi^T x_i \le \pi^T w_i$ and $x_i \in \mathbf{R}_+^n$.
- $\sum_i x_{ij} \le \sum_i w_{ij}$ for each good j.

The vector $x_i(\pi)$ that maximizes $u_i(x_i)$ subject to constraints is called the *demand* of agent i at price π. For good j, we define the *excess demand* $z_j(\pi) = \sum_i x_{ij}(\pi) - \sum_i w_{ij}$. The excess demand function satisfies

- Continuity.
- Homogeneity: $z(\pi) = z(\alpha\pi)$ for all $\alpha > 0$, where $z(\pi) = (z_1(\pi), \cdots, z_n(\pi))$.
- Walras' Law: $\sum_j \pi_j z_j(\pi) = 0$ for any price π since $\pi^T x_i(\pi) = \pi^T w_i$.

Theorem 1. *If the excess demand function satisfies the above conditions, then there exists an equilibrium price vector $\pi^* = (\pi_1^*, \cdots, \pi_n^*) \in \Delta_n$ (the standard simplex) such that for all j, $z_j(\pi^*) \leq 0$, and for all j with $\pi_j^* > 0$, we know that $z_j(\pi^*) = 0$.*

Proof. Let $\psi(\pi) = \arg\max_{\pi' \in \Delta_n} (\pi')^T z(\pi)$. By Brouwer's fixed point theorem, there exists π^* such that $\pi^* = \psi(\pi^*) = \arg\max_{\pi' \in \Delta_n} (\pi')^T z(\pi^*)$. For all $\pi' \in \Delta_n$, we have $(\pi')^T z(\pi^*) \leq (\pi^*)^T z(\pi^*) = 0$, where the equality follows from Walras' Law. Therefore, $z_j(\pi^*) \leq 0$ for all j, and if $\pi_j^* > 0$, then $z_j(\pi^*) = 0$. □

For any function $g : \Delta_n \to \Delta_n$, define $z(\pi) = g(\pi) - \lambda(\pi)\pi$ with $\lambda(\pi) = \frac{\sum \pi_j g_j(\pi)}{\sum \pi_j^2}$. It can be shown that $z(\pi)$ satisfies all three conditions provided g is positive homogeneous.

Theorem 2. *π^* is the equilibrium price for excess demand function z iff π^* is a fixed point of g.*

Proof. ($\Rightarrow$) We know that $z_j(\pi^*) \leq 0$, meaning that $g_j(\pi^*) \leq \lambda(\pi^*)\pi_j^*$. If $\pi_j^* = 0$, then $g_j(\pi^*) \leq 0$, but $g_j(\pi^*)$ is nonnegative, which means that $g_j(\pi^*) = 0 = \pi_j^*$. Now for $\pi_j^* > 0$, we have $z_j(\pi^*) = 0$, meaning that $g_j(\pi^*) = \lambda(\pi^*)\pi_j^*$. Since $g(\pi^*) \in \Delta_n$, we have $\sum g_j(\pi^*) = 1$, and from the fact that $\sum \pi_j^* = 1$, we know that $\lambda(\pi^*) = 1$. Therefore, $g_j(\pi^*) = \pi_j^*$ when $\pi_j^* > 0$. □