MSE 334 Computation of Equilibria Lecture 3 - 03/21/2009

Market and Game Dynamics

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1 Market Dynamics

1.1 Tatonnement Process

Recall from last lecture: There are n goods and m agents. The payoff of agent j, $U_j : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$ is strictly concave and strictly increasing. For any price vector p, we define the excess demand function as follows,

$$f_i(p) = \sum_{j=1}^{m} x^*_{ij}(p) - w_{ij} \quad 1 \leq i \leq n,$$

where $x^*_{ij}(p)$ is the optimum allocation of good i to agent j under the price vector p and w_{ij} is the j's endowment of good i. Last lecture, we showed that $f(p)$ has the following two properties:

- Homogeneity (H): $f(\alpha p) = f(p)$, $\forall \alpha > 0$
- Walras Law (W): $p^T f(p) = 0$

In this lecture, we introduce a dynamic process for updating the price vector which converges to the equilibrium price for all excess demand functions that satisfy the gross substitutability condition.

Gross Substitutability (GS): If $p_i \leq p'_i$, $\forall i$, and $\exists j$ such that $p_j < p'_j$, then

$$f_i(p) > f_i(p') \quad \forall i \neq j$$

in other words, $\frac{df_i}{dp_j} > 0, \quad i \neq j$.

Tatonnement Process: If the excess demand satisfies (H), (W), and (GS) then the following process, which is called Tatonnement process, converges to the market clearing price vector, $\overline{p}$.

$$\frac{dp_i}{dt} = G(f_i(p)),$$

where G can be any monotone sign-preserving function. In what follows, we prove that (H), (W), and (GS) are sufficient conditions for convergence of this process. However, it has been shown that these conditions are necessary as well.

1.2 Weak Axiom of Revealed Preferences

Theorem: (Arrow-Hurwicz 60's) If the excess demand, f, satisfies conditions (H), (W), and (GS) and $\overline{p}$ is the equilibrium price then for every non-equilibrium price p, we have:

$$\sum_{i=1}^{n} \overline{p}_i f_i(p) > 0$$

Note: This condition is called Weak Axiom of Revealed Preferences (WARP).

Implications: For GS markets, the equilibrium price, $\overline{p}$, can be computed in polynomial time.

Proof Sketch: Condition (W) implies that for any p:

$$\sum_{i=1}^{n} p_i f_i(p) = 0$$

therefore, p is the normal vector for the hyperplane which divides the space in half and the equilibrium price is in the half-space where $\sum_{i=1}^{n} x_i f_i(p) > 0$.

Proof of WARP: WLOG, by scaling the goods' units, we can assume that $\overline{p} = (1, 1, \ldots, 1)$. Therefore we need to show that for any non-equilibrium price, p,

$$\sum_{i=1}^{n} f_i(p) > 0.$$

WLOG assume,

$$p_1 \leq p_2 \cdots \leq p_n,$$

Clearly, for any non-equilibrium price $\exists v$ such that $p_v < p_{v+1}$ (otherwise p would be the equilibrium price). Consider the following sequence of price vectors:

$$\begin{aligned}
\pi^1 &= (p_1, p_1, \ldots, p_1) \\
\pi^2 &= (p_1, p_2, \ldots, p_2) \\
&\vdots \\
\pi^s &= (p_1, p_2, \ldots, p_s, p_s, \ldots, p_s) \\
&\vdots \\
\pi^n &= (p_1, p_2, \ldots, p_n)
\end{aligned}$$

$f(p)$ can be written as the following telescopic sum:

$$f(p) = f(\pi^n) = f(\pi^n) \pm f(\pi^{n-1}) \pm f(\pi^{n-2}) \cdots \pm f(\pi^1)$$

Note that $f(\pi^1) = 0$, hence we have:

$$f(\pi^n) = \sum_{s=1}^{n-1} \left[f(\pi^{s+1}) - f(\pi^s) \right]$$

Fixing s, let $y_i = f_i(\pi^{s+1})$ and $x_i = f_i(\pi^s)$. By condition (W),

$$\begin{aligned}
0 &= \sum_{i=1}^{n} \pi_i^{s+1} y_i - \sum_{i=1}^{n} \pi_i^s x_i \\
&= \sum_{i=1}^{s} \pi_i^{s+1}(y_i - x_i) + \sum_{i=s+1}^{n} p_{s+1} y_i - \sum_{i=s+1}^{n} p_s x_i \\
&\leq \sum_{i=1}^{s} p_{s+1}(y_i - x_i) + \sum_{i=s+1}^{n} p_{s+1}(y_i - x_i) + \sum_{i=s+1}^{n} (p_{s+1} - p_s) x_i \qquad (1) \\
&= \sum_{i=1}^{n} p_{s+1}(y_i - x_i) + \sum_{i=s+1}^{n} (p_{s+1} - p_s) x_i \qquad (2)
\end{aligned}$$

where in (1), we used the fact that $y_i \geq x_i,\ 1 \leq i \leq s$; since $\pi_j^{s+1} = p_{s+1} \geq \pi_j^s = p_s,\ j > s$, condition (GS) implies that $y_i = f_i(\pi^{s+1}) \geq x_i = f_i(\pi^s),\ 1 \leq i \leq s$. $y_i - x_i \geq 0,\ 1 \leq i \leq s$ and

$p_{s+1} \geq \pi_i^{s+1},\ 1 \leq i \leq s$, hence $\sum_{i=1}^{s} \pi_i^{s+1}(y_i - x_i) \leq \sum_{i=1}^{s} p_{s+1}(y_i - x_i)$. The second term in (2) is non-positive because (GS) implies that $x_i \leq 0,\ s+1 \leq i \leq n$ [1] and $p_{s+1} \geq p_s$. Therefore,

$$\begin{cases} \sum_{i=1}^{n} p_{s+1}(y_i - x_i) + \sum_{i=s+1}^{n} (p_{s+1} - p_s) x_i \geq 0 \\ \sum_{i=s+1}^{n} (p_{s+1} - p_s) x_i \leq 0 \end{cases} \Rightarrow \sum_{i=1}^{n} (y_i - x_i) \geq 0, \quad 1 \leq s \leq n$$

The inequality in (1) is strict for $s = v$ which results in:

$$f(\pi^n) = \sum_{s=1}^{n-1} \left[f(\pi^{s+1}) - f(\pi^s) \right] > 0 \quad \square$$

1.3 Convergence of Tatonnement Process

Convergence of Tatonnement process can be established using the following potential function,

$$\begin{aligned} V(p) &= \frac{1}{2} \sum_{i=1}^{n} (p_i - \overline{p}_i)^2 \\ \frac{dV}{dt} &= \sum_{i=1}^{n} (p_i(t) - \overline{p}_i) \frac{dp_i}{dt} \\ &= \sum_{i=1}^{n} p_i f_i(p(t)) - \sum_{i=1}^{n} \overline{p}_i f_i(p(t)) \\ &= -\sum_{i=1}^{n} \overline{p}_i f_i(p(t)) < 0 \end{aligned} \tag{3}$$

where in (3) we used (W) and (WARP). $V(t) \geq 0$, $\frac{dV}{dt} < 0$, and $V(p) = 0$ iff $p = \overline{p}$, hence the process is asymptotically stable.

1.3.1 Another Potential

The following potential function for the process of updating the price vector (suggested by Samuelson) provides another sufficient condition for convergence of Tatonnement Process.

$$\begin{aligned} V(p) &= \frac{1}{2} \sum_{i=1}^{n} f_i(p_1, p_2, \ldots, p_n)^2 \\ \frac{dV}{dt} &= \sum_{i=1}^{n} \sum_{r=1}^{n} f_i(p(t)) \frac{df_i}{dp_r} \frac{dp_r}{dt} \\ &= \sum_{i=1}^{n} \sum_{r=1}^{n} f_i(p(t)) \frac{df_i}{dp_r} f_r(p(t)) \\ &= f^T(p) Df(p) f(p) \end{aligned} \tag{4}$$

where Df is the Jacobian matrix of f. If $Df(p) + Df(p)^T$ is negative definite then $\frac{dV}{dt} < 0$. Moreover, $V(t) \geq 0$ and $V(p) = 0$ iff $p = \overline{p}$, therefor the process is asymptotically stable.

[1] $f_i(\pi^1) = 0,\ \forall i$, the price of good $i,\ i > s$ is non-decreasing in $\pi^1, \pi^2, \ldots \pi^s$, therefore, by (GS) the excess demand for these goods are non-positive in step s.

2 Game Dynamics

In the context of exchange economy, as shown in the previous section, the existence of a dynamic that converges to the corresponding equilibrium (Competitive Equilibrium which is the optimal allocation under $\overline{p}$) has been established under fairly general assumptions. However, much less is known in the context of Game dynamics. In this section, we illustrate an example for which the Nash equilibrium can be computed in polynomial time.

Setting: Consider a game with m players. Let X^i denote the action set of player i, $X = \prod_{i=1}^m X^i$, and $\sum_{i=1}^m n_i = n$. The payoff function $F : X \to \mathbb{R}^n$ is a continuous mapping that assigns player i a n_i-dimensional payoff vector, $(F_1^i(x), F_2^i(x), \ldots, F_{n_i}^i(x))$.

Example I: Let $m = 1$, think of a population game in which x_j represents the fraction of the players that choose strategy j. $F_j(x)$ is the payoff that a player receives by playing strategy j.

Definition: Let $TX^i = \{x^i - y^i : \; x^i, y^i \in X^i\}$ and $TX = \prod_{i=1}^n TX^i$. The game $F : X \to \mathbb{R}^n$ is *stable* if

$$(y - x)^T (F(y) - F(x)) \leq 0 \;\; \forall x, y \in X \tag{5}$$

Equivalently, in the case where F is differentiable, the game is stable iff $Df(x)$ is negative semidefinite with respect to TX and for all $x \in X$, i.e., $z^T Df(x) z \leq 0$, $\forall z \in TX$.

Definition: $x \in X$ is a Nash equilibrium if every *used* strategy is optimal. In other words,

$$x_k^i > 0 \Rightarrow F_k^i(x) \geq F_j^i(x) \;\; 1 \leq j \leq n_i$$

Equivalently $x \in X$ is NE iff,

$$x^T F(x) \geq y^T F(x) \;\; \forall y \in X. \tag{6}$$

Remark: In analogy with (WARP) for markets, we show that for a stable game,

$$\text{y is a non-equilibrium strategy} \Rightarrow y^T F(y) \leq x^T F(y) \tag{7}$$

where x is the NE. Note that (7) means any non-equilibrium y determines a half-space in which NE lies.

Proof of (7)**:** Inequalities (5) and (6) can be written as:

$$\begin{cases} (y-x)^T F(y) - (y-x)^T F(x) \leq 0 \\ -(y-x)^T F(x) \geq 0 \end{cases} \quad \Rightarrow (y-x)^T F(y) \leq 0$$