MSE 334 Computation of Equilibria Lecture 4 - 03/28/2009

Lemke-Howson Algorithm

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Consider a two person bimatrix game where the payoff matrices are $A_{m\times n}$ and $B_{m\times n}$. A pair of strategies $(\boldsymbol{x}, \boldsymbol{y})$ is a Nash equilibrium for game (A, B) if and only if

$$\forall\ 1 \le i \le m,\ x_i > 0 \Rightarrow (A\boldsymbol{y})_i = \max_k (A\boldsymbol{y})_k$$

$$\forall\ m+1 \le j \le m+n,\ y_j > 0 \Rightarrow (\boldsymbol{x}^T B)_j = \max_k (\boldsymbol{x}^T B)_k$$

Let $M = \{1, 2, \cdots, m\}$ and $N = \{m+1, m+2, \cdots, m+n\}$. Define the support of $\boldsymbol{x}$ by $S(\boldsymbol{x}) = \{i \mid x_i > 0\}$. Define the support of $\boldsymbol{y}$ similarly.

Definition 1. *A bimatrix game (A, B) is non-degenerate if and only if for every strategy $\boldsymbol{x}$ of the row player, $|S(\boldsymbol{x})|$ is at least the number of pure best responses to $\boldsymbol{x}$, and for every strategy $\boldsymbol{y}$ of the column player, $|S(\boldsymbol{y})|$ is bigger than or equal to the number of pure best responses to $\boldsymbol{y}$.*

An equivalent definition is: for any $\boldsymbol{y}'$ that is a best response to $\boldsymbol{x}$, $|S(\boldsymbol{x})| \ge |S(\boldsymbol{y}')|$, and for any $\boldsymbol{x}'$ that is a best response to $\boldsymbol{y}$, $|S(\boldsymbol{y})| \ge |S(\boldsymbol{x}')|$.

Also note that we can slightly perturb the payoff matrices to make the game non-degenerate. Therefore WLLOG (with little loss of generality!), we can assume that game (A, B) is non-degenerate.

The following proposition is directly implied by the definition:

Proposition 2. *If $(\boldsymbol{x}, \boldsymbol{y})$ is a Nash equilibrium of a non-degenerate bimatrix game, then $|S(\boldsymbol{x})| = |S(\boldsymbol{y})|$.*

Now consider the following Polytopes:

$$P = \{(u, \boldsymbol{x}) \mid x_i \ge 0, \sum x_i = 1,\ \boldsymbol{x}^T B \le u \cdot \mathbf{1}\}$$

$$Q = \{(v, \boldsymbol{y}) \mid y_j \ge 0, \sum y_j = 1,\ A\boldsymbol{y} \le v \cdot \mathbf{1}\}$$

By the above proposition it is easy to see that every Nash equilibrium can be described as a pair of corner points of P and Q. For simplicity of notation, consider the following transformations"

$$\overline{P} = \{\boldsymbol{x} \mid x_i \ge 0,\ \boldsymbol{x}^T B \le \mathbf{1}\}$$

and

$$\overline{Q} = \{\boldsymbol{y} \mid y_j \ge 0,\ A\boldsymbol{y} \le \mathbf{1}\}.$$

There is a one to one correspondence between the corners of P and $\overline{P}$, except the zero corner of $\overline{P}$. In fact, for each corner $(u, \boldsymbol{x})$ of P, $\boldsymbol{x}/u$ is a corner of $\overline{P}$; and for each nonzero corner $\boldsymbol{x}$ of $\overline{P}$, $(1/\sum x_i, \boldsymbol{x}/\sum x_i)$ is a corner of P. The same correspondence exists for Q and $\overline{Q}$.

The corner points of $\overline{P}$ and $\overline{Q}$ are of our interest because they correspond to special set of strategies of the players. $\boldsymbol{x}$ is a corner point of $\overline{P}$ implies some inequalities among $\{\boldsymbol{x} \mid x_i \ge 0,\ \boldsymbol{x}^T B \le \mathbf{1}\}$ bind. If $x_i = 0$, then row i is not used in the mixed strategy $\boldsymbol{x}$; if $(\boldsymbol{x}^T B)_j = 1$, then column j is a best response to row player's strategy $\boldsymbol{x}$. Next we give an explicit connection of the corner points of $\overline{P}, \overline{Q}$ and Nash equilibria.

Define graph G_1, G_2 as follows: The vertices of G_1, G_2 are the corner points of $\overline{P}$, $\overline{Q}$ respectively. There is an edge between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in G_1 if and only if $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are adjacent corner points of $\overline{P}$. Define the edges of G_2 similarly. Then label each vertex $\boldsymbol{x}$ of G_1 with the indices of the tight constraints in $\overline{P}$, i.e.

$$L(\boldsymbol{x}) = \{i \mid x_i = 0\} \cup \{j \mid (\boldsymbol{x}^T B)_j = 1\}$$

Label G_2 similarly. By the non-degeneracy of the game, $|L(\boldsymbol{x})| \leq m$ and $|L(\boldsymbol{y})| \leq n$. We have the following theorem.

Theorem 3. *A pair $(\boldsymbol{x}, \boldsymbol{y})$ is a Nash equilibrium if and only if $(\boldsymbol{x}, \boldsymbol{y})$ is completely labeled: $L(\boldsymbol{x}) \cup L(\boldsymbol{y}) = M \cup N = \{1, 2, \cdots, m+n\}$.*

Proof Suppose $L(\boldsymbol{x}) \cup L(\boldsymbol{y}) = \{1, 2, \cdots, m+n\}$. For each $i \in M$ that is in the label set of $\boldsymbol{x}$, row i is not used in $\boldsymbol{x}$, for each $j \in N$ that is in the label set of $\boldsymbol{x}$, column j for the other player is a best response to $\boldsymbol{x}$. These conclusions are symmetric for the label set of $\boldsymbol{y}$. Let $M_1 = \{i | x_i = 0\}$, $N_2 = \{j | (\boldsymbol{x}^T B)_j = 1\}$; $N_1 = \{j | y_j = 0\}$, $M_2 = \{i | (A\boldsymbol{y})_i = 1\}$. Since $|L(\boldsymbol{x})| \leq m$ and $|L(\boldsymbol{y})| \leq n$, then $L(\boldsymbol{x}) \cup L(\boldsymbol{y}) = M \cup N$ implies (M_1, M_2) is a partition of M and (N_1, N_2) is a partition of N. Therefore $\boldsymbol{x}$ consists of strategies only in M_2, and is a best response to $\boldsymbol{y}$, $\boldsymbol{y}$ consists of strategies only in N_2 and is a best response to $\boldsymbol{x}$.

On the other hand, if $(\boldsymbol{x}, \boldsymbol{y})$ is a pair of Nash equilibrium, then $M \backslash S(\boldsymbol{x})) \subset L$ because those rows are not used in $\boldsymbol{x}$, and $S(\boldsymbol{y}) \in L$ because those columns are best responses to $\boldsymbol{x}$. Note the game is non-degenerate, so $|S(\boldsymbol{x})| = |S(\boldsymbol{y})|$, then $L(\boldsymbol{x}) = (M \backslash S(\boldsymbol{x})) \cup S(\boldsymbol{y})$. Similarly, $L(\boldsymbol{y}) = (N \backslash S(\boldsymbol{y})) \cup S(\boldsymbol{x})$. Hence $L(\boldsymbol{x}) \cup L(\boldsymbol{y}) = M \cup N$. □

Finally, we use this connection of Nash equilibrium and graphs G_1, G_2 to give a combinatorial (albeit exponential-time) algorithm of finding a Nash equilibrium in a bimatrix game. The algorithm is by Lemke and Howson. The basic idea is to pivot alternatingly in $\overline{P}$ and $\overline{Q}$ until we find a pair that is completely labeled.

Let $G = G_1 \times G_2$, i.e., vertices of G are defined as $v = (v_1, v_2)$ where $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. There is an edge between $v = (v_1, v_2)$ and $v' = (v_1', v_2')$ in G if and only if $(v_1, v_1') \in E(G_1)$ or $(v_2, v_2') \in E(G_2)$. Then for each vertex $v = (v_1, v_2) \in V(G)$, define its label by $L(v) = L(v_1) \cup L(v_2)$. For each $k \in M \cup N$, define the set of "k-almost" completely labeled vertices by

$$U_k = \{v \in V(G) | L(v) \supseteq M \cup N \backslash \{k\}\}$$

We have the following key results of U_k:

Theorem 4. *For any $k \in M \cup N$,*

1. *$(0, 0)$ and all Nash equilibrium points belong to U_k. Furthermore, their degree in the graph induced by U_k is exactly one.*
2. *The degree of every other vertex in the graph induced by U_k is two.*

Proof First, note that the label set of $(0, 0)$ and any Nash equilibrium is exactly $M \cup N$, so $(0, 0)$ and all Nash equilibrium points are in U_k for any k. Furthermore, let $v = (v_1, v_2)$ be $(0, 0)$ or any Nash equilibrium point. Without loss of generality, suppose $k \in L(v_1)$, where v_1 is a corner point of the polytope $\overline{P}$. Among all edges in G_1 that v_1 is incident to, there is only one direction leading to a vertex v_1' without label k (i.e. loosing the binding constraint corresponding to label k). It is easy to see that $(v_1', v_2) \in U_k$, therefore there is only one neighbor of v in U_k.

For part (2), let $v = (v_1, v_2)$ be any other point in U_k. Then there must be a duplicated label in $L(v_1)$ and $L(v_2)$, denoted by l. Similarly to (2), there is exactly one direction of v_1's edges in $\overline{P}$ to drop the label l, and the new vertex v_1' has all labels v_1 has except l, so $(v_1', v_2) \in U_k$. It is symmetric for v_2. Hence there are two neighbors of v in U_k. □

In other words, in a non-degenerate bimatrix game (A, B) the set of k-almost completely labeled vertices in G and their induced edges consist of disjoint paths and cycles. The endpoints of the paths are the artificial equilibrium (0,0) and the equilibria of the game.

Corollary 5. *A non-degenerate bimatrix game has an odd number of Nash equilibria.*

Algorithm (Lemke-Howson)
Input: A Non-degenerate bimatrix game (A, B).
Output: One Nash equilibrium of the game.

1. Choose $k \in M \cup N$.
2. Start with $(x, y) = (0, 0) \in G$. Drop label k from (x, y) (from $x \in \overline{P}$ if $k \in M$, from $y \in \overline{Q}$ if $k \in N$).
3. Let (x, y) be the current vertex. Let l be the label that is picked up by dropping label k. If $l = k$, terminate and (x, y) is a Nash equilibrium of the game. If $l \neq k$, drop l in the other polytope and repeat this step.

The Lemke-Howson algorithm starts from the artificial equilibrium $(0, 0)$ and follows the path in U_k. Since the number of vertices of G is exponential in n and m, so the algorithm may take an exponential time to find a Nash equilibrium .

Reference

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[3]. B. V. Stengel, Equilibrium Computation for Two-Person Games in Strategic and Extensive Form. Chapter 3, Algorithmic Game Theory.