MS&E 334 Computation of Equilibria Lecture 5 - 05/05/2009

Leontief Economies Encode Nonzero Sum Two-Player Games

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1 Exchange Markets with Leontief Utilities

Suppose there are n players in the exchange market, each of them has a divisible good different from the others, i.e. player i is endowed with one unit of good i initially, and every player has a Leontief utility function, i.e. for player i, its utility function is

$$u_i(x) = \min_j \frac{x_{ij}}{a_{ij}}$$

where $a_{ij} > 0$ for all $1 \leq i, j \leq n$ (they are given as a square matrix called Leontief matrix), and x_{ij} is the units of good j acquired by player i.

To trace the origin of this Leontief utility function, recall that the CES (Constant Elasticity of Substitution) utility functions could be expressed as

$$u(x_1, \cdots, x_n) = (\sum_i \alpha_i x_i^\rho)^{1/\rho}$$

where $\alpha_i > 0$, $\sum_i \alpha_i = 1$ and $\rho \leq 1$. If we let $\rho \to -\infty$, meaning perfect complementarity, we are going to have

$$u(x_1, \cdots, x_n) = x_{i^*},\ i^* = argmin_i x_i,$$

and we call this asymptotic form of CES utility Leontief's utility.

Now we go back to the Leontief exchange economy, we find that if the price of good i is p_i, then for player i, he has a budget constraint, by the above setup and Walras' Law, as

$$\sum_j u_i a_{ij} p_j = p_i$$

2 Linear Complementarity Problem

Following the above argument, we derive the complementarity condition:

$$p_j(\sum_i u_i a_{ij} - 1) = 0$$

In order to write down the matrix form of the above complementarity condition, we define

$$\mathcal{B} = \{i : p_i > 0\},\ \mathcal{N} = \{i : p_i = 0\},\ A = (a_{ij}) \in \mathbb{R}^{n \times n},\ U = diag(u),$$

then

$$UAP = P,\ A^T U \leq e,\ P(e - A^T U) = 0,\ U, P \geq 0,$$

could be splitted as:

$$U_\mathcal{B} A_{\mathcal{B}\mathcal{B}} P_\mathcal{B} = P_\mathcal{B},\ A_{\mathcal{B}\mathcal{B}}^T U_B = e,\ (A^T)_{\mathcal{B}\mathcal{N}} U_\mathcal{N} \leq e,\ U_\mathcal{B}, P_\mathcal{B} > 0.$$

3 Solving LCP as Solving NEs to Non-zero Sum Games

Lemma 1. *If $A_{\mathcal{BB}}$ is reduciable, then $U_{\mathcal{B}}A_{\mathcal{BB}}P_{\mathcal{B}} = P_{\mathcal{B}}$ has a solution for $P_{\mathcal{B}}$.*

Proof: $A_{\mathcal{BB}}$ is reduciable means that the graph corresponding to $A_{\mathcal{BB}}$ is a simply connected one, and being pre-multiplied by the diagonal matrix $U_{\mathcal{B}}$ won't change this topological property (it can only change the weight of the edges), so by the existence of the solution as the existence of stationary distribution of a markov chain on a simply connected graph, we know that $U_{\mathcal{B}}A_{\mathcal{BB}}P_{\mathcal{B}} = P_{\mathcal{B}}$ has a solution for $P_{\mathcal{B}}$. Another proof is purely based on matrix analysis and could be found in [2].

So the irreducibility of $A_{\mathcal{BB}}$ is an important assumption here. Next we show how this is correlated with Nash equilibria of some two matrix games (i.e. non zero-sum games).

Lemma 2. *Suppose $(B_{m\times n}, C_{m\times n})$ is a bimatrix game, $H = \begin{bmatrix} 0 & B \\ C^T & 0 \end{bmatrix}$, for a simplex $\rho \in \Delta^{m+n}$, the followings are equivalent:*

1. *1. ρ is a symmetric Nash equilibrium of a symmetric game defined by H.*

2. *2. ρ could be written as $\begin{bmatrix} \alpha x \\ (1-\alpha)y \end{bmatrix}$ where (x, y) is an NE of game (B, C).*

3. *3. $\rho = C\rho'$ where ρ' is the solution of LCP:*

$$\rho' A + \gamma = 1, \ \ \rho, \gamma > 0.$$

Proof: *The equivalence of 1. and 2. are well proved as Theorem 4 in [1], and the equivalence of 2. and 3. are proved as Lemma 1 and Lemma 2 in [1]. To discuss in detail, we sketch the proofs showing 2. and 3. are equivalent: assume 2. holds, i.e. $\beta = (\beta_1, \cdots, \beta_{n+m})$ be the vector of the utility values at equilibrium prices for the two-groups Leontief economy (B, C), then $\sum_i h_{ij}\beta_j \leq 1$ for all $1 \leq i \leq n+m$, and $\beta_j > 0$ if and only if $\pi_j > 0$. In addition, $\beta_i > 0$ implies $\sum_i h_{ij}\beta_j = 1$. Thus β solves the above LCP; on the other hand, assume 3. holds, i.e. Let $w \neq 0$ be any complementarity solution to the above LCP. Partition the index set $\{1, 2, \cdots, m+n\} = P \sqcup Z$ where $P = \{j : \ w_j > 0\}$ and $Z = \{j : \ w_j = 0\}$. Then we could prove that there exists $\pi_j > 0$ for each $j \in P$ such that $w_j = \pi_j / \sum_{k\in P} h_{kj}\pi_k$ by analyzing the principal submatrix of H induced by the indices in P. Then we argue that π is an equilibrium: for $j \in P$, we have $w_j = \pi_j / \sum_{k\in P} h_{kj}\pi_k = \pi_j / \sum_k h_{kj}\pi_k$, however, for $j \in Z$, we have $w_j = \pi_j / \sum_{k\in P} h_{kj}\pi_k = 0$ as $\sum_k h_{kj}\pi_k > 0$, moreover, as w is a solution of LCP, we have for each good i such that $1 \leq i \leq n+m$, $\sum_j h_{ij}w_j \leq 1$. So the conditions for an equilibrium are satisfied.*

Therefore, we have the following conclusion:
Finding an NE to game (B, C) $\Leftrightarrow$ Finding an NE for $\boldsymbol{A}$ $\Leftrightarrow$ Solving the original LCP.

For more information, consult [1].

References

[1] B. Codenotti, A. Saberi, K. Varadarajan, Y. Ye, Leontief Economies Encode Nonzero Sum Two-Player Games, SODA 2006.

[2] Y. Ye, On The Complexity of Approximating a KKT Point of Quadratic Programming. Math. Program. 80, 195212 (1998)