**MS&E 334: Computation of Equilibria** **Lecture 6 - 05/12/2009**

Correlated Equilibria

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# 1 The Chicken-Dare Game

The *chicken-dare game* can be throught of as two drivers racing towards an intersection. A player can chose to *dare* ($d$) and pass through the intersection or *chicken out* ($c$) and stop. The game results in a draw when both players chicken out and the worst possible outcome if they both dare. A player wins when he dares while the other chickens out. The game has one possible payoff matrix given by

| | $d$ | $c$ |
|---|---|---|
| $d$ | $0,0$ | $4,1$ |
| $c$ | $1,4$ | $3,3$ |

with two pure strategy Nash equilibria $(d,c)$ and $(c,d)$ and one mixed equilibrium where each player mixes the pure strategies with probability 1/2 each.

Now suppose that prior to playing the game the players performed the following experiment. The players draw a ball labeled with a strategy, either ($c$) or ($d$) from a bag containing three balls labelled $c, c, d$. The players then agree to follow the strategy suggested by the ball. It can be verified that there is no incentive to deviate from such an agreement since the suggested strategy is best in expectation.

This experiment is equivalent to having the following strategy profile chosen for the players by some third party, a *correlation device.*

| | $d$ | $c$ |
|---|---|---|
| $d$ | 0 | 1/3 |
| $c$ | 1/3 | 1/3 |

This matrix above is not of rank one and so is not a Nash profile. And, the social welfare in this scenario is 16/3 which is greater than that of any Nash equilibrium.

# 2 Correlated Equilibrium

We consider players $p = 1, 2, \ldots, n$ each with strategy set $S_p$ defining the strategy profile

$$S = \prod_{p=1}^{n} S_p$$

with $S_{-q}$ denoting the profile for all players except player $q$. The payoffs for each player player $p$ are functions $u^p$ on the strategy profile $S$ into the nonnegative integers.

We define $x$ as a distribution on $S$ (i.e $x \geq 0$ and $\sum_{s\in S} x_s = 1$) where for $\bar{s} \in S_{-p}$ we denote by $x_{i,\bar{s}}$ the probability that player $p$ takes strategy $i$ while everyone else plays $\bar{s}$. Similarly, $u^p_{i,\bar{s}}$ is the payoff to player $p$ for taking strategy $i \in S_p$ while everyone else plays $\bar{s}$.

The distribution $x$ is a *correlated equilibrium* (CE) if and only conditioned on player $p$ accepting the recommended strategy $i$

$$\sum_{\bar{s}\in S_{-p}} u^p_{i,\bar{s}} x_{i,\bar{s}} \geq \sum_{\bar{s}\in S_{-p}} u^p_{j,\bar{s}} x_{i,\bar{s}} \quad \forall p \text{ and } \forall i,j \in S_p$$

i.e. the expected payoff from playing the recommended strategy is no worse than playing any other strategy.

Then, correlated equilibrium conditions can be written as the following linear program.

$$\begin{aligned} \sum_{\bar{s}\in S_{-p}} \left(u^p_{i,\bar{s}} - u^p_{j,\bar{s}}\right) x_{i,\bar{s}} \geq 0 \quad & \forall p \text{ and } \forall i,j \in S_p \\ x_s \geq 0 \quad \forall s \in S \quad & \qquad\qquad (CE) \\ \sum_{s\in S} x_s = 1 \quad & \end{aligned}$$

We can also optimize for some objective function, for example the social welfare, over these conditions. Notice that the number of variables in this LP is the cardinality of $S$. Therefore this LP is exponentially long. However, the number of constraints which include the utilities is $\sum_{p=1}^{n} \binom{|S_p|}{2}$ which is polynomial in the number of players and number of strategies per player.

## 3 Existence

We consider the following LP in the variables $x_s$ for $s \in S$ and its dual program

$$\begin{aligned} \max \sum_s x_s & & & \\ Ux \geq\ 0 & \qquad (P) \qquad & `U^T y \leq -1 & \qquad (D) \\ x \geq\ 0 & & y \geq 0 & \end{aligned}$$

where the constraint matrix $U$ are the utility coefficients from the definition of $(CE)$ i.e. row $i,j,p$ with $i,j \in S_p$ and column $s \in S$ entry $u^p_{i,s} - u^p_{j,s}$. With the exception of the trivial solution, which cannot correspond to any probability distribution, all solutions of $(P)$ are unbounded. Then by the Weak Duality theorem, the dual $(D)$ is infeasible.

Furthermore, for any probability distribution and any dual feasible $y$ the quantity $xU^T y$ clearly must be negative. We now show that there exists $x \neq 0,\ x \geq 0$ such that $x^T U^T y = 0$ for all $y \geq 0$. We first expand the quantity $x^T U^T y$ as follows.

$$\begin{aligned} x^T U^T y &= y^T U x \\ &= \sum_{i,j,p} y^p_{ij} \sum_{\bar{s}\in S_{-p}} (u^P_{i,\bar{s}} - u^p_{j,\bar{s}}) x_{i,\bar{s}} \\ &= \sum_{i,j,p} \sum_{\bar{s}\in S_{-p}} y^p_{ij} u^p_{i,\bar{s}} x_{i,\bar{s}} - \sum_{i,j,p} \sum_{\bar{s}\in S_{-p}} y^p_{ij} u^p_{j,\bar{s}} x_{i,\bar{s}} \qquad (*) \end{aligned}$$

Now consider $x$ which is a product distribution, that is $x_s = \prod_{p=1}^{n} x^p_{s^p}$ with $s = (s^1, s^2, \ldots, s^n) \in S$ and player $p$ strategy profile $x^p$. For player $p$ with strategy $k \in S_p$ and $\bar{s} \in S_{-p}$ the coefficient of utility $u^p_{k,\bar{s}}$ is obtained by fixing a player $p$ and strategy $k \in S_p$ in $(*)$. The coefficient is

$$\begin{aligned} \sum_{j\in S_p} y^p_{kj} x_{k,\bar{s}} - \sum_{i\in S_p} y^p_{ik} x_{i,\bar{s}} &= x_{k,\bar{s}} \sum_{j\in S_p} y^p_{kj} - \prod_{q\neq p} x^p_{s^q} \sum_{i\in S_p} y^p_{ik} x^p_i \\ &= \prod_{q\neq p} x^q_{s^q} \left[ x^p_k \sum_{j\in S_p} y^p_{kj} - \sum_{i\in S_p} y^p_{ik} x^p_i \right]. \end{aligned}$$

Since $x^T U^T y$ is linear in the utilities we can normalize the $y_{ij}^p$'s such that for each $p$ we have $\sum_j y_{ij}^p = 1$. We then have the coefficient

$$\prod_{q \neq p} x_{s^q}^q \left[ x_k^p - \sum_{i \in S_p} y_{ik}^p x_i^p \right]$$

multiplied by some normalizing factor.

Notice that these are equations for the stationary distribution $x^p$ of a Markov chain for player $p$ with transition matrix given by the normalized $y_{ij}^p$'s. Defining a product distribution $x$ using these stationary distributions for each player then ensures that $x^T U^T y = 0$. We conclude that $(P)$ is infeasible, then solutions to $(P)$ are unbounded and therefore a correlated equilibrium $(CE)$ exists.

# 4 Computation

The algorithm for computing a *correlated equilibrium* in polynomial time relies on the fact that the dual, unlike the primal, has polynomially number of variables. The steps of the algorithm are as follows.

- Run $k$ step of the ellipsoid method producing $k$ candidate points $y_i$.
- Compute distributions $x_1, x_2, \ldots, x_k$ such that $x_i^T U^T y_i = 0$.
- Let $X$ be a matrix of rows $x_i$ and compute $\alpha \geq 0$ such that $(UX^T)\ \alpha \geq 0$.

In the first step we attempt to solve $(D)$ with the ellipsoid method, which in polynomially many steps should determine that the program is infeasible. Each step the ellipsoid method produces a candidate solution $y_i$. Terminate after $k$ produced a sequence $y_1, y_2, \ldots, y_k$ and by the results of the previous section we can find $x_1, x_2, \ldots, x_k$ such that $x_i^T U^T y_i = 0$. Therefore, $(XU^T)y \leq -1$ is also an infeasible linear program, the dual of which is given by $(UX^T)\alpha \geq 0$. This linear program is unbounded with $X^T\alpha$, a distribution satisfying the original program $(P)$, implying a *correlated equilibrium.*

The first two steps of the algorithm are polynomial in the number of players and number of strategies per player. The final linear program is of polynomial size and $\alpha$ can be computed efficiently. However, the construction of matrix $UX^T$ is not, as the number of columns of $U$ is exponential. We now have the following theorem.

**Theorem:** *There is a solution of* $(CE)$ *that is a convex combination of polynomially many product distributions. Furthermore, an oracle for computing* $U^T X$ *yields a polynomial time algorithm for computing correlated equilibria.*

Such an oracle is available for many classes of succinct games including congestion games, graphical games, polymatrix games, symmetric games, etc [1].

# 5 Properties

An interesting property shown by [S. Hart, A. Mas-Colell, 2000] is that if players play the game repeatedly and are allowed to depart from strategies which they regret, the empirical distribution of play approaches a correlated equilibrium. That is, history can act as a correlation device with learning dynamics approaching the set of correlated equilibria $(CE)$.

We can also show that $(CE)$ has the following properties.

- Every Nash equilibrium is a correlated equilibrium.
- Every Nash equilibrium lies on the boundary of $(CE)$.

The first is trivially true since every Nash equilibrium will satisfy $(CE)$ conditions. For the second property, notice that the space of correlated equilibria is convex and nonempty described by equations $(CE)$. We can show that for every Nash equilibrium distribution at least one inequality in $(CE)$ is satisfied exactly.

If the one of the player's support does not include a strategy, then clearly the probability of the state $t$ where the player takes that strategy is zero. Thus $x_t = 0$ in $(CE)$ and the Nash equilibrium must be on the corresponding face of the convex polygon.

If all strategies of each player are in the supports then the utility equations must take the form

$$\sum_{\bar{s} \in S_{-p}} \left( u^p_{i,\bar{s}} - u^p_{j,\bar{s}} \right) x_{i,\bar{s}} = 0 \qquad \forall i.j \in S_p$$

i.e. the strategies should give the same payoff for the mixed Nash equilibrium. So, the Nash equilibrium is again on the boundary.

# References


[1] C.H. Papadimitriou, T. Roughgarden, *Computing Correlated Equilibria in Multiplayer Games*, Journal of the ACM, Vol. 55, No. 3, Article 14, July 2008.