Optimal Asset Management Contracts with Hidden Savings

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November 2018

Abstract

We characterize optimal asset management contracts in a classic portfolio-investment setting. When the agent has access to hidden savings, his incentives to misbehave depend on his precautionary saving motive. The contract dynamically distorts the agent’s access to capital to manipulate his precautionary saving motive and reduce incentives for misbehavior. We provide a sufficient condition for the validity of the first-order approach: if the agent’s precautionary saving motive weakens after bad outcomes, the contract is globally incentive compatible. We extend our results to incorporate market risk, hidden investment, and renegotiation.

1 Introduction

Delegated asset management is ubiquitous in modern economies, from fund managers investing in financial assets to CEOs or entrepreneurs managing real capital assets. To align incentives, agents must retain a risky stake in their investment activity. However, hidden savings pose a significant challenge. Agents can save to self insure against bad outcomes, undoing the incentive scheme. In this paper we characterize optimal dynamic asset management contracts when the agent has access to hidden savings.

This paper has two main contributions. First, we build a model of delegated asset management based on a classic environment of portfolio investment. An agent with CRRA preferences continuously invests in risky assets, but can secretly divert returns and has access to hidden savings. These assumptions lead to a tractable characterization of the optimal contract and the dynamic distortions induced by hidden savings. Second, on the

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methodological side, we provide a general verification theorem for global incentive compat-
ibility that is valid for a wide class of contracts. Global incentive compatibility is ensured as long as the agent’s precautionary saving motive weakens after bad outcomes.

When the agent has access to hidden savings, his incentives to divert funds depend on his precautionary motive. If the agent expects a risky consumption stream in the future, hidden savings which allow him to self insure are very valuable. This makes fund diversion to build up savings attractive. The optimal contract must therefore manage the agent’s precautionary saving motive by committing to limit his future risk exposure, especially after bad outcomes. This is accomplished by restricting the agent’s access to capital, since giving the agent capital to manage requires exposing him to risk in order to align incentives. By promising a future amount of capital that is ex-post inefficiently low, the principal makes fund diversion less attractive today and reduces the cost of giving capital to the agent up front. This is optimal ex-ante. This dynamic tradeoff leads to history-dependent distortions in the agent’s access to capital and a skewed compensation scheme. After good outcomes his access to capital improves, which allows him to keep growing rapidly. After bad outcomes he gets starved for capital and stagnates. The flip side is that the agent’s consumption is somewhat insured on the downside, and he is punished instead with lower consumption growth. This incentive scheme limits the agent’s precautionary motive.

One of the main methodological contributions of this paper is to provide an analytical verification of the validity of the first-order approach. This is an old, hard problem in the theory of dynamic contracts because the agent has such a rich action space. The first-order approach, introduced to the problem of hidden savings by Werning (2001), incorporates the agent’s Euler equation as a constraint in the contract design problem. This ensures the agent does not have a profitable deviation with savings alone. However, the agent may find it attractive to divert funds and save for when he is punished for poor performance in the future. We prove the validity of the first order approach analytically by establishing an upper bound on the agent’s continuation utility off-path, for any deviation and after any history. Global incentive compatibility is ensured if the agent’s precautionary saving motive becomes weaker after bad outcomes. Intuitively, if the contract becomes less risky after bad outcomes, hidden savings become less valuable to the agent after diverting funds. It’s worth stressing that this verification argument applies beyond the optimal contract, to any contract in which the precautionary saving motive weakens after bad outcomes.

Since the principal uses dynamic distortions to the agent’s access to capital to provide incentives, it is natural to ask how hidden investment and renegotiation affect results. Fortunately, both issues can be easily handled. If the agent is able to secretly invest in risky capital, the principal is limited in his ability to restrict his access to capital. The

\footnote{Kocherlakota (2004) provides a well-known example in which double deviations are profitable (and the first-order approach fails) when cost of effort is linear. To establish the validity of the first-order approach, the existing literature has often pursued the numerical option (e.g. see Farhi and Werning (2013)).}
constraint becomes binding only after sufficiently bad outcomes, when the principal would like to reduce capital managed by the agent so much that the agent would find it attractive to invest on his own (i.e. without sharing risk with the principal). We can add to our model observable market risk that commands a premium, and allow the agent to also invest his hidden savings in the market. Our agency model can therefore be fully embedded within the standard setting of continuous-time portfolio choice.

A second concern is that the optimal contract requires commitment. The principal relaxes the agent’s precautionary saving motive by promising an inefficiently low amount of capital in the future. It is therefore tempting at that point to renegotiate and start over. To address this issue, we also characterize the optimal renegotiation-proof contract. This leads to a stationary contract that restricts the agent’s access to capital in a uniform way.

Finally, we map our optimal contract into a classic portfolio-consumption problem with a dynamic leverage constraint that limits the agent’s risk exposure. This constraint reduces the agent’s precautionary motive and allows the agent’s incentive constraints to hold with a lower retained equity stake, improving risk sharing. Hidden savings therefore link retained equity constraints and leverage constraints, which are widely used in applied macro-finance work. The logic is completely different than that of models with limited commitment such as Hart and Moore (1994) and Kiyotaki and Moore (1997). The leverage constraint exists not because the agent can walk away, but because it limits risk and therefore helps relax the equity constraint.

Literature Review. This paper fits within the literature on dynamic agency problems. It builds upon the standard recursive techniques, but adds the problem of persistent private information. In our case, about the agent’s hidden savings. There is extensive literature that uses recursive methods to characterize optimal contracts, including Spear and Srivastava (1987), Phelan and Townsend (1991), Sannikov (2008), He (2011), Biais et al. (2007), and Clementi and Hopenhayn (2006). The agency problem we study is one of cash flow diversion, as in DeMarzo and Fishman (2007), DeMarzo and Sannikov (2006) and DeMarzo et al. (2012), but unlike their models we have CRRA rather than risk neutral preferences. With risk-neutral preferences, the optimal contracts with and without hidden savings are the same. Once concave preferences are introduced, hidden savings become binding. Rogerson (1985) shows that the inverse Euler equation, which characterizes the optimal contract without hidden savings, implies that the agent would save if he could. This opens the door to potential distortions to control the agent’s incentives to save, but also presents the problem of double deviations. In some settings distortions do not arise, e.g. the CARA settings, such as He (2011), and Williams (2013), where the ratio of the agent’s current utility to continuation utility is invariant to contract design.\footnote{Likewise, the dynamic incentive accounts of Edmans et al. (2011) exhibit no distortions either, as hidden action enters multiplicatively and project size is fixed.} However, when distortions
do arise, it is difficult to characterize the specific form they take, and the first-order approach may fail. We are able to provide a sharp characterization of the optimal contract and the distortions generated by the presence of hidden savings, and provide a verification theorem for global incentive compatibility which is valid for a wide class of contracts.

Our paper is related to the literature on persistent private information, since the agent has private information about savings. The growing literature in this area includes the fundamental approach of Fernandes and Phelan (2000), who propose to keep track of the agent’s entire off-equilibrium value function, and the first-order approach, such as He et al. (2017) and DeMarzo and Sannikov (2016) who use a recursive structure that includes the agent’s “information rent,” i.e., the derivative of the agent’s payoff with respect to private information. In all cases, in designing the contract we have to evaluate the agent’s payoff off-path, after actions which the agent is not supposed to take, in order to ensure that the agent’s incentives to refrain from those actions are designed properly. In our case, since we want to control the agent’s incentives to save secretly, the agent’s information rent is marginal utility of consumption (of an extra unit of savings). Key common issues are (1) the way that information rents enter the incentive constraint, (2) distortions that arise from this interaction and (3) forces that affect the validity of the first-order approach. In our case, we verify the validity of the first-order approach by characterizing an analytic upper bound, related to the CRRA utility function, on the agent’s payoff after deviations. The bound coincides with the agent’s utility on path (hence the first-order approach is valid), and its derivative with respect to hidden savings is the agent’s “information rent.” Farhi and Werning (2013) have verified the first-order approach numerically by computing the agent’s value function after deviations explicitly, in the context of insurance with unobservable skill shocks. Other papers that study the problems of persistent private information via a recursive structure that includes information rents include Garrett and Pavan (2015), Cisternas (2014), Kapička (2013), and Williams (2011).

We use a classic portfolio-investment environment widely used in macroeconomic and financial applications. Our model provides a unified account of equity and leverage constraints, which are two of the most commonly used financial frictions in the macro-finance literature in the tradition of Bernanke and Gertler (1989) and Kiyotaki and Moore (1997). Di Tella (2016) adopts a version of our setting without hidden savings to study optimal financial regulation policy in a general equilibrium environment. Our paper is also related to models of incomplete idiosyncratic risk sharing, such as Aiyagari (1994) and Krusell and Smith (1998). Here the focus is on risky capital income, as in Angeletos (2006) or Christiano et al. (2014) (rather than risky labor income). This affords us a degree of scale invariance

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that allows us to provide a sharp characterization of the optimal contract. In our setting, after good performance, the agent does not need to be retired nor outgrow moral hazard as in Sannikov (2008) or Clementi and Hopenhayn (2006) respectively. Likewise, since the project can be scaled down, neither will the agent retire after sufficiently bad outcomes as in DeMarzo and Sannikov (2006). Rather, the optimal contract dynamically scales the size of the agent’s fund with performance, taking into account his precautionary saving motive. Access to capital provides the principal with an important incentive tool. He is able to relax the incentive constraints and improve risk sharing by committing to distort project size below optimum over time and after bad performance. This result stands in contrast to Cole and Kocherlakota (2001), where project scale is fixed and the optimal contract is risk-free debt. We recover the result of Cole and Kocherlakota (2001) only in the special case when the agent can secretly invest on his own just as efficiently as through the principal, so the principal cannot control the scale of investment at all.

2 The model

We consider a setting in which an agent manages risky capital and contracts with a set of outside investors to raise funds and share risk. We may think of these investors collectively as “the principal”, whose objective is to maximize the market value of their payoff, so we may also refer to them as “the market”.4

The agent can manage capital and obtain a risky return, but may also divert returns to obtain a private monetary benefit. If the diversion rate is $a_t$, the observed return per dollar invested in capital is

$$dR_t = (r + \alpha - a_t) \, dt + \sigma \, dZ_t$$

(1)

where $r > 0$ is the risk-free rate, $\alpha > 0$ the excess return of his investment, and $\sigma > 0$ its idiosyncratic volatility. $Z$ is a Brownian motion that represents agent-specific idiosyncratic risk. If the agent is a fund manager, it represents the outcome of his particular investment/trading activity; if he is an entrepreneur it represents the outcome of his particular project. There is a complete financial market with equivalent martingale measure $Q$ where idiosyncratic risk is not priced, so $Q = P$. In the Online Appendix we allow for aggregate risk with a market price, such that $Q \neq P$.

Diversion of $a_t \geq 0$ gives the agent a flow of $\phi a_t k_t$ funds, where $k_t \geq 0$ is the capital the agent is managing. For each stolen dollar, the agent keeps only fraction $\phi \in (0, 1)$. If the agent also receives payments $c_t \geq 0$ from the principal and consumes $\tilde{c}_t \geq 0$, then his

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4Even if there isn’t a single principal, the agent himself would like to commit to a contract to then sell shares to dispersed outside investors.
hidden savings $h_t \geq 0$ evolve according to $^5$

$$dh_t = (rh_t + c_t - \tilde{c}_t + \phi k_t a_t) \ dt. \quad (2)$$

The agent invests his hidden savings at the risk-free rate $r$. Later we will introduce hidden investment, and allow the agent to also invest his hidden savings in risky capital.

The principal observes returns $R$ but not the agent’s diversion $a$, consumption $\tilde{c}$, or hidden savings $h$. The principal can commit to a fully history-dependent contract $C = (c, k)$ that specifies payments to the agent $c_t$ and capital $k_t$ as functions of the history of realized returns $R$ up to time $t$. After signing the contract $C$ the agent chooses a strategy $(\tilde{c}, a)$ that specifies $\tilde{c}_t$ and $a_t$, also as a function of the history of returns $R$.

The agent has CRRA preferences. Given contract $C$, under strategy $(\tilde{c}, a)$ the agent gets utility

$$U^\tilde{c},a_0 = \mathbb{E}\left[\int_0^\infty \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} e^{-\theta t} dt\right] \quad (3)$$

Given contract $C$, we say a strategy $(\tilde{c}, a)$ is feasible if 1) there is a finite utility $U^\tilde{c},a_0$, and 2) $h_t \geq 0$ always. Let $S(C)$ be the set of feasible strategies $(\tilde{c}, a)$ given contract $C$.

The principal pays for the agent’s consumption, but keeps the excess return $\alpha$ on the capital that the agent manages. He tries to minimize the cost of delivering utility $u_0$ to the agent

$$J_0 = \mathbb{E}^Q \left[\int_0^\infty e^{-rt} (c_t - k_t \alpha) dt\right] \quad (4)$$

A standard argument in this setting implies that the optimal contract must implement no stealing, i.e. $a = 0$. In addition, without loss of generality and for analytic convenience, we can restrict attention to contracts in which $h = 0$ and $\tilde{c} = c$, i.e. the principal saves for the agent.$^6$ Of course, the optimal contract has many equivalent and more natural forms, in which the agent maintains savings, but all these forms can be deduced easily from the optimal contract with $\tilde{c} = c$.

We say a contract $C = (c, k)$ is admissible if 1) there is a finite utility $U^{c,0}_0$, and 2)$^7$

$$\mathbb{E}^Q \left[\int_0^\infty e^{-rt} |c_t + k_t \alpha| dt\right] < \infty \quad (5)$$

$^5$We don’t allow negative hidden savings, $h_t \geq 0$. This is without loss of generality if the contract can exhaust the agent’s credit capacity.

$^6$Lemma O.1 in the Online Appendix establishes this in a more general setting with both aggregate risk and hidden investment.

$^7$This assumption plays the role of a no-Ponzi condition, making sure the principal’s objective function is well defined. It rules out exploding strategies where the present value of both consumption and capital is infinity.
We say an admissible contract $C$ is incentive compatible if

$$(c, 0) \in \arg \max_{(\tilde{c}, a) \in \mathcal{S}(C)} U_{0}^{\tilde{c}, a}$$

Let $\mathcal{I}C$ be the set of incentive compatible contracts. For an initial utility $u_0$ for the agent, an incentive compatible contract is optimal if it minimizes the cost of delivering initial utility $u_0$ to the agent

$$v_0 = \min_{(c, k)} J_0$$

subject to:

$$U_{0}^{c, 0} \geq u_0$$

$$(c, k) \in \mathcal{I}C$$

By changing $u_0$ we can trace the Pareto frontier for this problem.

To make the problem well defined and avoid infinite profits/utility, we assume throughout that

$$\rho > r(1 - \gamma).$$

(6)

If this condition fails the agent can obtain infinite utility simply by investing in a risk-free bond with return $r > 0$. Also, if $\alpha$ is too large, the principal’s profit can be infinite. We assume the following necessary condition throughout

$$\alpha \leq \bar{\alpha} \equiv \phi \sigma \gamma \sqrt{2/(1 + \gamma)} \sqrt{\rho - r(1 - \gamma)}/\gamma.$$  

(7)

If this fails, the principal can get infinite profit through simple stationary contracts described in Section 3. The exact bound on $\alpha$ where the profit from dynamic contracts is finite can be found only numerically, but the following conditions are sufficient,\(^8\)

$$\alpha \leq \begin{cases} 
\phi \sigma \sqrt{2} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma}} & \text{if } \gamma \geq 1/2 \\
\phi \sigma \gamma \sqrt{2(1 - \gamma)} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma}} & \text{if } \gamma \leq 1/2.
\end{cases}$$

Remark. Notice that we build the model with the aim of tractability, and to highlight our main object of interest, hidden savings. CRRA preferences together with a scalable investment technology give us scale invariance. In particular, if contract $(c, k)$ gives utility $u_0$ to the agent and has cost $v_0$ to the principal, then the scaled version $(\lambda c, \lambda k)$ has utility $\lambda^{1-\gamma} u_0$ and cost $\lambda^2 v_0$. If the former contract is optimal for utility $u_0$, then the latter is optimal for utility $\lambda^{1-\gamma} u_0$. The scale invariance property reduces the problem by one dimension—the utility dimension—and allows us to highlight the dimension central to our interest—the dimension of the precautionary savings motive, as the analysis of the next

\(^8\)See Lemma O.8 and Lemma O.5 in the Online Appendix for details.
section shows.

3 Solving the model

We solve the model as follows. We first derive necessary first-order incentive-compatibility conditions for the agent’s effort and savings choice, using two appropriate state variables: the agent’s continuation utility and consumption level. This allows us to formulate the principal’s relaxed problem, minimizing the cost subject to only first-order conditions, as a control problem, and characterize it with an HJB equation.

We then derive a sufficient condition for global incentive compatibility (against all deviations, not just local), which uses the same two state variables. The condition is on-path, i.e. for a particular recommended strategy of the agent, but it is sufficient because it allows us to bound the agent’s payoff off-path after arbitrary deviations. We show that the solution to the relaxed problem satisfies the sufficient condition, thereby proving it is the optimal contract. More generally, the sufficient condition identifies a whole class of globally incentive compatible contracts, and is useful in a broader context as we show in the next section.

Incentive compatibility

We use the continuation utility of the agent as a state variable for the contract

\[ U_{c,0}^t = \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(s-t)} \frac{c_{s}^{1-\gamma}}{1-\gamma} \, ds \right] \]

First we obtain the law of motion for the agent’s continuation utility.

**Lemma 1.** For any admissible contract \( C = (c,k) \), the agent’s continuation utility \( U_{c,0}^t \) satisfies

\[ dU_{c,0}^t = \left( \rho U_{c,0}^t - \frac{c_t^{1-\gamma}}{1-\gamma} \right) \, dt + \Delta_t \left( dR_t - (\alpha + r) \, dt \right) \]

for some stochastic process \( \Delta \).

Faced with this contract, the agent might consider stealing and immediately consuming the proceeds, i.e. following a strategy \((c + \phi ka, a)\) for some \( a \), which results in savings \( h = 0 \). The agent adds \( \phi k_t a_t \) to his consumption, but reduces the observed returns \( dR_t \), and therefore his continuation utility \( U_{c,0}^t \), by \( \Delta_t a_t \). Incentive compatibility therefore requires

\[ 0 \in \arg \max_{a \geq 0} \frac{(c_t + \phi k_t a)^{1-\gamma}}{1-\gamma} - \Delta_t a \]
Taking FOC yields
\[ \Delta_t \geq c_t^{-\gamma} \phi k_t \] (10)
which is positive. We need to give the agent some “skin in game”, which exposes him to risk. This is costly because the principal is risk-neutral with respect to \( Z \) so he would like to provide full insurance to the agent.

Notice how the private benefit of the hidden action depends on the marginal utility of consumption, \( c_t^{-\gamma} \). The principal will therefore take into account that giving the agent consumption not only delivers utility, as captured by (8), but also relaxes the IC constraint (10). As a result, the principal would like to control the agent’s consumption. Without hidden savings, the principal would impose the Inverse Euler equation, i.e. \( e^{(r-\rho)t}c_t^{-\gamma} \) would be a martingale. But the agent can secretly save and consume when the marginal utility of consumption is higher, so the optimal contract must respect the agent’s Euler equation: the discounted marginal utility \( e^{(r-\rho)t}c_t^{-\gamma} \) must be a supermartingale.

The following lemma summarizes all the necessary conditions for incentive compatibility and their implication on the path of the agent’s consumption. We present sufficient conditions in the next subsection.

**Lemma 2.** If \( C = (c,k) \) is an incentive compatible contract, then (10) must hold and the agent’s consumption must satisfy
\[ \frac{dc_t}{c_t} = \left( \frac{r-\rho}{\gamma} + \frac{1+\gamma}{2} (\sigma_c^2) \right) dt + \sigma_c^2 \frac{1}{\sigma} (dR_t - (\alpha + r) dt) + dL_t \] (11)
for some stochastic process \( \sigma^c \) and a weakly increasing stochastic process \( L \).

Equation (11) imposes a lower bound on the growth rate of the agent’s consumption. The first term \( \frac{r-\rho}{\gamma} \) captures the benefit of postponing consumption without risk, given by the risk-free rate \( r \), the discount rate \( \rho \), and the elasticity of intertemporal substitution \( 1/\gamma \). The second term \( \frac{1+\gamma}{2} (\sigma_c^2) \) captures the agent’s precautionary saving motive. A risky consumption profile induces the agent to postpone consumption to self-insure, resulting in a steeper consumption profile.

It’s worth emphasizing the link between the precautionary motive and incentives to divert funds, which will play a central role in the optimal contract. Exposing the agent to risk today helps provide incentives for good behavior today. But exposing the agent to risk in the future creates a precautionary motive that makes fund diversion today more attractive. The intuition is that when faced with risk, the agent places a large value on hidden savings he can use to self-insure, which makes fund diversion more attractive. The

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9 The agent can expect lower marginal utility in the future, because he can’t borrow. However, as it turns out, in the optimal contract the Euler equation holds as an equality, i.e. \( e^{(r-\rho)t}c_t^{-\gamma} \) is a martingale.
dynamic behavior of the optimal contract arises out of the principal’s attempt to manipulate the agent’s precautionary motive in order to relax the IC constraints.

**State space**

It is convenient to work with the following transformation of the state variables

\[ x_t = \left( (1 - \gamma) U_t^c, 0 \right)^{\frac{1}{1 - \gamma}} > 0 \]

\[ \hat{c}_t = \frac{c_t}{x_t} \geq 0 \]

Variable \( x \) is just a monotone transformation of continuation utility, but it is measured in consumption units (up to a constant). As a result, \( \hat{c} \) measures how front loaded the agent’s consumption is. The state \( \hat{c} \) is related to the agent’s precautionary saving motive for saving. If the agent faces risk looking forward, he will want to postpone consumption in an attempt to self insure (low \( \hat{c} \)). As a result, while \( x_t \) can take any positive value, \( \hat{c}_t \) has an upper bound.

**Lemma 3.** For any incentive compatible contract \( C \), we have for all \( t \)

\[ \hat{c}_t \leq \mathbb{E}_{\tilde{P}} \left[ \int_t^\infty e^{-\int_t^s \left( \frac{r - \gamma}{\gamma} - \frac{1 - 2\sigma^c s}{2} \right) ds} \right]^{-\frac{1}{1 - \gamma}} \leq \hat{c}_h \]  

(12)

where \( \tilde{P} \) is an equivalent measure such that \( Z_t - \int_0^t (1 - \gamma) \sigma^c ds \) is a \( \tilde{P} \)-martingale, and

\[ \hat{c}_h \equiv \left( \frac{\rho - r (1 - \gamma)}{\gamma} \right)^{\frac{1}{1 - \gamma}} > 0 \]  

(13)

If ever \( \hat{c}_t = \hat{c}_h \), then the continuation contract satisfies \( k_{t+s} = 0 \) and \( \hat{c}_{t+s} = \hat{c}_h \) at all future times \( t + s \) and gives the agent a unique deterministic consumption path with growth \( (r - \rho) / \gamma \). The continuation contract has cost \( \hat{v}_h x_t \) to the principal, where \( \hat{v}_h \equiv \hat{c}_h^2 \).

**Remark.** The first inequality in (12) comes from the term \( dL_t \geq 0 \) in (11) because the agent can’t secretly borrow, \( h_t \geq 0 \). In the optimal contract \( dL_t = 0 \) and the first inequality becomes an equality. This expression formalizes the notion that \( \hat{c}_t \) captures the agent’s precautionary motive coming from his exposure to risk in the future, computed under an appropriate change of measure. In the special case where \( \sigma^c_t = \sigma^c \) is constant, we have

\[ \hat{c}_t = \left( \frac{\rho - r (1 - \gamma)}{\gamma} - \frac{1 - 2\sigma^c \gamma}{2} \right)^{\frac{1}{1 - \gamma}} \]  

which is decreasing in \( \sigma^c \).

The upper bound \( \hat{c}_h \) corresponds to autarky, where the agent saves and consumes on his own without managing any capital. This contract minimizes the agent’s precautionary motive because it is fully safe. It minimizes the cost of the agent’s consumption, but is very costly because it doesn’t give any capital to the agent and so doesn’t take advantage
of the excess return $\alpha > 0$. $\hat{c} < \hat{c}_h$ means the agent expects to manage capital and be exposed to risk in the future, and we may think of $\hat{c}$ as indexing how risky the continuation contract is for the agent.

Using Ito’s lemma we can obtain laws of motion for $x_t$ and $\hat{c}_t$ from (8) and (11). Using the normalization $\Delta_t \sigma / U^{c_0}_t = (1 - \gamma) \sigma_x^2$, we obtain

$$\frac{dx_t}{x_t} = \left( \frac{\rho - \hat{c}_t^{-1} - \gamma}{1 - \gamma} + \frac{\gamma}{2} (\sigma_x^2)^2 \right) dt + \sigma_x dZ_t$$

(14)

and

$$\frac{d\hat{c}_t}{\hat{c}_t} = \left( \frac{\hat{c}_t^{1-\gamma} - \hat{c}_h^{1-\gamma}}{1 - \gamma} + \frac{\gamma (\sigma_x^2)^2}{2} + \gamma \sigma_x^2 \sigma_{\hat{c}}^2 + \frac{1 + \gamma (\sigma_x^2)^2}{2} \right) dt + \sigma_{\hat{c}} dZ_t + dL_t$$

(15)

with $\sigma_{\hat{c}} = \sigma_{\hat{c}} - \sigma_x^2$. The constraint (10) can be rewritten as

$$\sigma_x^2 \geq \hat{c}_t^{-\gamma} \hat{k}_t \phi \sigma_x$$

(16)

where $\hat{k}_t = \frac{k_t}{x_t}$. It will always be binding because conditional on $\sigma_x^2$ it is always better to give the agent more capital to manage. The IC constraint (16) establishes a link between the agent’s exposure to risk $\sigma_x^2$ and the amount of capital he manages $\hat{k}_t$ mediated by the marginal utility of consumption $\hat{c}_t^{-\gamma}$, which captures the agent’s precautionary saving motive.

The relaxed problem

The relaxed problem uses only the first-order conditions for incentive compatibility. It is a stochastic control problem with laws of motion for the state variables $x_t$ and $\hat{c}_t$ given by (14) and (15) with absorption at $\hat{c}_h$, controls $\sigma_x^2$ and $\sigma_{\hat{c}}^2$, and cost flow

$$c_t - k_t \alpha = x_t \left( \hat{c}_t - \hat{c}_t^{\gamma} \frac{\alpha}{\phi \sigma_x^2} \right)$$

(17)

This cost flow already incorporates the IC constraint (16).

The initial value of $x_0$ for this problem is determined by the initial utility $u_0$, but there is no corresponding constraint for $\hat{c}_0$—the principal can set $\hat{c}_0$ freely to minimize the cost. A low $\hat{c}$ means the agent expects a large exposure to risk in the future, so it determines the principal’s “budget for risk exposure”. If $\hat{c}_t$ ever reached the upper bound $\hat{c}_h$, the principal would have to give the agent the perfectly safe autarky contract without any capital, which is a costly way to deliver utility. On the other hand, a low $\hat{c}$ means the agent has large incentives to divert funds, as captured by the IC constraint (16). Hence, the choice of $\hat{c}_0$ takes into account the cost of committing to lower risk in the future and the benefit of
improved incentives now.

After $\hat{c}_0$ is set the principal chooses $\sigma^x_t$ and $\sigma^\hat{c}_t$ dynamically. Higher $\sigma^x_t$ today, i.e. exposing the agent to more risk, allows the principal to give more capital to the agent and earn the excess return $\alpha$. However, this is costly because a) the agent is risk averse and must be compensated with more utility in the future—the drift of $x_t$ is increasing in $\sigma^x_t$—and b) it eats up the budget for risk exposure in the future—the term $(\sigma^x_t)^2/2$ raises the drift of $\hat{c}_t$. The principal can choose $\sigma^\hat{c}_t$ to mitigate the latter effect: when $\sigma^x_t > 0$, setting $\sigma^\hat{c}_t < 0$ reduces the drift of $\hat{c}_t$, preserving the budget for risk exposure. The intuition is that a lower volatility of consumption, $\sigma^c_t = \sigma^x_t + \sigma^\hat{c}_t$, weakens the agent’s precautionary motive. This is the first of two forces that guide the dynamic choice of $\sigma^\hat{c}_t$.

The second force is that the benefit of a larger budget of risk exposure is proportional to $x_t$ due to scale invariance, so the principal prefers to have a larger budget for risk exposure when $x_t$ goes up. This also implies $\sigma^\hat{c}_t < 0$. As we show below analytically, $\sigma^\hat{c}_t < 0$ is a key property of the optimal contract: it implies that the contract gets safer after poor outcomes. In addition, we also show that $\sigma^\hat{c}_t < 0$ is a sufficient condition that ensures global incentive compatibility, so that the solution of the relaxed problem is indeed the optimal contract.

The HJB equation and the optimal contract

Because preferences are homothetic and the principal’s objective is linear, we know the principal’s cost function in the relaxed problem takes the form $v(x, \hat{c}) = \hat{v}(\hat{c}_t)x_t$ with $\hat{v}(\hat{c}_t) = \hat{v}_h > 0$. Notice that since we could always raise $\hat{c}_t$ using $dL_t$, we know that $\hat{v}(\hat{c})$ must be weakly increasing. In fact, we show below that $\hat{v}(\hat{c})$ increases strictly over the interval in which $\hat{c}_t$ stays over the course of the optimal contract, so we can drop the term $dL_t$ from what follows. We will also sometimes write $\hat{v}_t = \hat{v}(\hat{c}_t)$, and $\hat{v}$ instead of $\hat{v}(\hat{c})$.

The HJB equation associated with this problem is

$$r \hat{v} x = \min_{\sigma^x, \sigma^\hat{c}} \left( \hat{c} - \hat{k} \alpha \right)x + \mathbb{E}_t^Q \left[ d \left( \hat{v}_t x_t \right) \right]$$

subject to (14), (15), and (16), and $\hat{k} \geq 0$. Using Ito’s lemma and canceling the $x$ on both sides, we get

$$r \hat{v} = \min_{\sigma^x, \sigma^\hat{c}} \hat{c} - \sigma^x \sigma^\gamma \frac{\alpha}{\sigma} + \hat{v} \left( \frac{\beta - \hat{c}_1 - \gamma}{1 - \gamma} + \frac{\gamma}{2} (\sigma^x)^2 \right)$$

$$+ \hat{v}' \hat{c} \left( \frac{\hat{c}_1 - \gamma - \hat{c}_h - \gamma}{1 - \gamma} + \frac{(\sigma^x)^2}{2} + (1 + \gamma) \sigma^x \sigma^\hat{c} + \frac{1 + \gamma}{2} (\sigma^\hat{c})^2 \right) + \hat{v}'' \hat{c}^2 (\sigma^\hat{c})^2$$

Even though we have two state variables, $\hat{c}$ and $x$, the HJB equation boils down to a second order ODE in $\hat{c}$. This is a feature of homothetic preferences and linear technology that makes the problem more tractable.

The following Theorem characterizes the optimal contract in the relaxed problem. Fig-
ures 1 and 2 show the cost function and the drift and volatility of state variables \( x \) and \( \hat{c} \) for a numerical solution.

Theorem 1. The cost function of the relaxed problem \( \hat{v}(\hat{c}) \) has a flat portion on \([0, \hat{c}_l]\) and a strictly increasing portion on \([\hat{c}_l, \hat{c}_h]\), for some \( \hat{c}_l \in (0, \hat{c}_h) \). The HJB equation (18) holds with equality for \( \hat{c} \geq \hat{c}_l \). For \( \hat{c} < \hat{c}_l \), we have \( \hat{v}(\hat{c}) = \hat{v}(\hat{c}_l) \equiv \hat{v}_l \) and the HJB holds as an inequality

\[
A(\hat{c}, \hat{v}_l) \equiv \min_{\sigma^x} \hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi \sigma} - r\hat{v}_l + \hat{v}_l \left( \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x)^2 \right) > 0 \quad \forall \hat{c} < \hat{c}_l.
\] (19)

At \( \hat{c}_l \) the cost function \( \hat{v}(\hat{c}) \) satisfies \( \hat{v}'(\hat{c}_l) = 0 \), \( \hat{v}''(\hat{c}_l) > 0 \), and \( A(\hat{c}_l, \hat{v}_l) = 0 \). The cost function is always below the inverse of the marginal utility of consumption, \( \hat{v}(\hat{c}) < \hat{c}^\gamma \) for all \( \hat{c} \in [\hat{c}_l, \hat{c}_h] \), with \( \hat{v}(\hat{c}_h) = \hat{c}_h^\gamma \).

The state variables \( x_t \) and \( \hat{c}_t \) follow the laws of motion (14) and (15) with \( \sigma_t^x > 0 \) and \( \sigma_t^x < 0 \) for all \( t > 0 \), and \( dL_t = 0 \) always, so the Euler equation holds as an equality. The optimal contract starts at \( \hat{c}_0 = \hat{c}_l \) and immediately moves into the interior of the domain never reaching either boundary, that is, \( \hat{c}_t \in (\hat{c}_l, \hat{c}_h) \) for all \( t > 0 \).

At \( t = 0 \) we have \( \mu_0^x > 0 \) and \( \sigma_0^x = 0 \), and \( \sigma_0^x \) is chosen myopically, without taking into account its effect on the agent’s precautionary saving motive, to maximize

\[
\sigma^x \hat{c}_t^\gamma \frac{\alpha}{\phi \sigma} - \hat{v}(\hat{c}_t) \left( \frac{2}{2}(\sigma^x)^2 \right)
\] (20)

Let us highlight some features of the cost function and the optimal contract, drawing from Theorem 1 and the numerical examples in Figures 1 and 2. The shape of the cost function, with a flat portion up to \( \hat{c}_l \) followed by an increasing portion, reflects the optimal choice of \( \hat{c}_0 = \hat{c}_l \). When \( \hat{c} \) is inefficiently low, the agent expects a large exposure to risk in the future, and has a high precautionary savings motive that harms today’s incentives. Since the principal can raise \( \hat{c} \) to the optimal level of \( \hat{c}_l \) using the process \( dL_t \geq 0 \), the cost function is not decreasing but flat over \([0, \hat{c}_l]\). As \( \hat{c} \) moves above \( \hat{c}_l \), the contract must give the agent an inefficiently low risk exposure, with the end point of \( \hat{c}_h \) that corresponds to the safe autarky contract with no capital. As a result, the cost function \( \hat{v}(\hat{c}) \) is increasing over \([\hat{c}_l, \hat{c}_h]\). The marginal cost of delivering utility is below the inverse marginal utility of consumption, \( \hat{v} < \hat{c}^\gamma \), because higher \( \hat{c} \) not only delivers utility to the agent, but also relaxes the IC constraint.

Figure 2 illustrates several other properties, which are consistent across all numerical examples we computed. The bottom right panel shows that the agent’s risk exposure \( \sigma^x \) declines with \( \hat{c} \). This is intuitive: the principal exposes the agent to less risk when the “budget for risk exposure” gets depleted. Lower risk \( \sigma^x \), in combination with higher consumption \( \hat{c} \), imply a lower expected growth in the agent’s utility \( x_t \), as illustrated in the
Figure 1: The cost function $\hat{v}(\hat{c})$ of the optimal contract. The starting point of the optimal contract is indicated by the blue dot. Parameters: $\rho = r = 5\%, \alpha = 1.7\%, \gamma = 1/3, \phi\sigma = 0.2$.

Figure 2: The drift, $\mu^{\hat{c}}$ and $\mu^{x}$, and volatility, $\sigma^{\hat{c}}$ and $\sigma^{x}$, of the state variables $\hat{c}$ and $x$ under the optimal contract. The starting point of the optimal contract is indicated by the blue dot. Parameters: $\rho = r = 5\%, \alpha = 1.7\%, \gamma = 1/3, \phi\sigma = 0.2$. 
From the FOC for $\sigma^x$ we see that there is a myopic and a dynamic motive for exposing the agent to risk:

$$\sigma^x = \frac{\alpha}{\gamma (\hat{v}c^{-\gamma} \phi \sigma)} - \frac{\hat{v}'}{\hat{v} \gamma} (1 + \gamma)\sigma^c + \sigma^x$$

(21)

The myopic motive trades off the excess return on capital, $\alpha > 0$, against the higher future utility required to compensate the agent for the risk. The dynamic component takes into account that higher $\sigma^x$ eats up the budget for risk exposure (increases the drift of $\hat{c}$). This is costly because the cost $\hat{v}(\hat{c})$ is increasing in $\hat{c}$. At $t = 0$ we know that $\hat{v}' = 0$ because $\hat{c}_t$ is chosen optimally, so the dynamic motive disappears and we have myopic optimization. While we don’t have a proof, in all numerical simulations the dynamic motive reduces $\sigma^x$ below its myopically optimal level and $\sigma^x$ is decreasing in $\hat{c}$.

An important feature of the optimal contract is that $\sigma^c < 0$, which can be seen in the bottom left panel. After bad performance $\hat{c}$ rises, so the agent’s risk exposure $\sigma^x$ declines, together with the expected growth rate of utility and assets, $\mu^x$. As explained above, there are two reasons to set $\sigma^c < 0$. First, it reduces the volatility of the agent’s consumption, $\sigma^c = \sigma^x + \sigma^c$, and therefore helps relax the precautionary motive. Second, since the cost is proportional to promised utility, $\hat{v}(\hat{c})x$, the principal prefers to use the least costly continuation contracts with low $\hat{c}$ after good shocks when he must deliver more utility to the agent. This also implies $\sigma^c < 0$. The overall effect is that after bad outcomes the agent’s consumption is somewhat insured, relative to his continuation utility $x$, and instead he is punished with lower access to capital and lower expected growth.

The top left panel shows the drift of $\hat{c}$, which is positive at $\hat{c}_t$. This means that while the contract starts at the optimal $\hat{c}_0 = \hat{c}_t$, it immediately moves into the interior of the domain. The reason for this is that, since the Euler equation is forward-looking, promising an inefficiently safe contract at a future time $t$ (higher $\hat{c}_t$) relaxes the IC constraint for every period $s < t$, so it makes sense to backload distortions. This intertemporal tradeoff allows the principal to relax the IC constraints today at the cost of an inefficiently safe contract in the future.

Given the behavior of $\mu^c$, one may suspect that the stationary distribution of $\hat{c}_t$ is concentrated near the point of zero drift. It turns out that this is not the case. Even though we can show analytically that the drift is negative near $\hat{c}_h$, the asymptotic properties of $\mu^c$ and $\sigma^c$ near $\hat{c}_h$ are such that the process $\hat{c}_t$ slows down and gets progressively more and more delayed near $\hat{c}_h$ (but never reaches $\hat{c}_h$). Theorem 2 formalizes this asymptotic result. There is no stationary distribution. The principal backloads distortions in order to relax the agent’s IC constraints, so the contract starts at $\hat{c}_t$ and in the long-run spends most of the time near the upper boundary corresponding to autarky $\hat{c}_h$. 

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Theorem 2. The optimal contract of the relaxed problem does not have a stationary distribution. In the long-run, it spends almost all the time near $\hat{c}_h$,

$$\frac{1}{t} \int_0^t 1_{\{\hat{c}_t > \hat{c}_h - \epsilon\}}(\hat{c}_s) ds \to 1 \text{ a.s. \ \forall \epsilon > 0}$$

Remark. However, $P\{\hat{c}_t \to \hat{c}_h\} = 0$. No matter how close any path gets to $\hat{c}_h$, it will always leave that neighborhood at some point in the future, but this happens progressively less and less often for any path.

To sum up, the contract starts at $\hat{c}_0 = \hat{c}_l$, where the principal exposes the agent to the most risk and gives him the most capital, relative to the agent’s utility, compared to any point in the future. The level of $\hat{c}_0$ reflects the agent’s expectation of lower risk in the future, which lowers the agent’s precautionary saving motive and improves incentives ex-ante. The optimal way of providing incentives is to give the agent an inefficiently safer contract with lower growth after bad performance. In the long run, distortions progressively accumulate and the contract is close to the safe autarky contract.

Verification theorems

We know that the principal’s cost function in the relaxed problem satisfies the HJB equation, but how do we know that the equation has no other solutions? And how do we know that we have identified the true (non-relaxed) optimal contract? The following theorem shows that if an appropriate solution to the HJB has been found, e.g. numerically, then it must be the true cost function of the non-relaxed problem and we can use it to build a globally incentive compatible optimal contract.

Theorem 3 (Verification Theorem). Let $\hat{v}(\hat{c}) : [\hat{c}_l, \hat{c}_h] \to [\hat{v}_l, \hat{v}_h]$ be a strictly increasing $C^2$ solution to the HJB equation (18) for some $\hat{c}_l \in (0, \hat{c}_h)$, such that $\hat{v}_l \equiv \hat{v}(\hat{c}_l) \in (0, \hat{v}_h]$, $\hat{v}'(\hat{c}_l) = 0$, $\hat{v}''(\hat{c}_l) > 0$ and $\hat{v}(\hat{c}_h) = \hat{v}_h$. If $\gamma < \frac{1}{2}$, suppose in addition that

$$1 - \hat{v}_1(\hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma-1} \alpha^2 (\phi \sigma)^{-2} \hat{v}_1^{-2}) \leq 0$$

Then,

1) For any incentive compatible contract $C = (c, k)$ that delivers at least utility $u_0$ to the agent, we have $\hat{v}(\hat{c}_l) ((1 - \gamma)u_0)^{\frac{1}{1-\gamma}} \leq J_0(C)$.

2) Let $C^*$ be a contract generated by the policy functions of the HJB. Specifically, the state variables $x^*$ and $\hat{c}^*$ are solutions to (14) and (15) with $dL_t = 0$, with initial values $x_0^* = ((1 - \gamma)u_0)^{\frac{1}{1-\gamma}}$ and $\hat{c}_0^* = \hat{c}_l$. If $C^*$ is admissible, and $\sigma^*$ is bounded, then $C^*$ is an optimal contract, with cost $J_0(C^*) = \hat{v}(\hat{c}_l) ((1 - \gamma)u_0)^{\frac{1}{1-\gamma}}$. 

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The HJB equation can be solved as an ODE by plugging in the FOCs. We only need to verify condition (22) in case $\gamma < \frac{1}{2}$, and that the contract generated by the HJB $C^*$ is admissible. The following sufficient condition can be useful.

**Lemma 4.** If the candidate contract $C^*$ constructed in Theorem 3 has $\mu^{\ast} < r$, then $C^*$ is admissible and delivers utility $u_0$ to the agent.

**Global incentive compatibility.** An important step in Theorem 3 is to verify that the solution to the relaxed problem is indeed globally incentive compatible. Here we provide sufficient conditions for global incentive compatibility of any contract that satisfies the local constraints on savings (15) and stealing (16). These conditions are guaranteed to hold in the optimal contract, which proves the validity of the first-order approach, but are more general than that and can be used to check incentive compatibility of many other contracts of interest below.

While (15) and (16) ensure that neither stealing and immediately consuming, nor secretly saving without stealing are attractive on their own, they leave open the possibility that a double deviation—stealing and saving the proceeds for later—could be attractive to the agent. To see how this can happen, notice that since stealing makes bad outcomes more likely, it increases the expected marginal utility of consumption in the future $\mathbb{E}_t \left[ e^{(r-\rho)t} c_t^{\ast} \gamma \right]^{10}$ Saving the stolen funds for consumption later could therefore be very attractive. However, hidden savings have decreasing marginal value (the first dollar yields $c_t^{-\gamma}$, the second one less than that), which depends on the agent’s precautionary saving motive. This observation allows us to derive a sufficient condition to rule out profitable double deviations.

**Theorem 4.** Let $C = (c, k)$ be an admissible contract with associated processes $x$ and $\hat{c}$ satisfying (14) and (15) and (16), with bounded $\mu^x$, $\mu^\hat{c}$, and $\sigma^\hat{c}$, and with $\hat{c}$ uniformly bounded away from zero and bounded above by $\hat{c}_h$. Suppose that the contract satisfies the following property

$$\sigma^\hat{c}_t \leq 0$$

Then for any feasible strategy $(\hat{c}, a)$, with associated hidden savings $h$, we have the following upper bound on the agent’s utility, after any history

$$U_t^{\hat{c}, a} \leq \left( 1 + \frac{h_t}{x_t} \right)^{1-\gamma} U_t^{c, 0}$$

In particular, since $h_0 = 0$, for any feasible strategy $U_0^{\hat{c}, a} \leq U_0^{c, 0}$, and the contract $C$ is therefore incentive compatible.

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10In other words, even if $e^{(r-\rho)t} c_t^{-\gamma}$ is a martingale under $P$, it might be a submartingale under $P^a$. 
Theorem 4 shows that condition (23) is sufficient by providing a closed-form explicit upper bound (24) on the agent’s off-equilibrium payoff for any savings level \( h_t \geq 0 \). According to (24), if the agent does not have any hidden savings, \( h_t = 0 \), the most utility he could get is \( U_t^{c,0} \), i.e. the utility level he obtains from “good behavior” \((c,0)\). Hence, the contract is incentive compatible. However, if the agent had somehow accumulated hidden savings in the past, he would want to deviate from \((c,0)\) in the future, at the very least to increase his consumption and attain a greater utility. Inequality (24) bounds the utility the agent can get, and the bound tightens as \( \hat{c}_t \) rises and the agent’s precautionary saving motive decreases. The bound is consistent with the intuition that the marginal value of hidden savings becomes lower as the agent’s precautionary saving motive decreases (however, remember that this is just an upper bound on achievable utility).

The sufficient condition \( \sigma_t^c \leq 0 \) can be understood as follows. Hidden savings become more valuable when the agent faces more risk, i.e. has a higher precautionary saving motive. With \( \sigma_t^c \leq 0 \) the contract becomes less risky for the agent after bad outcomes (hidden savings becomes less valuable). Since stealing makes bad outcomes more likely, if the agent steals and saves for later, he expects to have a hidden dollar when it is least valuable to him. This makes double deviations unprofitable.

As shown above, the optimal contract has the property that \( \sigma_t^c \leq 0 \) because the principal wants to contain the agent’s precautionary saving motive and the most efficient way to do this is by reducing the agent’s risk exposure after bad outcomes. As it happens it is the same property that is sufficient for global incentive compatibility.

We would like to emphasize once more that Theorem 4 identifies \( \sigma_t^c \leq 0 \) as a general sufficient condition for incentive compatibility, without even assuming that the contract is recursive in variables \( x \) and \( \hat{c} \). These variables are well-defined for an arbitrary contract, and their laws of motion do not need to be Markov for Theorem 4 to apply. Condition \( \sigma_t^c \leq 0 \) can be used to verify global incentive compatibility of other contracts. For example, it implies that stationary contracts that we discuss below are also all globally incentive compatible.

**Discussion: hidden savings and dynamic distortions**

To understand the dynamic distortions induced by hidden savings, it is useful to go over a series of benchmark contracts. Front-loading the agent’s consumption relaxes the IC constraint (16). Without hidden savings this can be done independently of the agent’s exposure to risk. With hidden savings, however, the principal must distort the agent’s exposure to risk to manipulate his precautionary motive. The Online Appendix has technical details and formal results.

First let’s see what happens without hidden savings, so we are not constrained by the Euler equation. Instead, the optimal contract will satisfy the Inverse Euler equation, i.e.
Figure 3: The cost function \( \hat{v}(\hat{c}) \) solid in blue for the optimal contract, dashed in red for stationary contracts \( \hat{v}_r(\hat{c}) \), dotted in yellow for the Inverse Euler equation, and dashed in black for contracts with myopic optimization over \( \sigma^x \). Parameters: \( \rho = r = 5\% \), \( \alpha = 1.7\% \), \( \gamma = 1/3 \), \( \phi \sigma = 0.2 \).

\( e^{(\rho-r)t}c^x_1 \) would be a martingale.\(^\text{11}\) The Inverse Euler equation does not pin down \( \hat{c} \); it simply links \( \hat{c} \) and \( \sigma^x \) in the following way (the constant environment implies constant \( \hat{c} \) without hidden savings):

\[
\sigma^x = \sigma^x_{\text{inv}}(\hat{c}) = \sqrt{2 \left( \frac{\hat{c}^{1-\gamma} - \hat{c}^{1-\gamma}}{1 - \gamma} \right) \frac{1}{1 - 2\gamma}} \tag{25}
\]

The green dotted line in Figure 3 shows the cost of all stationary contracts consistent with the inverse Euler equation, and the optimal contract without hidden savings is the one that minimizes the cost, indicated with the green dot. Since the agent does not have access to hidden savings, the principal can optimize myopically over \( \sigma^x \) without taking into account the effect of risk on \( \hat{c} \):

\[
\sigma^x_n = \frac{\alpha}{\gamma(\hat{v}_n \hat{c}^{\gamma}_n \phi)\sigma}
\]

The dashed black line shows the cost of contracts that optimize myopically over \( \sigma^x \), for any given \( \hat{c} \), given by equation (19). The optimal contract without hidden savings is at the tangency point of both lines.

Notice that in contrast to the classic Rogerson (1985) setting, here the marginal cost of delivering utility to the agent is lower than the inverse of the marginal utility of consumption, \( \hat{v}_n < \hat{c}^{\gamma}_n \), which can be seen clearly in Figure 3. This reflects the fact that giving

\(^\text{11}\)The optimal contract without hidden savings only exists if \( \gamma < 1/2 \). If \( \gamma > 1/2 \) the principal can obtain infinite profits for any \( \alpha > 0 \). See Di Tella (2016). Figure 3 shows results for \( \gamma < 1/2 \), but hidden savings constrain the principal and makes the problem well defined for any \( \gamma \).
consumption to the agent not only delivers utility to him, but also relaxes the IC constraint. It may seem surprising, then, that the Inverse Euler equation holds here. The reason is that while consumption today relaxes the IC constraint today, consumption tomorrow relaxes the IC constraint tomorrow, so this is not a reason to depart from the Inverse Euler equation.\footnote{To be more precise, the marginal cost of utility and \( c_t^\gamma \) are in a fixed relationship. The logic of the Inverse Euler equation implies that \( e^{(\rho-r)t}c_t^\gamma/(1+\gamma^2(\sigma_x^2)^2c_t^{-1}) \) should be a martingale, and in this constant environment this implies that \( e^{(\rho-r)t}c_t^\gamma \) is a martingale.}

As is well understood, the inverse Euler equation implies that the agent would like to save if he could. If the agent has access to hidden savings, we are constrained by the Euler equation. Let’s focus first on the family of stationary contracts with a constant \( \hat{c} \). These are interesting for two reasons. First, they illustrate clearly how the principal can relax the IC constraint by distorting the agent’s exposure to risk (restricting his access to capital), and are easy to contrast to the case without hidden savings. Second, stationary contracts are renegotiation proof (the optimal renegotiation-proof contract is the best stationary contract, as we will show in Section 5), and will play an important role when we introduce hidden investment in Section 4.

To keep \( \hat{c} \) constant while respecting the Euler equation, we must set \( \sigma^{\hat{c}} = 0 \) and pick \( \sigma^x \) to satisfy:

\[
\sigma^x = \sigma_x^x(\hat{c}) \equiv \sqrt{2} \sqrt{\frac{\hat{c}_h^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma}} \tag{26}
\]

so that \( \mu^{\hat{c}} = 0 \) in (15). The take-away from expression (26) is that if we want to convince the agent to front-load his consumption (higher \( \hat{c} \)) in order to relax the IC constraint, we need to expose him to less risk (lower \( \sigma^x \)) in order to weaken his precautionary motive. The red dashed line in Figure 3 shows the cost of stationary contracts, and the optimal one is indicated by the red dot.\footnote{The cost curves associated with the Euler and Inverse Euler equations intersect, but remember that the \( \sigma^x \) in the background are different, given by (25) and (26). Two different suboptimal contracts that happen to have the same cost.}

Consider first what would happen if we ignored the effect of \( \hat{c} \) on the IC constraint. In this case the cost of utility would be equal to the inverse of the marginal utility of consumption, \( \hat{v} = \hat{c}^\gamma \), and myopic optimization over \( \sigma^x \) taking \( \hat{c} \) as given would be optimal. The resulting contract lies at the tangency point of the curve of stationary contracts and the curve of myopic optimization, indicated by a black dot. As it turns out this myopic stationary contract corresponds to the solution of a simple consumption-portfolio problem where the agent consumes and invests on his own while retaining a fraction \( \phi \) of the risk.\footnote{That is, it coincides with the solution to the problem}

\[
\max_{(c,k)} U(c) \quad \text{s.t.} \quad dw_t = (rw_t + \alpha k_t - c_t)dt + k_t(\phi \sigma)dZ_t, \quad w_t \geq 0
\]

for a given \( w_0 \). This is a classic consumption-portfolio problem where the risky asset has Sharpe ratio \( \alpha/(\phi \sigma) \). It is well understood that the solution to this problem is characterized by the Euler equation and
and satisfies the well known formula:

\[
\hat{c}_p = \left( \hat{c}_h^{1-\gamma} - (1 - \gamma) \frac{1}{2} \left( \frac{\alpha}{\gamma \phi \sigma} \right)^2 \right)^{\frac{1}{1-\gamma}}, \quad \sigma^x_p = \frac{\alpha}{\gamma \phi \sigma}, \quad \hat{v}_p = \hat{c}_p^\gamma
\]  \hspace{1cm} (27)

Myopic optimization does not take into account how lower \(\sigma^x\) can raise \(\hat{c}\) through the Euler equation and relax the IC constraint. Suppose we reduce \(\sigma^x\) in a uniform way, i.e. within the family of stationary contracts. Since we start from a plan that is optimal taking the IC constraint as given, with \(\hat{v} = \hat{c}^\gamma\) and myopic optimization over \(\sigma^x\), the losses of reducing \(\sigma^x\) are second order, but the benefit of relaxing the binding IC constraint itself are first order. As a result, the optimal stationary contract is always to the right of the myopic one, \(\hat{c}_{min}^r > \hat{c}_p\), which means that the agent is exposed to less risk, \(\sigma^x_r(\hat{c}_{min}^r) < \sigma^x_p\). For all stationary contracts with \(\hat{c} > \hat{c}_p\), we depart from myopic optimization, \(\sigma^x_r(\hat{c}) < \frac{\alpha}{\gamma (\hat{v}_r(\hat{c}) \hat{c}^\gamma - \gamma \phi) \sigma}\), and the marginal cost of delivering utility to the agent is less than the inverse of the marginal utility of consumption, \(\hat{v}_r(\hat{c}) < \hat{c}^\gamma\), reflecting how higher \(\hat{c}\) relaxes the IC constraint.

The optimal contract with hidden savings does better than the optimal stationary contract, but worse that the optimal contract without hidden savings, and has features of both. It recognizes that higher \(\hat{c}\) can relax the IC constraint, so the marginal cost of delivering utility is below the inverse of the marginal utility of consumption, \(\hat{v}(\hat{c}) < \hat{c}^\gamma\) for all \(\hat{c} \in \{\hat{c}_l, \hat{c}_h\}\). But instead of reducing the agent’s exposure to risk uniformly, the optimal contract uses dynamic distortions that reduce the exposure to risk in the future and after bad outcomes. As described above, this is a less costly way of manipulating the agent’s precautionary motive that takes advantage of the fact that the Euler equation is forward-looking. In particular, reducing the exposure to risk at time \(t = 0\) does not relax the agent’s precautionary motive, which is forward looking, so the optimal contract optimizes myopically over \(\sigma^x\) at \(t = 0\), just like the optimal contract without hidden savings. As a result, the optimal contract starts on the curve of myopic optimization too.\(^{15}\)

4 Hidden investment

The previous results show how the principal can use dynamic distortions to the agent’s access to capital to manipulate his precautionary motive. This suggests that hidden investment and renegotiation could be important constraints. Fortunately, we can incorporate both into our setting and study the role each play. In this section we study the role of hidden investment, and in Section 5 the role of renegotiation. Here we focus on the main myopic optimization over \(k\) (corresponding to myopic optimization over \(\sigma^x\) in our setting). This yields the expressions in (27). See Section 6.

\(^{15}\)While the optimal contract without hidden savings and the myopic stationary contract actually have a constant \(\hat{c}\), the optimal contract with hidden savings does not. However, because \(\hat{v}'(\hat{c}_l) = 0\) the cost of the optimal contract coincides with that of a contract with a fixed \(\hat{c}_l = \hat{c}_l\) and myopic optimization.
economic insights. The Online Appendix has all the formal results and also extends the setting to incorporate aggregate risk.

If the agent can secretly invest in his private technology, his hidden savings follow the law of motion
\[ dh_t = (rh_t + z_t h_t \alpha + c_t - \tilde{c}_t + \phi k_t a_t) dt + z_t h_t \sigma dZ_t \] (28)
The new term \( z_t \geq 0 \) is the portfolio weight on his private technology in his hidden savings. If he invests in his private technology he gets the excess return \( \alpha \) but is exposed to risk. A valid interpretation for hidden investment is that the principal can give the agent an amount of capital \( k_t \), but the agent can secretly invest more.

A contract \( C = (c, k) \) specifies the contractible payments \( c \) and capital \( k \), contingent on returns \( R \). After signing the contract the agent can choose a strategy \( (\tilde{c}, a, z) \) to maximize his utility. The agent’s utility and the principal’s objective function are still given by (3) and (4). As in the baseline setting, it is without loss of generality to look for a contract where the agent does not steal, has no hidden savings, and no hidden investment. A contract is therefore incentive compatible if the agent’s optimal strategy is \( (c, 0, 0) \).

The law of motion of the state variables \( x_t \) and \( \hat{c}_t \) are unchanged, (14) and (15), as well as the “skin in the game” constraint (16). But the agent’s discounted marginal utility \( e^{(r-\rho)t} c_t^{-\gamma} \) must be a supermartingale under any valid trading strategy. Otherwise, the agent may want to save a dollar, invest it in his private technology, and consume it later. As a result, we have a new IC constraint:
\[ \sigma_x^T + \sigma_{\hat{c}}^T \geq \frac{\alpha}{\gamma \sigma} \] (29)
The interpretation (29) is simple. The agent can obtain a premium \( \frac{\alpha}{\sigma} \) for his idiosyncratic risk. If his exposure to risk is not \( \sigma_x^T \geq \frac{\alpha}{\gamma \sigma} \), he would benefit from deviating by secretly investing and taking on risk. The excess return \( \alpha \) would more than compensate the extra exposure to risk at the margin.

Hidden investment restricts the principal’s ability to provide incentives. In particular, the principal cannot promise to give the agent a perfectly deterministic consumption. If he tried to do this, the agent would just secretly invest and take risk on his own. This is costly because the principal would like to use a promise of future safety to relax the agent’s precautionary saving motive, and is reflected in a smaller upper bound \( \hat{c}_h \). In the baseline where the agent cannot invest in his private technology, we have \( \hat{c}_h = \left( \frac{\rho - r (1 - \gamma)}{\gamma} \right)^{\frac{1}{1 - \gamma}} \), corresponding to the \( \hat{c} \) in autarky where the agent cannot invest in capital. If the agent can invest in his private technology the upper bound is
\[ \hat{c}_h = \left( \frac{\rho - r (1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\sigma \gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \]
which is lower and corresponds to the \( \hat{c} \) in autarky where the agent invests in his private technology on his own (so he can’t get any risk sharing).\(^{16}\)

The HJB equation is the same as in the baseline, but we have an extra constraint:

\[
0 = \min_{\sigma^x, \sigma^{\hat{c}}} \hat{c} - r\hat{v} - \sigma^x \hat{c}^{\gamma} \frac{\alpha}{\phi \sigma} + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma^x)^2 \right) + \hat{v}'\hat{c} \left( \frac{\hat{c}^{1-\gamma} - \hat{c}^*}{1-\gamma} \right) + \frac{(\sigma^x)^2}{2} + (1 + \gamma) \sigma^x \sigma^{\hat{c}} + \frac{1 + \gamma}{2} (\sigma^{\hat{c}})^2 + \frac{\hat{v}''}{2} \sigma^{\hat{c}} (\sigma^{\hat{c}})^2 \tag{30}
\]

subject to \( \sigma^x \geq 0 \) and (29).

Figure 4 shows the optimal contract with hidden investment. It has the same features as in the baseline setting. The contract starts at some \( \hat{c}_l \) and then moves immediately into the interior of the domain \((\hat{c}_l, \hat{c}_h)\). If the new IC constraint (29) is not binding, the FOC for \( \sigma^x \) and \( \sigma^{\hat{c}} \) are the same as in the baseline. But with hidden investment the IC constraint (29) can be binding in some region of the state space. In Figure 5, the IC constraint (29) is binding near the upper bound \( \hat{c}_h \). In this case \( \sigma^{\hat{c}} = \frac{\alpha}{\sigma^x} - \sigma^x \) and the FOC for \( \sigma^x \) is

\[
\sigma^x = \frac{\hat{c}^{\gamma} \alpha}{\phi \sigma} + \frac{\hat{v}'' \sigma^2}{2} \frac{\alpha}{\sigma^{\hat{c}}}
\]

We can show that \( \sigma^x_t > \frac{\alpha}{\sigma^x} \) and therefore \( \sigma^{\hat{c}}_t < 0 \) for all \( t > 0 \). Contract dynamics are therefore qualitatively the same as in the baseline. The contract starts at \( \hat{c}_l \) with myopic optimization over \( \sigma^x \) and becomes less risky over time and after bad outcomes, never revisiting \( \hat{c}_l \) or reaching autarky \( \hat{c}_h \).\(^{17}\)

Just as in the baseline setting without hidden savings, the marginal cost of utility is lower than the inverse of the marginal utility of consumption, \( \hat{v}(\hat{c}) < \hat{c}^{\gamma} \), and the optimal contract optimizes myopically over \( \sigma^x \) at \( t = 0 \). After that, the principal distorts the agent’s access to capital to manipulate his precautionary motive. How can the principal do this, if the agent has access to hidden investment? The answer is that if the agent invests on his own he is the residual claimant and must bear the whole risk, while if he invests through the principal he gets to share some risk because \( \phi < 1 \). The principal is really restricting the amount of capital he is willing to share risk on. The agent would like to invest more if he could share the risk on the extra capital with the principal, but not if he must bear the whole risk on his own. For the same reason, although \( \hat{c}_h \) corresponds to the consumption profile under autarky, there are still gains from trade at that point. Since \( \phi < 1 \), the principal can give the agent the same consumption process that he would get in autarky, but more capital. As a result, the cost to the principal is lower than what the agent could

\(^{16}\)The parameter restrictions (6) and (7) ensure \( \hat{c}_h \) is positive in both cases.

\(^{17}\)There is one subtle difference. The optimal contract without hidden investment does not have a stationary distribution, but instead limits in the long-run to autarky in the sense of Theorem 2. With hidden investment, there is a stationary distribution \( \psi(\hat{c}) \). See Theorem O.2 in the Online Appendix.
Figure 4: The cost function $\hat{v}(\hat{c})$ solid in blue without hidden investment, dotted green with hidden investment. For reference, the cost of stationary contracts dashed in red, and myopic optimization $A(\hat{c}, \hat{v}) = 0$ dashed in black. The starting point without hidden investment is indicated by the blue dot, with hidden investment by the green dot, the myopic contract by the black dot, autarky by the purple dot, and the optimal renegotiation-proof contract by the red dot (see Section 5). Parameters: $\rho = r = 5\%$, $\alpha = 1.7\%$, $\gamma = 1/3$, $\phi = 0.8$, $\sigma = 0.2/0.8$.

Figure 5: The drift, $\mu^{\hat{c}}$ and $\mu^{x}$, and volatility, $\sigma^{\hat{c}}$ and $\sigma^{x}$, of the state variables $\hat{c}$ and $x$ with hidden investment. The dotted part is where the hidden investment IC constraint is binding. The black dashed lines indicate $\hat{c}_l$ and $\hat{c}_h$ with hidden investment. Parameters: $\rho = r = 5\%$, $\alpha = 1.7\%$, $\gamma = 1/3$, $\phi = 0.8$, $\sigma = 0.2/0.8$.
get in autarky, \( \hat{v}(\hat{c}_h) = \hat{v}_r(\hat{c}_h) < \hat{c}_h^\gamma \). Figure 4 shows the cost of autarky as a purple dot.

It is useful to ask under what conditions the gains from trade are completely exhausted, and the optimal contract corresponds to autarky. Lemma O.9 in the Online Appendix shows this is true in the special (but salient) case with hidden investment and \( \phi = 1 \). In this case, the optimal contract, the myopic contract, and autarky coincide. Without hidden investment there would still be gains from trade. While the agent can save on his own, the principal can still control his access to capital to provide incentives, and can therefore provide some risk sharing. However, with hidden investment, the optimal contract coincides with autarky. Intuitively, the agent can both save and invest on his own, so the principal cannot provide any risk sharing in an incentive compatible way. We can see this case as a limit in Figure 4. If we let \( \phi \to 1 \), while also adjusting \( \sigma \) so that \( \phi \sigma \) is constant, all the curves corresponding to the case with no hidden investment remain unchanged. The optimal contract with hidden investment, however, becomes progressively worse (the green curve shifts up), because the agent finds investing on his own more attractive. This is reflected in a falling \( \hat{c}_h \approx \hat{c}_p = \left( \frac{p-r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2} \left( \frac{\alpha}{\gamma \sigma} \right)^2 \right) \).

We close this Section by considering validity of the first-order approach with hidden investment. The agent’s ability to invest his hidden savings makes the verification of global incentive compatibility potentially more difficult. The agent could find it attractive to steal and save the proceeds for later, while investing them in his own private technology that’s correlated with his observed returns. Fortunately, we can extend the results of Theorem 4 to deal with hidden investment. Theorem O.4 in the Online Appendix shows that \( \sigma \hat{c} \leq 0 \) is still sufficient to ensure global incentive compatibility even when the agent has hidden investment in both his private technology and aggregate risk. Since \( \sigma \hat{c} \leq 0 \) is guaranteed to hold in the optimal contract with hidden investment, this shows the validity of the first-order approach, but it’s more general than that and can be used to verify global incentive compatibility of a wide class of contracts.

5 Renegotiation

The optimal contract requires commitment. The principal can relax the agent’s precautionary saving motive by promising an inefficiently low amount of capital and risk over time and after bad outcomes. This suggests that the agent and principal could be tempted to renegotiate the contract and “start over”, undoing the whole incentive scheme. Here we formalize a notion of renegotiation, and characterize the optimal renegotiation-proof contract. As it turns out, the optimal renegotiation-proof contract is the best stationary contract, so even without commitment we are able to do better than under myopic optimization. This section is consistent with the presence of aggregate risk and hidden investment introduced in Section 4 and the Online Appendix.
After signing an incentive compatible contract $C = (c, k)$, the principal can at any time offer a new continuation contract that leaves the agent at least as well off (the offer is “take it or leave it”). The question is, what kind of contracts can he offer—what is a valid “challenger” to the original contract? Here we use a notion of internal consistency. If $C$ is renegotiation proof, then surely appropriately scaled parts of it should be a valid challenger. This means that at any stopping time $\tau$, the principal can renegotiate and get a continuation cost $x_\tau \times \inf \hat{v}(\omega, t)$. With this in mind, we say that an incentive compatible contract is renegotiation-proof (RP) if

$$\infty = \arg \min_{\tau} \mathbb{E}^Q \left[ \int_0^\tau e^{-r_t} (c_t - k_t \alpha) dt + e^{-r_\tau} x_\tau \inf \hat{v} \right]$$

The optimal contract with hidden savings is not renegotiation proof, because after any history $\hat{v}_t > \hat{v}_l = \inf \hat{v}(\omega, s)$, so the principal is always tempted to “start over”.

It is easy to see that an incentive compatible contract is renegotiation-proof if and only if the continuation cost $\hat{v}_t$ is constant. Stationary contracts have a constant $\hat{v}$, because $\hat{c}$ is constant. But they are not the only renegotiation-proof contracts. They were built using $dL_t = 0$. There are other contracts with a constant $\hat{c}$ that use $dL_t > 0$, i.e. the drift of $\hat{c}$ would be negative without $dL_t$. In addition, there could be non-stationary contracts with a constant cost $\hat{v}(\hat{c})$ for all $\hat{c}$ in the domain. However, Theorem O.5 in the Online Appendix shows that the optimal renegotiation-proof contract is the optimal stationary contract, with cost $\hat{v}_r^{\min} = \min_{\hat{c} \in (\hat{c}_r, \hat{c}_h]} \hat{v}_r(\hat{c})$, shown in Figure 4 with a red dot. This result shows that even without full commitment, hidden savings generate distortions in how much capital the agent receives. The principal can do better than myopic optimization.

Remark. It is possible that $\hat{v}_r^{\min} = \hat{c}_h$ if the agent can invest his hidden savings and $\phi$ is close enough to 1. In the special case with hidden investment and $\phi = 1$, we have $\hat{v}_r^{\min} = \hat{c}_p = \hat{c}_h$, as show in Lemma O.9.

6 Fund Capital Structure

In this section, we show how dynamic distortions become reflected in fund leverage, payout rate and the division of risk between the agent and principal (i.e. outside investors). This perspective is useful because these quantities are of applied interest. Also, they connect our results to the classic consumption-portfolio problem.

As a starting point, notice that without a long-term contract, the agent could simply shift fraction $1 - \phi$ of risk to the outside market and allocate his wealth between the risk-free asset and capital, as in the standard optimal consumption and portfolio choice problem. This risk sharing is incentive compatible because insurance does not cover more than the cost of cash diversion, hence the agent does not want to steal. The resulting contract is the
myopic contract of Section 3, which satisfies the Euler equation and myopic optimization over risky capital (taking into account that only a fraction $\phi$ of the risk is retained by the agent).

Relative to this simple benchmark, distortions in the optimal contract take the form of dynamic deviations from myopic optimization for capital: the contract restricts the funds invested in capital (i.e. the ratio $k_t/x_t$), particularly after bad outcomes. That is, while the Euler equation for the risk-free asset holds (meaning that the agent does not want to have hidden savings), the agent would like to allocate more funds to capital if he had the freedom to do so.\footnote{This principal’s ability to control capital is key. Even with hidden investment, the principal can still control capital somewhat as long as $\phi < 1$. If the agent could secretly invest in capital with the same efficiency (i.e. if $\phi = 1$), the principal could not improve upon the simple portfolio benchmark. This reflects the result in Section 4.}

These distortions allow the principal to improve risk sharing. That is, the agent can shift a greater fraction of risk than $1 - \phi$ to the principal and still respect incentive compatibility. First, because the restricted access to capital makes the marginal value of funds invested in capital greater than the marginal utility of consumption. Second, because after bad outcomes the restriction becomes tighter. As we move away from the simple optimal portfolio benchmark, distorting the agent’s access to capital has a second order negative effect on welfare, but shifting a larger fraction of risk on the principal has a first order positive effect.

To see this in more detail, let’s be more explicit about the mapping from the optimal contract to the consumption-portfolio setting. At any time, there is capital $k_t$ in the fund. Since the cost of compensating the agent is $J_t = \mathbb{E}_t \left[ \int_t^{\infty} e^{-r(s-t)} (c_s - \alpha k_s) ds \right]$, the value of the principal’s stake in the fund is $k_t - J_t$. This gives us a specific division of the value of capital $k_t$: the principal gets $k_t - J_t$ and the agent consequently gets $J_t$. Let us refer to $J_t$ as the agent’s wealth (recall that the agent needs to contribute $J_0$ at time 0 in order for the principal to break even).

What about the division of risk? $J_t$ follows the law of motion:

$$dJ_t = (rJ_t - c_t + \alpha k_t) \, dt + \tilde{\phi}_t \sigma k_t \, dZ_t$$

for some stochastic process $\tilde{\phi}_t$. This equation is a classic dynamic budget equation, with the allocation of funds to the risk-free asset, capital (of which risk $1 - \tilde{\phi}_t$ is insured) and consumption $c_t$. Since the principal is risk neutral with respect to idiosyncratic risk, he doesn’t charge anything for insurance. As a result, the agent keeps all of the excess return of capital $\alpha$, but only a fraction $\tilde{\phi}_t$ of the risk. We call this the agent’s \textit{retained equity stake}. Ideally, we would set $\tilde{\phi}_t = 0$ to obtain perfect risk-sharing, but the retained equity stake is important to give the agent some “skin in the game” and ensure incentive compatibility.

For any contract that is recursive in $\tilde{c}_t$ and scale invariant to $x_t$, with a cost function
\( \hat{v}(\hat{c}_t)x_t \), we can match terms to obtain

\[
k_t/J_t = \frac{\sigma_t^2}{\hat{v}_t \hat{c}_t^{-\gamma} \phi \sigma}, \quad \tilde{\phi}_t = \left(1 + \frac{\hat{v}_t^2}{\sigma_t^2} (\hat{v}_t \hat{c}_t^{-\gamma}) \phi \right), \quad c_t/J_t = \frac{\hat{c}_t}{\hat{v}_t}
\]

where \( \sigma_t^0 = \hat{v}'(\hat{c}_t)/\hat{v}(\hat{c}_t) \sigma_t^2 \hat{c}_t \) is the volatility of \( \hat{v}_t \).

The optimal contract distorts the agent’s portfolio-weight on capital, or leverage, \( k/J \), to weaken the agent’s incentives to steal. This is reflected in a lower retained equity stake,

\[
\tilde{\phi}_t = \left(1 + \frac{\sigma_t^0}{\sigma_t^2} (\hat{v}_t \hat{c}_t^{-\gamma}) \phi \right) < \phi
\]

which improves risk sharing. The retained equity stake \( \tilde{\phi}_t \) is below \( \phi \) for two reasons mentioned above. First, the marginal value of consumption is below the marginal value of funds that can be invested in capital, \( \hat{v}_t \hat{c}_t^{-\gamma} < 1 \). Second, after bad outcomes the agent is punished not only with less wealth, but also with tighter constrains, \( \sigma_t^0/\sigma_t^2 < 0 \). Finally, facing less risk, the agent’s consumption rate out of wealth, \( c_t/J_t \), is above the level corresponding to the simple benchmark consumption-portfolio problem (the myopic contract).

Figure 6 shows these quantities in our optimal contract. Fund leverage \( k_t/J_t \) is decreasing with \( \hat{c}_t \), the agent’s share of risk \( \tilde{\phi}_t \) is below \( \phi \), and the consumption rate \( c_t/J_t \) is above the level of the optimal portfolio benchmark. Notice that the optimal contract uses future distortions to relax the retained equity stake, \( \tilde{\phi}_0 < \phi \), but optimizes myopically over capital at \( t = 0 \), so fund leverage is higher at \( t = 0 \) than under the simple portfolio benchmark. The retained equity stake \( \tilde{\phi} \) and the agent’s consumption rate are also tied to the dynamics of the contract and the incentive constraints. Specifically, the principal uses dynamic distortions to the fund leverage to relax the agent’s retained equity stake and improve risk sharing.
7 Discussion: dynamic private information

Our model is related to a whole class of problems with the broad title “dynamic adverse selection”. These are dynamic environments where the agent has private information. In our case, information about savings. In these environments the agent’s deviation payoff plays a central role, also called his information rent. This term originates from the static auction environment of Myerson (1981), where agents whose type (valuation) is higher can mimic lower types, and their higher valuation earns them higher utility or rents. Although the two problems may appear unrelated, at the core of both lie distortions that the principal uses to control the payoff of a deviating type.

In the static Myerson (1981) setting the distortion can take the form of a reserve price which reduces the rents of high valuation bidders and helps the principal screen. In dynamic models of adverse selection distortions are more complex. In our setting the principal distorts the agent’s access to capital ex-post to reduce his precautionary motive and therefore reduce the value of hidden savings, which improves ex-ante incentives against stealing.

Ideally we would completely characterize the agent’s off-path value function. This is the approach in Fernandes and Phelan (2000), but it leads to an infinite-dimensional state space which makes it difficult to apply. Instead, we use the first-order approach, which focuses on local incentives, and verify global incentive compatibility using an upper bound of the agent’s off-path value function. To understand where this bound comes from, it’s useful to consider a setting in which our upper bound is tight. To this end, suppose that $\phi = 1$ and the agent can choose any real $a_t$ (i.e. the agent can use hidden savings to boost observed returns), and consider contracts with $\hat{c}_t = 0$. This class of contracts includes the stationary contracts of Section 3, as well as any contract in which distortions are reflected in a deterministic path for $\hat{c}_t$. We know from Theorem 4 that $x_t + \hat{c}_t^{-\gamma} h_t$ is an upper bound on the off-path utility of the agent, and we can see from the proof that if $\sigma_t^\hat{c} = 0$ then this remains an upper bound if the agent can use his hidden savings to boost observed returns, $a_t \leq 0$. In this case the agent can actually achieve the upper bound. To see this, consider the utility he would get if he immediately put all his hidden savings back into the contract,

$$x_t + \int_0^h \hat{c}(h')^{-\gamma} dh'$$

(33)

where $\hat{c}(h')$ is the level of $\hat{c}$ that results after the agent uses $h'$ of savings to boost returns. This expression resembles the bound used in DeMarzo and Sannikov (2016) and Pavan et al. (2014). With deterministic contracts, $\hat{c}(h') = \hat{c}_t$ for all $h'$ because $\hat{c}_t$ does not depend on the history of observed returns, so the agent obtains utility $x_t + \hat{c}_t^{-\gamma} h_t$, achieving the upper bound. So in this special case we actually know exactly what the agent’s off-path utility is, and we can therefore show that contracts with $\sigma_t^\hat{c} = 0$ are globally incentive compatible.

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19Pavan et al. (2014), Battaglini (2005), DeMarzo and Sannikov (2016), He et al. (2017).
even if we allow \( a_t \leq 0 \).

In our model, \( x_t + \hat{c}_t \gamma h_t \) is only an upper bound. The assumption that \( a_t \geq 0 \) is necessary for this result, since otherwise the agent can obtain utility of at least (33) by immediately using all his hidden savings to boost returns. When \( \hat{c}_t \) declines with reported returns (i.e. when \( \sigma_t^\hat{c} < 0 \)), this utility is greater than the bound, \( x_t + \int_0^h \hat{c}(h')^{-\gamma} dh' > x_t + \hat{c}_t^{-\gamma} h_t \). The restriction \( a_t \geq 0 \) implies that \( x_t + \hat{c}_t^{-\gamma} h_t \) is an upper bound on the agent’s deviation payoff, which allows us to prove global incentive compatibility for all contracts with \( \sigma_t^\hat{c} \leq 0 \).

We don’t know if the first-order approach fails when \( a_t \leq 0 \) is allowed. We finish by considering what would happen if it did fail. The principal can always reduce the agent’s off-equilibrium utility to our bound, \( x_t + \hat{c}_t^{-\gamma} h_t \), by using deterministic contracts. This means that solving the relaxed problem with the extra constraint \( \sigma_t^\hat{c} = 0 \) provides an upper bound on the cost of the optimal contract, while the solution to the relaxed problem with free \( \sigma_t^\hat{c} \) provides a lower bound. This is an example of how non-local distortions may be required to provide incentives.

8 Conclusions

We study the role of hidden savings in a classic portfolio-investment problem with fund diversion. The agent’s precautionary saving motive plays central role in his incentives to divert funds. If the agent expects a large exposure to risk in the future, he places a large value on hidden savings that he can use to self insure. As a result, the principal must manipulate the agent’s precautionary saving motive by committing to limit his exposure to risk in the future, especially after bad outcomes. Since giving capital to the agent requires exposing him to risk to align incentives, this leads to dynamic distortions in the agent’s access to capital and a skewed compensation scheme. After good outcomes the agent’s access to capital improves, allowing his to keep growing rapidly. After bad outcomes his access to capital is restricted and he stagnates. In exchange, his consumption is somewhat insured on the downside, and he is punished instead with lower growth. We also extend our environment to incorporate hidden investment and renegotiation, and show how it can be mapped into a classic consumption-portfolio problem with a retained equity constraint and a leverage constraint.

An important methodological contribution is to provide a sufficient analytical condition for the validity of the first-order approach. If the agent’s precautionary saving motive is weaker after bad outcomes, the contract is globally incentive compatible. This condition holds in the optimal contract and in a wider class of contracts beyond the optimal one. In fact, the sufficient condition does not even require a recursive structure, and it is valid even with aggregate risk and hidden investment.
References


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Appendix

Lemma 1
Consider
\[ Y_t = \mathbb{E}_t \left[ \int_0^\infty e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right] = \int_0^t e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds + e^{-\rho t} U_t^{c,0} \]
Since \( Y \) is an \( \mathbb{F} \)-adapted \( P \)-martingale, and \( \mathbb{F} \) is generated by \( Z \), we can apply a martingale representation theorem to obtain
\[ dY_t = e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} - \rho e^{-\rho t} U_t^{c,0} \, dt = e^{-\rho t} \Delta_t \sigma \, dZ_t \]
for some stochastic process \( \Delta \) also adapted to \( \mathbb{F} \). Dividing by \( e^{-\rho t} \) and rearranging we get (8).

Lemma 2
We will show that if (9) fails then the agent has a profitable deviation with stealing. Equation (10) is simply the first-order condition necessary for (9). After that, we show that (11) follows from the standard Euler equation, which must necessarily hold to rule out deviations with savings (without stealing).

First, suppose that (9) fails on a set of positive measure. Let \( a_t \) be the action that achieves the maximum of (9) on \([0, \bar{a}]\), where \( \bar{a} \) is an arbitrary bound. Since the objective in (9) is concave in \( a \), we have that \( a_t > 0 \) whenever (9) fails. Consider the strategy \( (c + \phi k a, a) \), which results in zero savings.

From the representation (8), under the strategy \( (c + \phi k a, a) \) the process
\[ V_t = \int_0^t e^{-\rho s} \frac{(c_s + \phi k_s a_s)^{1-\gamma}}{1-\gamma} ds + e^{-\rho t} U_t^{c,0} \]
has drift
\[ e^{-\rho t} \left( \frac{(c_t + \phi k_t a_t)^{1-\gamma}}{1-\gamma} - \rho U_t^{c,0} + \rho U_t^{c,0} - \frac{c_t^{1-\gamma}}{1-\gamma} - \Delta_t a_t \right) \]
since \( dR_t - (\alpha + r) \, dt \) has drift \(-a_t\). By our construction of \( a_t \), the drift is non-negative and positive on a set of positive measure. Hence
\[ U_0^{c,0} = V_0 \leq \mathbb{E}_0^a[V_{\tau^n}] = \mathbb{E}_0^a \left[ \int_0^{\tau^n} e^{-\rho s} \frac{(c_s + \phi k_s a_s)^{1-\gamma}}{1-\gamma} ds + e^{-\rho t} U_t^{c,0} \right] \]
for some increasing sequence of stopping times \( \{\tau^n\}_{n \in \mathbb{N}} \to \infty \) a.s., and for \( n \) large enough the inequality is strict. The last expectation is the agent’s payoff from following \( (c + \phi k a, a) \) until time \( \tau^n \) and then reverting to \((c,0)\), and this strategy is a profitable deviation over
following \((c, 0)\) throughout.

For (11), notice that \(e^{-\langle \rho - r \rangle t} c_t^{-\gamma}\) must be a supermartingale to ensure that the agent does not want to save. Then we can write \(e^{-\langle \rho - r \rangle t} c_t^{-\gamma} = M_t - A_t\), where \(M_t\) is a local martingale and \(A\) a weakly increasing process. By the Martingale Representation Theorem, \(M_t = \int_0^t \sigma_t^M \, dZ_t\) for some process \(\sigma_t^M\). Define \(\sigma_t^c\) by \(\sigma_t^M = -\gamma \sigma_t^c e^{-\langle \rho - r \rangle t} c_t^{-\gamma}\). Then using Ito’s lemma we obtain (11) with a weakly increasing process \(L\).

**Lemma 3**

For the bound, let \(y_t = ct^{1-\gamma}\) and use the law of motion of \(c_t\) in equation (11) to compute

\[
dy_t = (1 - \gamma) y_t \left( \frac{r - \rho}{\gamma} + \frac{1 + \gamma}{2} (\sigma_t^c)^2 \right) dt + (1 - \gamma) y_t \sigma_t^c dZ_t + dA_t
\]

where \(A\) is a weakly increasing process (coming from the term \(dL_t\)). This is a linear SDE with solution

\[
y_s \geq y_t e^{\int_t^s \left( \frac{r - \rho}{\gamma} - \frac{1}{2} (\sigma_u^c)^2 \right) \, du} e^{\int_t^s (1 - \gamma) \sigma_u^c dZ_u - \frac{1}{2} (1 - \gamma)^2 (\sigma_u^c)^2 \, du}
\]

for \(s > t\), where the inequality comes from the term \(dA_t \geq 0\). Now compute

\[
U_t = \mathbb{E}_t \left[ \int_t^\infty e^{-\rho s} y_s \, ds \right] \geq y_t \mathbb{E}_t \left[ \int_t^\infty e^{-\rho s} e^{\int_t^s \left( \frac{r - \rho}{\gamma} - \frac{1}{2} (\sigma_u^c)^2 \right) \, du} e^{\int_t^s (1 - \gamma) \sigma_u^c dZ_u - \frac{1}{2} (1 - \gamma)^2 (\sigma_u^c)^2 \, du} \, ds \right]
\]

\[
= y_t \mathbb{E}_t \left[ \int_t^\infty e^{-\int_t^s \left( \frac{r - \rho(1 - \gamma)}{\gamma} - \frac{1}{2} (\sigma_u^c)^2 \right) \, du} e^{\int_t^s (1 - \gamma) \sigma_u^c dZ_u - \frac{1}{2} (1 - \gamma)^2 (\sigma_u^c)^2 \, du} \, ds \right]
\]

Since \(e^{\int_t^\infty (1 - \gamma) \sigma_u^c dZ_u - \frac{1}{2} (1 - \gamma)^2 (\sigma_u^c)^2 \, ds}\) is a martingale, it defines an equivalent measure \(\tilde{P}(\sigma^c)\) where \(\tilde{Z}_t = Z_t - \int_0^t (1 - \gamma) \sigma_s^c dZ_s\) is a \(\tilde{P}\)-martingale. Therefore,

\[
U_t \geq y_t \mathbb{E}_t^{\tilde{P}} \left[ \int_t^\infty e^{-\int_t^s \left( \frac{r - r(1 - \gamma)}{\gamma} - \frac{1}{2} (\sigma_u^c)^2 \right) \, du} \, ds \right]
\]

Now compute

\[
x_t = ((1 - \gamma)U_t)^{\frac{1}{1-\gamma}} \geq c_t \mathbb{E}_t^{\tilde{P}} \left[ \int_t^\infty e^{-\int_t^s \left( \frac{r - r(1 - \gamma)}{\gamma} - \frac{1}{2} (\sigma_u^c)^2 \right) \, du} \, ds \right]^{\frac{1}{1-\gamma}}
\]

\[
\implies \frac{c_t}{x_t} \leq \mathbb{E}_t^{\tilde{P}} \left[ \int_t^\infty e^{-\int_t^s \left( \frac{r - r(1 - \gamma)}{\gamma} - \frac{1}{2} (\sigma_u^c)^2 \right) \, du} \, ds \right]^{-\frac{1}{1-\gamma}}
\]

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Regardless of the probability measure, the maximum value for the rhs is achieved with $\sigma^c_s = 0$ for all $s > t$, so we obtain

$$\hat{c}_t \leq \mathbb{E}^P_t \left[ \int_t^\infty e^{-\int_t^s \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2} (\sigma^c_u)^2 \right) du} ds \right]^{\frac{1}{1-\gamma}} \leq \left( \frac{\rho - r(1-\gamma)}{\gamma} \right)^{\frac{1}{1-\gamma}} = \hat{c}_h$$

In addition, since the upper bound can only be achieved with a deterministic consumption, $U^c, 0$ is deterministic too. This implies that both $c_t$ and $x_t$ grow at rate $r - \rho \gamma$, so $\hat{c}_h$ is an absorbing state. In light of (10), we must have $k_{t+u} = 0$ in the continuation contract, so we have the autarky contract with cost $\hat{v}_h, x_t$. This completes the proof.

**Theorem 1**

The proof is split into parts.

1) The cost function must be bounded above by $\hat{v}_h$ since we can always just give consumption to the agent without any capital, and obtain cost $\hat{v}_h$. It must be strictly positive because if $\hat{v}(\hat{c}) = 0$ for any $\hat{c} \in [0, \hat{c}_h]$, then we can scale up the contract and give infinite utility to the agent at zero cost, or else achieve infinite profits. Also $\hat{v}(\hat{c}) \to \hat{v}_h$ as $\hat{c} \to \hat{c}_h$ because $\hat{v}(\hat{c})$ corresponds to minimizing the cost subject to the constraint $\hat{c} \leq \hat{c}_0 \leq \mathbb{E}^P_t \left[ \int_t^\infty e^{-\int_t^s \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2} (\sigma^c_u)^2 \right) du} ds \right]^{\frac{1}{1-\gamma}}$, from Lemma 3. As $\hat{c} \to \hat{c}_h$, we converge to $\sigma^c_t = 0$ for all $t$, which corresponds to the autarky contract with cost $\hat{v}_h = \hat{c}^\gamma_h$.

2) Because we can always move $\hat{c}$ up using $dL_t$, we know that $\hat{v}$ must be weakly increasing. It is useful to write the function $A(\hat{c}; \hat{v})$ from (19) as

$$A(\hat{c}, \hat{v}) \equiv \hat{c} - rv - \frac{1}{2} \left( \frac{\hat{c} \gamma}{\hat{v} \sigma} \right)^2 + \hat{v} \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma}$$

which is the HJB equation when $\hat{v}(\hat{c})$ is flat. The region where the HJB equation holds cannot have any flat parts, because this would mean that $A(\hat{c}, \hat{v}(\hat{c})) = 0$. We know however from Lemma O.11 that this function can have at most two roots, so $\hat{v}$ must be strictly increasing in the region where the HJB holds.

The only way the HJB could not hold is if the optimal contract never spends any time there, i.e. if we ever found ourselves there, it would be optimal to immediately jump out using $dL_t$. The value in that region then must be constant and equal to the value at the destination point (the upper end of the flat region). The HJB should hold as an inequality with $A(\hat{c}; \hat{v}(\hat{c})) \geq 0$, since otherwise we could improve by lingering in the flat region for a while before jumping. Since $\hat{v}(\hat{c})$ is strictly increasing when the HJB equation holds, the contract will start with $\hat{c}_0$ at the upper end of a flat region. It cannot be that $\hat{c}_0 = 0$ because of Inada conditions, so we must have at least one flat interval $[0, \hat{c}_l]$. 
3) We would like to show that this is the only flat interval, and the HJB equation holds as an equality in the strictly increasing region $[\hat{c}_l, \hat{c}_h]$, and both regions are connected with smooth pasting, i.e. $\hat{v}'(\hat{c}_1) = 0$. Suppose then that there is a region $[\hat{c}_1, \hat{c}_2] \subset (0, \hat{c}_h)$ where the cost function is flat, and it’s strictly increasing immediately above it (possibly, $\hat{c}_1 = 0$). Let’s show that as $\hat{c} \searrow \hat{c}_2$, $A(\hat{c}, \hat{v}(\hat{c})) \to 0$ and $\hat{v}'(\hat{c}) \to 0$ (i.e. we have smooth pasting). Towards contradiction, imagine there is a kink at $\hat{c}_2$, i.e. the right-derivative of $\hat{v}(\hat{c})$ is strictly positive. This can only happen if $\sigma^\hat{c}(\hat{c}) = 0$ and $\mu^\hat{c}(\hat{c}_2) \geq 0$, since otherwise we would cross into the flat region where the HJB doesn’t hold (with $\hat{v}'(\hat{c}) > 0$ we must have $dL = 0$). First consider the case with $\mu^\hat{c}(\hat{c}_2) > 0$. We can contemplate the following deviation: start at $\hat{c}_2 - \delta$, with the same $\sigma^x$ and $\sigma^\hat{c} = 0$: by continuity $\mu^\hat{c} > 0$ along this plan, so after some time we will end up at $\hat{c}_2$ and we can go back to the optimal contract and obtain continuation cost $\hat{v}(\hat{c}_2)$. The value of this strategy for $\hat{c} < \hat{c}_2$ extends the cost function $\hat{v}(\hat{c})$ below $\hat{c}_2$ and satisfies the HJB equation (with $\sigma^\hat{c} = 0$, so it’s a first order ODE with boundary condition given by $\hat{v}(\hat{c}_2)$ at $\hat{c}_2$). However, because $\hat{v}'(\hat{c}_2) > 0$, we obtain a lower cost at $\hat{c}_2 - \delta$. This is therefore an attractive deviation, which is a contradiction, so $\mu^\hat{c}(\hat{c}_2) = 0$ is the only option left. In this case, since we also have $\sigma^\hat{c}(\hat{c}) = 0$, it means we have a stationary contract, and because we are at a local minimum of the cost function, we must also be at a local minimum of the cost of stationary contracts $\hat{v}_r(\hat{c})$ given by (O.29), implying $\hat{v}'(\hat{c}_2) = 0$. However, because $\hat{v}'(\hat{c}_2) > 0$, we get $\hat{v}(\hat{c} + \epsilon) > \hat{v}_r(\hat{c} + \epsilon)$, which cannot be. We conclude that $\hat{v}'(\hat{c}_2) = 0$.

We still need to show that $A(\hat{c}, \hat{v}(\hat{c})) \to 0$ as $\hat{c} \searrow \hat{c}_2$. To see this it’s useful to use the FOC for $\sigma^x$ conditional on $\sigma^\hat{c}$ to obtain

$$\sigma^x = \frac{\alpha \hat{c} \gamma}{\phi} - \hat{v}'(1 + \gamma)\sigma^\hat{c}$$

and plug it into the HJB equation. We can then re-write the HJB

$$0 = \min_{\sigma^\hat{c}} \bar{A} + \bar{B} \sigma^\hat{c} + \frac{1}{2} \bar{C}(\sigma^\hat{c})^2$$

with

$$\bar{A} = \hat{c} - r\hat{v} - \frac{1}{2} \hat{v}' \left( \frac{\alpha \hat{c} \gamma}{\phi} \right)^2 + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} \right) - \hat{v}' \hat{c} \left( \frac{\rho - r}{\gamma} + \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} \right)$$

$$\bar{B} = \hat{v}'(1 + \gamma) \left( \frac{\alpha \hat{c} \gamma}{\phi} \right) \geq 0$$

$$\bar{C} = \gamma \hat{v}'(1 + \gamma) \left( \frac{\hat{v} - \hat{v}' \hat{c}}{\phi} + \hat{v}'' \hat{c} \right)^2$$

For the HJB to have a minimum, it must be that $\bar{C} \geq 0$. If $\bar{C} = 0$ we can only have a minimum if $\bar{B} = 0$ as well, in which case $\bar{A} = 0$, and with $\hat{v}'(\hat{c}_2) = 0$ we get $A(\hat{c}, \hat{v}(\hat{c})) \to 0$.
as \( \hat{c} \searrow \hat{c}_2 \) as desired. If instead \( \hat{C} > 0 \), then

\[
\sigma^* = -\frac{\bar{B}}{\hat{C}} \leq 0
\]  

(39)

With \( \hat{v}'(\hat{c}_2) = 0 \) we must have \( \hat{v}''(\hat{c} + \epsilon) \geq 0 \) for small \( \epsilon \) (or else \( \hat{v}'(\hat{c} + \epsilon) < 0 \)). We can then show that \( \frac{\hat{C}}{\hat{B}^2} \to \infty \) as \( \hat{c} \searrow \hat{c}_2 \), which implies \( \frac{\hat{B}^2}{\hat{C}} \to 0 \) and therefore \( \hat{A} \to 0 \), and therefore \( A(\hat{c}, \hat{v}(\hat{c})) \to 0 \) as \( \hat{c} \searrow \hat{c}_2 \), as desired. Since this is true in particular when \( \hat{c}_1 = 0 \) and \( \hat{c}_2 = \hat{c}_1 \), we have proven the smooth pasting condition.

Now since at \( \hat{c}_2 \) we have \( A(\hat{c}, \hat{v}(\hat{c})) = 0 \), this is a root of \( A \). Below that we have \( A(\hat{c}, \hat{v}(\hat{c})) \geq 0 \). From Lemma O.11 we know that this can only be the case if \( \hat{c}_2 \) is the first root of \( A(\hat{c}, \hat{v}(\hat{c})) \). We would like to show that it cannot be the case that for \( \hat{c} < \hat{c}_1 < \hat{c}_2 \) the cost function is lower, i.e. \( \hat{v}(\hat{c}_1 - \delta) < \hat{v}(\hat{c}_1) = \hat{v}(\hat{c}_2) \). To see this, imagine the same problem with a smaller \( \alpha' < \alpha \), and cost function \( \hat{v}_{\alpha'}(\hat{c}) \geq \hat{v}(\hat{c}) \). We can pick \( \alpha' \) small enough that \( \hat{v}_{\alpha'}(\hat{c}_1) = \hat{v}(\hat{c}_2) \), where \( \hat{c}_1 \) is the upper end of the first flat region for the contract with \( \alpha' \) (and it minimizes \( \hat{v}_{\alpha'}(\hat{c}) \)). We can do this because \( \hat{v}_{\alpha'}(\hat{c}) \) is continuously increasing in \( \alpha' \) for any \( \hat{c} \), and has \( \hat{v}_{\alpha'}(\hat{c}) = \hat{v}_h \) for all \( \hat{c} \leq \hat{c}_h \) if \( \alpha' = 0 \). It must be that \( \hat{c}_1 \leq \hat{c}_2 \), because to the right of \( \hat{c}_2 \) \( \hat{v}_{\alpha'}(\hat{c}) \geq \hat{v}(\hat{c}) > \hat{v}(\hat{c}_2) \) for all \( \hat{c} > \hat{c}_2 \). However, looking at \( A(\hat{c}, \hat{v}(\hat{c}_2)) \) we notice it is decreasing in \( \alpha \), so \( A_{\alpha'}(\hat{c}_1, \hat{v}(\hat{c}_2)) > A(\hat{c}_{\alpha'}, \hat{v}(\hat{c}_2)) \geq 0 \). But the previous argument shows that \( A_{\alpha'}(\hat{c}_{\alpha'}, \hat{v}(\hat{c}_2)) = A_{\alpha'}(\hat{c}_{\alpha'}, \hat{v}_{\alpha'}(\hat{c}_{\alpha'})) = 0 \) because \( \hat{c}_{\alpha'} \) is the upper end of the first flat region of the optimal contract for \( \alpha' \). This is a contradiction, so we cannot have \( \hat{v}(\hat{c}_1 - \delta) < \hat{v}(\hat{c}_1) = \hat{v}(\hat{c}_2) \).

Putting all of this together, we only have one flat region, \([0, \hat{c}_1]\), where the HJB equation holds as an inequality \( A(\hat{c}, \hat{v}(\hat{c})) > 0 \) (the inequality is strict because of Lemma O.11 and the fact that \( \hat{c}_1 \) must be the first root), and a strictly increasing region \([\hat{c}_1, \hat{c}_h]\) where the HJB equation holds with equality. In the flat region we have \( \hat{v}(\hat{c}) = \hat{v}(\hat{c}_1) = \hat{v}_l \) for all \( \hat{c} \leq \hat{c}_1 \). At \( \hat{c}_1 \) we have \( \hat{v}'(\hat{c}_1) = 0 \) and \( A(\hat{c}_1; \hat{v}_l) = 0 \).

4) Now we want to show that \( \hat{v}''(\hat{c}_1) > 0 \). First suppose the \( A(\hat{c}, \hat{v}(\hat{c}_1)) \) is strictly positive below \( \hat{c}_1 \) and strictly negative above it, so that \( A(\hat{c}_1, \hat{v}_l) < 0 \) (we will show that this must indeed be the case below). Consider the first order ODE that results from fixing \( \sigma^* = 0 \) and \( \sigma^x = \frac{\alpha}{\sigma^* \sigma^y \phi} \) in the HJB equation:

\[
\hat{c} - \sigma^* \frac{\alpha}{\phi \sigma} + \hat{v}_f o \left( \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma^x)^2 \right) + \hat{v}_f o \frac{\epsilon^{1-\gamma} - \epsilon^{1-\gamma}_h}{1-\gamma} + \frac{(\sigma^x)^2}{2} = 0
\]

and consider the solution with boundary condition \( \hat{v}_fo(\hat{c}_1) = \hat{v}(\hat{c}_1) \). Since we already know that \( A(\hat{c}_1, \hat{v}(\hat{c}_1)) = 0 \), we must have \( \hat{v}'_fo(\hat{c}_1) = 0 \) because the term in parenthesis is \( \mu^\hat{c} \) if \( \sigma^* = 0 \), and Lemma O.13 shows this is strictly positive under these conditions. Furthermore, we must have \( \hat{v}'_fo(\hat{c}_1) \leq \hat{v}'o(\hat{c}_1) \). To see this, if \( \hat{v}'_fo(\hat{c}_1) > \hat{v}'o(\hat{c}_1) \geq 0 \), then \( \hat{v}'_fo(\hat{c}_1 + \epsilon) > \hat{v}'(\hat{c}_1 + \epsilon) \).
while \( \hat{v}_{fo}(\hat{c}_t + \epsilon) = \hat{v}(\hat{c}_t + \epsilon) + o(\epsilon) \) (because both have first derivatives equal to zero) for some small \( \epsilon \). By continuity, it will still be the case that \( \mu^\hat{c} > 0 \) for \( \hat{c}_t + \epsilon \), so starting at \( \hat{c}_t + \epsilon - \delta \) we will eventually get to \( \hat{c}_t + \epsilon \). We can then solve the first order ODE backwards with boundary condition \( \hat{v}_{fo}(\hat{c}_t + \epsilon) = \hat{v}(\hat{c}_t + \epsilon) \) and we will obtain a lower cost for \( \hat{c}_t + \epsilon - \delta \), i.e. \( \hat{v}_{fo}(\hat{c}_t + \epsilon - \delta) < \hat{v}(\hat{c}_t + \epsilon - \delta) \), because by continuity \( \hat{v}_{fo}(\hat{c}_t + \epsilon) > \hat{v}'(\hat{c}_t + \epsilon) \) for this solution as well. This cannot be, so we must have \( \hat{v}''_{fo}(\hat{c}_t) \leq \hat{v}''(\hat{c}_t) \).

Differentiating the first order ODE with respect to \( \hat{c} \) we obtain

\[
0 = \partial_{\hat{c}} A(\hat{c}, \hat{v}_t)|_{\hat{c}=\hat{c}_t} + \hat{v}''_{fo}(\hat{c}_t)\hat{c}_t \left( \frac{\hat{c}_t^{1-\gamma} - \hat{c}_0^{1-\gamma}}{1-\gamma} + \frac{(\sigma^x)^2}{2} \right)
\]

where we have used \( \hat{v}''_{fo}(\hat{c}_t) = 0 \) and the envelope theorem to compute the derivative \( \partial_{\hat{c}} A(\hat{c}, \hat{v}_t)|_{\hat{c}=\hat{c}_t} \). It follows that since \( A_2(\hat{c}_t, \hat{v}_t) < 0 \), then either \( \hat{v}''_{fo}(\hat{c}_t) > 0 \) and \( \hat{c}_t \) always and \( \sigma^x = \frac{\alpha}{\sigma^\gamma} \hat{c}_t^{\gamma} \). This requires \( \mu^\hat{c} = \sigma^\hat{c} = 0 \). But Lemma O.13 shows that for \( \sigma^\hat{c} = 0 \) and \( \sigma^x = \frac{\alpha}{\sigma^\gamma} \hat{c}_t^{\gamma} \), and \( A(\hat{c}_t, \hat{v}_t) = 0 \) we get \( \mu^\hat{c}(\hat{c}_t) > 0 \). Since the optimal contract must be incentive compatible and achieve cost \( \hat{v}(\hat{c}_t) \), \( A_2(\hat{c}_t, \hat{v}(\hat{c}_t)) = 0 \) cannot be. So we have \( \hat{v}''(\hat{c}_t) > 0 \).

5) \( \hat{v}''(\hat{c}_t) > 0 \) implies \( \sigma^\hat{c}(\hat{c}_t) = 0 \) and \( \mu^\hat{c}(\hat{c}_t) > 0 \). From (39) we get \( \sigma^\hat{c}(\hat{c}) \leq 0 \), and (35) implies \( \sigma^x(\hat{c}) \geq 0 \). It also implies that at \( \hat{c}_t \), since \( \hat{v}''(\hat{c}_t) = 0 \), the choice of \( \sigma^x \) maximizes (20). We want to show that for \( \hat{c} \in (\hat{c}_t, \hat{c}_h) \) we have \( \sigma^\hat{c}(\hat{c}) < 0 \) and \( \sigma^x(\hat{c}) > 0 \). First, from (39) we learn that \( \sigma^x = 0 \) requires \( \hat{v}' = 0 \). If this happens in the interior, because we know that \( \hat{v}' \geq 0 \) always, we get \( \hat{v}'' = 0 \) too. Optimality requires \( \hat{C} \geq 0 \), from (38), so we get \( \hat{v}'' = 0 \) too. Now take the derivative of the HJB equation with respect to \( \hat{c} \), using the envelope theorem, and using \( \hat{v}' = \hat{v}'' = \hat{v}''' = 0 \) to obtain \( A_2(\hat{c}, \hat{v}) = 0 \). But the same argument as in part 4 shows that this cannot be. Therefore we cannot have \( \sigma^\hat{c}(\hat{c}) = 0 \) in \( (\hat{c}_t, \hat{c}_h) \). (35) then shows that \( \sigma^x(\hat{c}) > 0 \) in the interior.

We also want to show that \( \hat{v} < \hat{c}^\gamma \) for all \( \hat{c} \in [\hat{c}_t, \hat{c}_h] \). We know that at \( \hat{c}_t \) we have \( \hat{v}_t \leq \hat{v}_p, \sigma^p = 0, \sigma^x = \frac{\alpha}{\sigma^\gamma} \hat{c}_t^{\gamma} \), and \( A(\hat{c}_t, \hat{v}_t) = 0 \), then from Lemma O.13 we have \( \hat{v}(\hat{c}_t) < \hat{c}_t^{\gamma} \). We also know that \( \hat{v}(\hat{c}) \leq \hat{v}_p(\hat{c}) \), so from Lemma O.14 if ever \( \hat{v}(\hat{c}) = \hat{c}^\gamma \) for some \( \hat{c} > \hat{c}_t \), it must be either that \( \hat{c} = \hat{c}_h \); or that \( \hat{c} \leq \hat{c}_p \) and therefore \( A(\hat{c}, \hat{c}^\gamma) = A(\hat{c}, \hat{v}(\hat{c})) \geq 0 \) and \( \partial_1 A(\hat{c}, \hat{v}(\hat{c})) < 0 \) (because \( \hat{c} \geq \hat{c}_t > 0 \)). From Lemma O.11 we know that \( A(\hat{c}, \hat{v}) \) is positive near 0 and either has one root in \( \hat{c} \) if \( \gamma \geq 1/2 \), or is convex with at most two roots if \( \hat{c} \leq 1/2 \).
This means that $A(\hat{c} - \delta, \hat{c}^\gamma) > 0$ for all $\delta \in (0, \hat{c}]$. Now we’ll use the same reasoning as in part 3. We can pick an $\alpha' < \alpha$ so that $\hat{v}_{\alpha'}(\hat{c}'_t) = \hat{v}(\hat{c})$, and $\hat{c}'_t < \hat{c}$, because $\hat{v}_{\alpha'}(\hat{c})$ is decreasing in $\alpha'$. However, $A_\alpha(\hat{c}, \hat{c}^\gamma)$ is decreasing in $\alpha$, so we get $A_{\alpha'}(\hat{c}'_t, \hat{c}^\gamma) > A(\hat{c}'_t, \hat{c}^\gamma) > 0$, which contradicts $A_{\alpha'}(\hat{c}'_t, \hat{v}_{\alpha'}(\hat{c}'_t)) = 0$. So we conclude that $\hat{v}(\hat{c}) < \hat{c}^\gamma$ for all $\hat{c} \in [\hat{c}_l, \hat{c}_h]$.

6) We want to show that $\hat{c}_l$ and $\hat{c}_h$ are inaccessible from $(\hat{c}_l, \hat{c}_h)$, so that $\hat{c}_l \in (\hat{c}_l, \hat{c}_h)$ for all $t > 0$, $L_t = 0$ always, and the Euler equation holds as an equality. First note that we already have proven that $\sigma^\hat{c}(\hat{c}) < 0$ in the interior, so $\hat{c}_l$ is a regular diffusion. For $\hat{c}_l$, use (39) and replace $\hat{v}' \approx \hat{v}'' \epsilon$ to obtain $\sigma^\hat{c}(\hat{c}_l + \epsilon) \approx K \epsilon$, with $K = -\sigma^z(\hat{c}_l)(1 + \gamma)\hat{c}_l^{-1} < 0$, and recall $\mu^\hat{c}(\hat{c}_l)\hat{c}_l = \bar{\mu} > 0$. Compute the scale function\(^{20}\)

$$S(\hat{c}) = \int_{\hat{c}}^{\hat{c}_l} \exp \left( - \int_{\hat{c}}^{y} \frac{2\mu^\hat{c}(z)z}{\sigma^z(z)^2} dz \right) dy$$

Using the approximation near $\hat{c}_l$ we obtain $S(\hat{c}) = -\infty$, so $\hat{c}_l$ is inaccessible, $P\{\tau_{\hat{c}_l} < \infty\} = 0$, and non-attracting, $P\{\hat{c}_l \rightarrow \hat{c}_l\} = 0$. The speed function is

$$m(\hat{c}) = \frac{1}{(\sigma^\hat{c}(\hat{c})\hat{c})^2} \exp \left( \int_{\hat{c}}^{\hat{c}_l} \frac{2\mu^\hat{c}(z)z}{\sigma^z(z)^2} dz \right)$$

and using the approximation we evaluate

$$\int_{\hat{c}_l}^{\hat{c}_l} S(\hat{c})m(\hat{c})d\hat{c} < \infty$$

So we conclude that $\hat{c}_l$ is an entrance boundary. $\hat{c}$ starts at $\hat{c}_l$ and immediately moves into the interior and never returns.

To show that $\hat{c}_h$ is inaccessible, we use the approximation in Lemma O.16, $\mu^\hat{c}\hat{c} = (4\gamma - 6(1 + \gamma)^2)\hat{c}_h^{-1}\hat{c}_h - \hat{c})^2$ and $(\sigma^\hat{c}\hat{c})^2 = 8(1 + \gamma)^2\hat{c}_h^{-1}\hat{c}_h - \hat{c})^3$. Plug into the expression for the scale function to obtain $S(\hat{c}) = const \times \left( \frac{-1}{K + 1} \right) (\hat{c}_h - \hat{c})^K + 1$, where $K = \gamma \left( \frac{1}{(1 + \gamma)^2} - \frac{3}{2} \right) < -1$ for any $\gamma > 0$. We take $\hat{c} \rightarrow \hat{c}_h$ and obtain $S(\hat{c}_h) = \infty$. This means $\hat{c}_h$ is inaccessible and non-attracting too. In fact, we can also show that $\hat{c}_h$ is a natural boundary because

$$\int_{\hat{c}_h}^{\hat{c}_h} S(\hat{c})m(\hat{c})d\hat{c} = \infty$$

**Theorem 2**

Using the same approximation near $\hat{c}_h$ in Lemma O.16,

$$\mu^\hat{c}\hat{c} = (4\gamma - 6(1 + \gamma)^2)\hat{c}_h^{-1}\hat{c}_h - \hat{c})^2$$

\[(\sigma \dot{c})^2 = 8(1 + \gamma)^2 \epsilon^{-\gamma}(\epsilon - \dot{c})^3\]

we compute the speed measure

\[m(\dot{c}) = (\sigma \dot{c} \epsilon)^{-2} \exp\left(\int \frac{2 \mu \dot{c} \epsilon}{(\sigma \dot{c} \epsilon)^2} \frac{dz}{\dot{c}}\right) = \text{const} \times (\dot{c} - \dot{c})^{-K-3}\]

where \(K = \frac{2}{(1 + \gamma)^2} - \frac{3}{2} < -1\), so that \(-K + 3 = -\left(\frac{2}{(1 + \gamma)^2} + \frac{3}{2}\right) < -3/2\). If there is a stationary distribution it must be proportional to \(m(\dot{c})\) and integrate to 1. But

\[\int \dot{c} m(y) dy = \frac{1}{-K - 2} (\dot{c} - \dot{c})^{-K-2}\]

When we take \(\dot{c} \rightarrow \dot{c}_h\) we find that \(\int \dot{c}_h m(y) dy = \infty\) because \(-K + 2 < 0\), which means there cannot be a stationary distribution.

The same computation near \(\dot{c}_l\) shows that \(\int \dot{c}_l m(y) dy < \infty\). This means that

\[\frac{1}{t} \int_0^t 1_{\dot{c}_l < \dot{c}_l - \epsilon} (\dot{c}_s) ds \rightarrow \frac{\int \dot{c}_l - \epsilon m(\dot{c}) d\dot{c}}{\int \dot{c}_l m(\dot{c}) d\dot{c}} = 0 \quad \text{a.s.} \quad \forall \epsilon > 0\]

which in turn implies that

\[\frac{1}{t} \int_0^t 1_{\dot{c}_l > \dot{c}_l - \epsilon} (\dot{c}_s) ds \rightarrow 1 \quad \text{a.s.} \quad \forall \epsilon > 0\]

**Theorem 3**

First let’s extend the function \(\hat{v}(\dot{c})\) as described above, with \(\hat{v}(\dot{c}) = \hat{v}_l \equiv \hat{v}(\dot{c}_l)\) for all \(\dot{c} < \dot{c}_l\) (we always have \(\dot{c} \in [0, \dot{c}_h]\)). The HJB holds as an equality for \(\dot{c} \geq \dot{c}_l\), but we need to check that it holds as an inequality for \(\dot{c} < \dot{c}_l\). Using the version of \(A(\dot{c}; \hat{v})\) in (34), notice \(A(\dot{c}_l, \hat{v}_l) = 0\), so \(\dot{c}_l\) is a root of \(A(\dot{c}; \hat{v})\). If \(\gamma \geq \frac{1}{2}\), Lemma O.11 says that it can have at most one root, and it’s positive for small \(\dot{c}\), so \(A(\dot{c}; \hat{v}_l) \geq 0\) for all \(\dot{c} < \dot{c}_l\). For \(\gamma < \frac{1}{2}\), it’s convex and can have at most two roots. Condition (22) guarantees that the derivative is negative, so \(\dot{c}_l\) is the smaller root, and we also have \(A(\dot{c}; \hat{v}_l) \geq 0\) for all \(\dot{c} < \dot{c}_l\). Notice that we only need to check (22) if \(\gamma < \frac{1}{2}\).

Consider any incentive compatible contract \(C = (c, k)\) that delivers utility of at least \(u_0\) to the agent, with associated state variables \(x\) and \(\dot{c}\). Because \(\hat{v}'(\dot{c}_l) = 0\) we can use Ito’s lemma\(^{21}\) and the HJB equation to obtain

\[e^{-r} x \hat{v}(\dot{c}_l x) x \hat{v} \geq \hat{v}(\dot{c}_0 x) x - \int_0^x e^{-r} (\dot{c}_l - \dot{k}_l \alpha) x dt\]

\(^{21}\)Notice \(\hat{v}''\) is discontinuous at \(\dot{c}_l\), but this doesn’t change Ito’s formula.
\[
+ \int_0^{\tau_n} e^{-rt} \dot{\hat{v}}(\hat{c}_t)x_t \left( \frac{\dot{\hat{v}}'(\hat{c}_t)}{\hat{v}(\hat{c}_t)} \dot{\hat{c}}_t \sigma^x_t + \sigma^x_t \right) \, dZ_t
\]

for localizing sequence of stopping times \( \{\tau_n\} \to \infty \text{ a.s.} \) such that the stopped integrals are martingales. Take expectations to obtain

\[
\mathbb{E}^Q_0 \left[ e^{-r\tau_n} \hat{v}(\hat{c}_{\tau_n})x_{\tau_n} \right] \geq \hat{v}(\hat{c}_0)x_0 - \mathbb{E}^Q_0 \left[ \int_0^{\tau_n} e^{-rt} (c_t - k_t \alpha) \, dt \right]
\]

(40)

Now we would like to take the limit \( n \to \infty \), but we need to use the dominated convergence theorem. First,

\[
\left| \int_0^{\tau_n} e^{-rt} (c_t - k_t \alpha) \, dt \right| \leq \int_0^\infty e^{-rt} |c_t - k_t \alpha| \, dt \leq \int_0^\infty e^{-rt} (|c_t + k_t \alpha|) \, dt
\]

which is integrable because the contract is admissible. Second, for an admissible contract

\[
0 \leq \lim_{n \to \infty} \mathbb{E}^Q_0 \left[ e^{-r\tau_n} \hat{v}(\hat{c}_{\tau_n})x_{\tau_n} \right] \leq \lim_{n \to \infty} \mathbb{E}^Q_0 \left[ e^{-r\tau_n} \hat{v}_h x_{\tau_n} \right] = 0
\]

To see why the last equality holds, notice that since \( \hat{v}_h x \) is the cheapest way of delivering utility to the agent without capital, the cost of consumption on the contract is

\[
\infty > \mathbb{E}^Q_0 \left[ \int_0^\infty e^{-rt} c_t \, dt \right] \geq \mathbb{E}^Q_0 \left[ \int_0^{\tau_n} e^{-rt} c_t \, dt + e^{-r\tau_n} \hat{v}_h x_{\tau_n} \right]
\]

Taking the limit \( n \to \infty \) and using the monotone convergence theorem, we obtain \( 0 \leq \lim_{n \to \infty} \mathbb{E}^Q_0 \left[ e^{-r\tau_n} \hat{v}_h x_{\tau_n} \right] \leq 0 \).

Upon taking the limit \( n \to \infty \) in (40), we obtain

\[
\mathbb{E}^Q_0 \left[ \int_0^\infty e^{-rt} (c_t - k_t \alpha) \, dt \right] \geq \hat{v}(\hat{c}_0)x_0
\]

Using \( \hat{v}(\hat{c}_t) \leq \hat{v}(\hat{c}_0) \) and \( x_0 \geq ((1 - \gamma)u_0)^{\frac{1}{1 - \gamma}} \) we obtain the first result.

For the second part, first let’s show that \( C^* \) is incentive compatible. We already know that for the HJB to have a solution with well defined policy functions it must be the case that \( \hat{B} \geq 0 \) and \( \hat{C} > 0 \), defined in (37) and (38), and therefore \( \sigma^{x*} \leq 0 \) and \( \sigma^{x*} \geq 0 \). If \( \hat{c}_t^* \in [\hat{c}_l, \hat{c}_h] \), because \( \sigma^{x*} \) is bounded, so is \( \sigma^{x*} \) and therefore so is \( \mu^{x*} \). We can then use Theorem 4 and the fact that \( h_0 = 0 \) to show that

\[
U^{c*,a}_0 \leq U^{c*,0}_0 = u_0
\]

for any feasible strategy \( (\hat{c}, a) \), and therefore \( C^* \) is indeed incentive compatible. To show that the cost of the contract is \( \hat{v}_l x_0^* \), we can use the HJB. If \( \hat{c}_t^* \in [\hat{c}_l, \hat{c}_h] \) always, where the HJB holds, then the same argument as in the first part shows the cost is \( \hat{v}_l x_0^* \). Notice
that with \( \hat{\nu}'(\hat{c}_l) = 0 \) and \( \hat{\nu}''(\hat{c}_l) > 0 \) we get that as \( \hat{c} \downarrow \hat{c}_l \), \( \hat{B} \rightarrow 0 \) and \( \hat{C} \rightarrow \hat{C}_l \), so \( \sigma^{\hat{\nu}} \rightarrow 0 \), \( \sigma^{x^*} \rightarrow \frac{\sigma_l^{x^*}}{\sigma \gamma_l^{x^*}} \), and \( A(\hat{c}_l; \hat{v}_l) = 0 \). From the first part we know \( \hat{\nu}(\hat{c}) \) is weakly below the true cost function, which is also bounded above by \( \hat{\nu}_h \), so there is a finite cost function and \( \hat{v}_l \) is weakly below it, and therefore \( \hat{v}_l \leq \hat{v}_p \) where \( \hat{v}_p \) is defined in (27). Lemma O.13 shows that under these conditions \( \mu(\hat{c}_l) > 0 \). Also, \( \hat{\nu}''(\hat{c}_l) > 0 \) implies \( \hat{\nu}'(\hat{c}_l + \epsilon) \approx \hat{\nu}'' \epsilon \), so \( \sigma^{\hat{\nu}}(\hat{c}_l + \epsilon) \approx K \epsilon \), with \( K = -\sigma^x(\hat{c}_l)(1 + \gamma)\hat{c}_l^{-1} < 0 \). As a result, \( \hat{c}_l \) is inaccessible from the interior of the domain, and therefore \( \hat{c}_l^* \in [\hat{c}_l, \hat{c}_h] \). Notice that the candidate contract does indeed deliver utility \( u_0 \) to the agent. To see this let \( U^* = \frac{(x^*)^{1-\gamma}}{1-\gamma} \), so using the law of motion of \( x^* \), (14), we get

\[
U_0^* = \mathbb{E} \left[ \int_0^{\tau_n} e^{-\rho t} \frac{c_{l}^{1-\gamma}}{1-\gamma} dt + e^{-\rho \tau_n} U_{\tau_n}^* \right]
\]

with some sequence \( \tau_n \rightarrow \infty \) a.s. Use the monotone convergence theorem and notice that

\[
\lim_{n \rightarrow \infty} \mathbb{E} \left[ e^{-\rho \tau_n} U_{\tau_n}^* \right] = 0
\]

because \( \rho - (1 - \gamma)(\mu^{x^*} - \frac{\gamma}{2}(\sigma^{x^*})^2) = \hat{c}^{1-\gamma} \geq \min\{\hat{c}_l^{1-\gamma}, \hat{c}_h^{1-\gamma}\} > 0 \). We then get that \( U_0^{\tau_n,0} = U_0^* = u_0 \). This completes the proof.

**Lemma 4**

We know from the proof of Theorem 3 that \( \hat{c}_l^* \in [\hat{c}_l, \hat{c}_h] \) and recall that \( \hat{c}_l > 0 \). Then an upper bounded \( \mu^{x^*} < r \) implies a bounded \( 0 \leq \sigma^{x^*} \leq \hat{\sigma}_X \). Then

\[
\mathbb{E}Q \left[ \int_0^\infty e^{-rt}(|c_l^{x^*}| + |k_l^{x^*} \alpha|)dt \right] \leq 2 \max \left\{ \hat{c}_h, \frac{\hat{\sigma}_X c_l^{x^*} \alpha}{\hat{\sigma}} \right\} \mathbb{E}Q \left[ \int_0^\infty e^{-rt}x_l^{x^*}dt \right] < \infty
\]

where the last inequality follows from \( \mu^{x^*} < r \). Let \( U^* = \frac{(x^*)^{1-\gamma}}{1-\gamma} \), so using the law of motion of \( x^* \), (14), we get

\[
U_0^* = \mathbb{E} \left[ \int_0^{\tau_n} e^{-\rho t} \frac{c_l^{1-\gamma}}{1-\gamma} dt + e^{-\rho \tau_n} U_{\tau_n}^* \right]
\]

with \( \tau_n \rightarrow \infty \) a.s. Use the monotone convergence theorem and notice that

\[
\lim_{n \rightarrow \infty} \mathbb{E} \left[ e^{-\rho \tau_n} U_{\tau_n}^* \right] = 0
\]

because \( \rho - (1 - \gamma)(\mu^{x^*} - \frac{\gamma}{2}(\sigma^{x^*})^2) = \hat{c}^{1-\gamma} \geq \min\{\hat{c}_l^{1-\gamma}, \hat{c}_h^{1-\gamma}\} > 0 \). We then get that \( U_0^{\tau_n,0} = U_0^* = u_0 \). We conclude that the contract is indeed admissible.
Theorem 4

We’ll do the proof for \( dL_t = 0 \) to keep it simple; it can be easily generalized for \( dL_t \neq 0 \). First write the bound \( (1 + \hat{h}_t c_t^{-\gamma})^{1-\gamma} U_t^{c,0} = \frac{\tilde{x}_t^{1-\gamma}}{1-\gamma} \), where \( \tilde{x}_t = x_t + h_t \bar{c}_t^{-\gamma} \), and define \( \hat{c}_t = \tilde{c}_t/\tilde{x}_t \).

For any feasible strategy \((\tilde{c},a)\), we can write the difference between the utility of that strategy, \( U_t^{\tilde{c},a} \), and the bound \( \frac{\tilde{x}_t^{1-\gamma}}{1-\gamma} \) as

\[
e^{-\rho t} \left( U_t^{\tilde{c},a} - \frac{\tilde{x}_t^{1-\gamma}}{1-\gamma} \right) = E_t \left[ \int_t^{\tau^n} e^{-\rho u} \frac{\tilde{c}_u^{1-\gamma}}{1-\gamma} du + \int_t^{\tau^n} d \left( e^{-\rho u} \frac{\tilde{c}_u^{1-\gamma}}{1-\gamma} \right) + e^{-\sigma^n} \left( U_t^{\tilde{c},a} - \frac{\tilde{x}_t^{1-\gamma}}{1-\gamma} \right) \right]
\]

for a localizing sequence \( \{\tau^n\} \) with \( \tau^n \to \infty \) a.s. We would like to show that this is always non-positive.

First, take the first two terms on the rhs and write them:

\[
E_t \left[ \int_t^{\tau^n} e^{-\rho u} \frac{\tilde{c}_u^{1-\gamma}}{1-\gamma} du + \int_t^{\tau^n} d \left( e^{-\rho u} \frac{\tilde{c}_u^{1-\gamma}}{1-\gamma} \right) \right] \tag{41}
\]

where \( \mu^\hat{c} \) and \( \sigma^\hat{c} \) are the geometric drift and volatility of \( \hat{x} \), and satisfy

\[
\hat{x}_t \mu^\hat{c}_t = x_t \left( \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma^\hat{c}_t)^2 - \frac{\sigma^\hat{c}_t}{\sigma} a_t \right) + \hat{c}_t^{-\gamma} (r h_t + c_t - \hat{c}_t + \phi k_t a_t) + h_t \hat{c}_t^{-\gamma} \left( \rho - r - \hat{c}_t^{1-\gamma} - \frac{\gamma}{2} (\sigma^\hat{c}_t)^2 - \gamma^2 \sigma^\hat{c}_t a_t \right)
\]

\[
\hat{x}_t \sigma^\hat{c}_t = \sigma^\hat{c}_t x_t - h_t \hat{c}_t^{-\gamma} \gamma \sigma^\hat{c}_t
\]

We will show that the rhs of (41) is non-positive. To do this, we will split the integrand into three parts,

\[
\frac{\tilde{c}_u^{1-\gamma}}{1-\gamma} \tilde{x}_u + \tilde{x}_u \mu^\hat{c}_u - \tilde{x}_u \frac{\gamma}{2} (\sigma^\hat{c}_u)^2 = A_t a_t + B_t + C_t
\]

and we will show that \( A_t \), \( B_t \), and \( C_t \) are always non-positive.

The term \( A_t \) collects the terms that multiply \( a_t \):

\[
A_t = \hat{c}_t^{-\gamma} \phi k_t - x_t \frac{\sigma^\hat{c}_t}{\sigma} + h_t \hat{c}_t^{-\gamma} \gamma \sigma^\hat{c}_t \leq 0 \quad \text{(0 by IC)}
\]

\[
\leq 0 \quad \text{(wlog.)}
\]

---

\[22\] If \( dL_t \neq 0 \) we would get an extra negative term on the rhs; \( dL_t > 0 \) pushes \( \hat{x} \) down because \( h_t \geq 0 \). Since we want to prove that the rhs is non-positive, we can take \( dL_t = 0 \) wlog.
where the second term is non-positive because \( h_t \geq 0 \) and \( \sigma_t^x \leq 0 \). Since \( a_t \geq 0 \), we can wlog take \( a_t = 0 \).

The term \( B_t \) collects the remaining terms that don’t have volatilities

\[
B_t = x_t \frac{\rho - \frac{c^1}{1 - \gamma}}{1 - \gamma} + \hat{c}_t \left( r h_t + \hat{c}_t x_t - \bar{c}_t \bar{x}_t \right) + h_t \hat{c}_t \left( \rho - r - \frac{c^1}{1 - \gamma} \right) + \bar{x}_t \frac{c^1}{1 - \gamma} - \rho
\]

This expression is maximized for \( \hat{c}_t = \dot{c}_t \), and then it simplifies to

\[
B_t \leq \dot{c}_t \left( -\dot{c}_t h_t \dot{c}_t \right) + h_t \dot{c}_t \left( -\frac{c^1}{1 - \gamma} \right) + h_t \dot{c}_t \frac{c^1}{1 - \gamma} = 0
\]

Finally, \( C_t \) collects the terms involving volatilities

\[
C_t = x_t \frac{\gamma}{2} (\sigma_t^x)^2 + h_t \dot{c}_t \left( -\frac{\gamma}{2} (\sigma_t^x)^2 - \gamma (\sigma_t^x)^2 \right) - \bar{x}_t \frac{\gamma}{2} (\sigma_t^x)^2
\]

\[
C_t = \frac{\gamma}{2} \left( x_t (\sigma_t^x)^2 - h_t \dot{c}_t \gamma (\sigma_t^x)^2 - h_t \dot{c}_t \gamma \sigma_t^x + \frac{1}{\bar{x}_t} (\bar{x}_t \sigma_t^x)^2 \right)
\]

\[
C_t = \frac{\gamma}{2} \left( x_t (\sigma_t^x)^2 - h_t \dot{c}_t \gamma (\sigma_t^x)^2 - h_t \dot{c}_t \gamma 2 \sigma_t^x \sigma_t^x - \frac{1}{\bar{x}_t} \left( x_t (\sigma_t^x)^2 + (h_t \dot{c}_t \gamma \sigma_t^x)^2 - x_t \sigma_t^x h_t \dot{c}_t \gamma \sigma_t^x \right) \right)
\]

\[
C_t = \frac{\gamma}{2} \left( \frac{1}{\bar{x}_t} \left( -\left( h_t \dot{c}_t \gamma \right)^2 (\sigma_t^x)^2 - (h_t \dot{c}_t \gamma)^2 2 \gamma \sigma_t^x \sigma_t^x - (h_t \dot{c}_t \gamma \sigma_t^x)^2 \right) \right)
\]

\[
C_t = \frac{\gamma}{2} \left( \frac{(h_t \dot{c}_t \gamma)^2}{\bar{x}_t} \left( (\sigma_t^x)^2 + 2 \gamma \sigma_t^x \sigma_t^x + (\gamma \sigma_t^x)^2 \right) = \frac{\gamma}{2} \left( \frac{(h_t \dot{c}_t \gamma)^2}{\bar{x}_t} \left( \sigma_t^x + \gamma \sigma_t^x \right)^2 \right) \leq 0
\]

Putting this together and plugging into (41), we get

\[
e^{-\rho t} \left( U_t^{\hat{c}, a} - \frac{\bar{x}_t^{1 - \gamma}}{1 - \gamma} \right) \leq E_t^a \left[ e^{-\rho r n} \left( U_t^{\hat{c}, a} - \frac{\bar{x}_t^{1 - \gamma}}{1 - \gamma} \right) \right]
\]

Taking the limit \( n \to \infty \), it only remains to show that the tail term \( \lim_{n \to \infty} E_t^a \left[ e^{-\rho r n} \left( U_t^{\hat{c}, a} - \frac{\bar{x}_t^{1 - \gamma}}{1 - \gamma} \right) \right] \leq 0.\)

\( ^{23} \text{Notice that if } \phi = 1 \text{ the expression for } A_t \text{ is valid also if we allow } a_t \in \mathbb{R}. \) As a result, in this case contracts with \( \sigma_t^x = 0 \) are IC even if we allow \( a_t \in \mathbb{R}. \).
Since the agent’s strategy \((\tilde{c}, a)\) is feasible, we have that

\[
\lim_{n \to \infty} \mathbb{E}_t^a \left[ e^{-\rho \tau_n U_{\tau_n}^{\tilde{c}, a}} \right] = 0
\]

For \(\gamma < 1\) we have \(\frac{\bar{x}_t^{1-\gamma}}{1-\gamma} \geq 0\), so it follows that

\[
\lim_{n \to \infty} \mathbb{E}_t^a \left[ e^{-\rho \tau_n \frac{\bar{x}_t^{1-\gamma}}{1-\gamma}} \right] \geq 0
\]

As a result, when we take \(n \to \infty\) we get \(\lim_{n \to \infty} \mathbb{E}_t^a \left[ e^{-\rho \tau_n (U_{\tau_n}^{\tilde{c}, a} - \frac{\bar{x}_t^{1-\gamma}}{1-\gamma})} \right] \leq 0\), and therefore \(U_{\tau_n}^{\tilde{c}, a} \leq \bar{x}_t^{1-\gamma}\) as desired.

For \(\gamma > 1\), if \(\lim_{n \to \infty} \mathbb{E}_t^a \left[ e^{-\rho \tau_n \frac{\bar{x}_t^{1-\gamma}}{1-\gamma}} \right] = 0\) for any feasible strategy \((\tilde{c}, a)\), then we are done. To show this, notice that the law of motion of \(\bar{x}\) satisfies

\[
d\bar{x}_t \leq (\lambda_1 \bar{x}_t - \lambda_2 \hat{c}_t) dt + \sigma_1 \bar{x}_t dZ_t^a
\]

where \(\lambda_2 = c_h^{-\gamma} > 0\), and \(\lambda_1 = \tilde{\mu} x + r + \hat{c}^{1-\gamma} + \gamma|\tilde{\mu}^{-}\hat{c}| + \frac{1}{2}\gamma(1 + \gamma)(\tilde{\sigma}^{-\hat{c}})^2\). Here’s where we use the assumption that \(\mu^x, \tilde{\mu}^{-}, \sigma^c\) are bounded and \(\hat{c} \leq \hat{c}_h\) is uniformly bounded away from 0; \(\mu^x, \tilde{\mu}^{-}, c, \sigma^c\) are appropriate bounds on each process. Notice that stealing only reduces the drift of \(\bar{x}_t\), since the change in the drift of \(x\) and \(h\) cancel out, and it increases the drift of \(\hat{c}\). Since \(\bar{x}_t > 0\) always, Lemma O.12 and its corollary ensure the desired limit.
Online Appendix

This Online Appendix extends the results of Di Tella and Sannikov (2016) to incorporate hidden investment, aggregate risk, and renegotiation. The case with no hidden investment and price of aggregate risk $\pi = 0$ yields the expressions in the paper.

Aggregate risk and hidden investment

We introduce aggregate risk and hidden investment into the baseline setting in the paper. The observed return is:

$$dR_t = (r + \pi \tilde{\sigma} + \alpha - a_t) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t$$

where $Z$ and $\tilde{Z}$ are independent Brownian motions that represents idiosyncratic and aggregate risk. There is a complete financial market with equivalent martingale measure $Q$. The risk-free rate is $r$, aggregate risk has market price $\pi$, and idiosyncratic risk is not priced. Capital has a loading $\sigma$ on idiosyncratic risk and $\tilde{\sigma}$ on aggregate risk, so the excess return on capital for the agent is $\alpha$, as in the baseline.

The agent receives cumulative payments $I$ from the principal and manages capital $k$ for him. Payments $I$ can be any semimartingale (it could be decreasing if the agent must pay the principal). This nests the relevant case where the contract gives the agent only what he will consume, i.e. $dI_t = c_t dt$. As in the baseline setting, the agent can steal from the principal at rate $a \geq 0$ and decide when to consume $\tilde{c} \geq 0$. He can invest his hidden savings in the same way the principal would, not only in a risk-free asset, but also in aggregate risk $\tilde{Z}$. In addition, the agent may be able to invest his hidden savings in his private technology. His hidden savings follow the law of motion

$$dh_t = dI_t + (rh_t + z_t h_t (\alpha + \pi \tilde{\sigma})) dt + z_t h_t \left( \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t \right) + \tilde{z}_t h_t d\tilde{Z}_t$$

where $z$ is the portfolio weight on his own private technology, and $\tilde{z}$ the weight on aggregate risk. While the agent can chose any position on aggregate risk, $\tilde{z}_t \in \mathbb{R}$, for his hidden private investment we consider two cases: 1) no hidden private investment, $z_t \in H = \{0\}$, and 2) hidden private investment, $z_t \in H = \mathbb{R}^+$.\textsuperscript{24}

The agent’s utility is

$$U_0 = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right]$$

\textsuperscript{24}We can also study other cases where the agent may not be able to invest in aggregate risk, or only take a positive position, which requires small modifications to the relevant incentive compatibility constraints. We focus on the economically most relevant case, where the agent can always invest his hidden savings in the market in the same way the principal would.
and the cost to the principal is

\[ J_0 = \mathbb{E}^Q \left[ \int_0^\infty e^{-rt} \left( dI_t - (\alpha - a_t)k_t \right) dt \right] \]

A contract \( C = (I, k, \tilde{c}, a, z, \tilde{z}) \) specifies the contractible payments \( I \) and capital \( k \), and recommends the hidden action \((\tilde{c}, a, z, \tilde{z})\), all contingent on the history of observed returns \( \bar{R} \) and the aggregate shock \( \tilde{Z} \). After signing the contract the agent can choose a strategy \((\tilde{c}, a, z, \tilde{z})\) to maximize his utility (potentially different from the one recommended by the principal). Given contract \( C \), a strategy is feasible if 1) there is a finite utility \( U_0^{\tilde{c},a,z,\tilde{z}} \), and 2) hidden savings \( h_t \geq 0 \) always. Since the agent can secretly invest in his private technology, we also impose the No-Ponzi condition on him 3) \( \mathbb{E}^Q \left[ \int_0^\infty e^{-rt} (\tilde{c}_t + \alpha z_t h_t) dt \right] < \infty \). Let \( S(C) \) be the set of feasible strategies given contract \( C \).

A contract \( C = (I, k, \tilde{c}, a, z, \tilde{z}) \) is admissible if 1) \((\tilde{c}, a, z, \tilde{z})\) is feasible given \( C \), and 2) \( \mathbb{E}^Q \left[ \int_0^\infty e^{-rt} (dI_t + k_t \alpha dt + a_t k_t dt) \right] < \infty \) (O.2)

An admissible contract \( C = (I, k, \tilde{c}, a, z, \tilde{z}) \) is incentive compatible if the agent’s optimal feasible strategy given \( C \) is \((\tilde{c}, a, z, \tilde{z})\), as recommended by the principal. Let \( \mathbb{I}C \) be the set of incentive compatible contracts. An incentive compatible contract is optimal if it minimizes the principal’s cost

\[ v_0 = \min_{C} J_0(C) \equiv \mathbb{E}^Q \left[ \int_0^\infty e^{-rt} (dI_t - (\alpha - a_t)k_t) dt \right] \]

st : \( U_0^{\tilde{c},a,z,\tilde{z}} \geq u_0 \)

\( C \in \mathbb{I}C \)

To incorporate aggregate risk into the setting, we need to slightly modify the parameter restrictions. We assume throughout that

\[ \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 > 0 \]

\[ \alpha < \tilde{\alpha} \equiv \frac{\phi \sigma \gamma}{\sqrt{1 + \gamma}} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2} \]

No stealing or hidden savings in the optimal contract

**Lemma O.1.** It is without loss of generality to look only at contracts that induce no stealing \( a = 0 \), no hidden savings, \( h = 0 \), and no hidden investment, \( z = \tilde{z} = 0 \).

**Remark.** This lemma is also valid for the baseline setting without aggregate risk or hidden
hidden savings, and no hidden investment, i.e. \( h = 0 \). Let \( k^h = zh \) and \( \tilde{k}^h = \tilde{z}h \) be the agent’s absolute hidden positions in his private technology and aggregate risk respectively. We will show that we can offer a new contract \( C' = (I', k', dI', 0, 0, 0) \) under which it is optimal for the agent to choose not to steal, no hidden savings, and no hidden investment, i.e. \( \tilde{c}' = dI', a = z = \tilde{z} = 0 \). The new contract has \( I'_t = \int_0^t \tilde{c}_s ds \) and \( k'_t = k(R^{a'}) + k^h \) (to simplify notation, we suppress dependence on \( \tilde{Z} \)).

If the agent now chooses \( \tilde{c}' = dI', a = z = \tilde{z} = 0 \), he gets hidden savings \( h' = 0 \) and consumption \( \tilde{c} \), so he gets the same utility as under the original contract and this strategy is therefore feasible under the new contract. If instead he chooses a different feasible strategy \( (\tilde{c}', a', z', \tilde{z}') \), he gets the utility associated with \( \tilde{c}' \). We will show that he could achieve this utility under the original contract by picking consumption \( \tilde{c}' \), stealing \( dR - dR^a(R^{a'}) \), hidden investment in private technology \( k^h(R^{a'}) + (k^h)' \), and hidden investment in aggregate risk \( \tilde{k}^h(R^{a'}) + (\tilde{k}^h)' \). Since the strategy \( (\tilde{c}', a', z', \tilde{z}') \) is feasible under the new contract \( C' \), and \( (\tilde{c}, a, z, \tilde{z}) \) feasible under the old contract \( C \), then in order to ensure the new strategy is feasible under the original contract we only need to show that hidden savings remain non-negative always

\[
h'_t = \int_0^t e^{r(t-s)} \left( dI_t(R^a(R^{a'})) - \tilde{c}'_t dt + \phi k'_t(R^a(R^{a'})) (dR_t - dR^a_t(R^{a'}))+(k^h(R^{a'}))dR_t + (\tilde{k}^h(R^{a'})) (\pi dt + d\tilde{Z}_t) \right)
\]

To show this is always non-negative, we will show it’s greater or equal to the sum of two non-negative terms. First, the hidden savings under the original contract, following the original feasible strategy, had \( R^{a'} \) been the true return

\[
A_t = \int_0^t e^{r(t-s)} \left( dI_t(R^a(R^{a'})) - \tilde{c}_t(R^{a'}) dt + \phi k_t(R^a(R^{a'})) (dR_t - dR^a_t(R^{a'}))+(k^h(R^{a'}))dR_t + (\tilde{k}^h(R^{a'})) (\pi dt + d\tilde{Z}_t) \right) \geq 0
\]

Second, hidden savings under the new contract, following the feasible new strategy

\[
B_t = \int_0^t e^{r(t-s)} \left( \tilde{c}_t(R^{a'}) dt - \tilde{c}'_t dt + \phi (k_t(R^a(R^{a'})) + k^h(R^{a'})) (dR_t - dR^a_t) +(k^h)'(dR_t) + (\tilde{k}^h)'(\pi dt + d\tilde{Z}_t) \right) \geq 0
\]

If \( \phi = 1 \) then \( h'_t = A_t + B_t \geq 0 \). With \( \phi < 1 \), we have \( h'_t \geq A_t + B_t \geq 0 \), because \( dR_t - dR^a_t = a'dt \geq 0 \) and \( k^h(R^{a'}) \geq 0 \). This means that \( \tilde{c}' = \tilde{c}, a = z = \tilde{z} = 0 \) is the agent’s optimal choice under the new contract \( C' \), since any other choice delivers an utility that he could have obtained - but chose not to - under the original contract \( C \).
We can now compute the principal’s cost under the new contract

\[ J'_0 = \mathbb{E}^Q \left[ \int_0^{\tau^*} e^{-rt} \left( \tilde{c}_t - \alpha \left( k_t(R^a) + k^h_t \right) \right) dt + e^{-r\tau^*} J'_{\tau^*} \right] = \]

\[
\mathbb{E}^Q \left[ \int_0^{\tau^*} e^{-rt} \left( dI_t(R^a) - \alpha \left( k_t(R^a) + k^h_t \right) \right) dt + e^{-r\tau^*} J'_{\tau^*} \right] = \mathbb{E}^Q \left[ \int_0^{\tau^*} e^{-rt} \left( dI_t(R^a) - \alpha \left( k_t(R^a) + k^h_t \right) \right) dt \right]

On the rhs, the first term is the cost under the original contract; the second term the destruction produced by stealing under the original contract, which is non-negative; and the third term is \( \mathbb{E}^Q \left[ e^{-r\tau^*} h_{\tau^*} \right] \geq 0 \), where \( h \) is the agent’s hidden savings under the original contract. To see this, write

\[ dh_t = h_t r dt + dI_t(R^a) - \tilde{c}_t dt + \phi k_t(R^a) a_t dt + k^h_t ((\alpha + \pi \tilde{\sigma}) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t) + k^h_t (\pi dt + d\tilde{Z}_t) \]

So

\[ d(e^{-rt} h_t) = e^{-rt} dh_t - re^{-rt} h_t dt \]

\[ = e^{-rt} \left( dI_t(R^a) - \tilde{c}_t dt + \phi k_t(R^a) a_t dt + k^h_t ((\alpha + \pi \tilde{\sigma}) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t) + k^h_t (\pi dt + d\tilde{Z}_t) \right) \]

Now take expectations under \( Q \), choosing the localizing process appropriately to get

\[ \mathbb{E}^Q \left[ \int_0^{\tau^*} d(e^{-rt} h_t) \right] = \mathbb{E}^Q \left[ \int_0^{\tau^*} e^{-rt} \left( dI_t(R^a) - \tilde{c}_t dt + \phi k_t(R^a) a_t dt + k^h_t \alpha dt \right) \right] \]

\[ = \mathbb{E}^Q \left[ e^{-r\tau^*} h_{\tau^*} - h_0 \right] \geq 0 \]

Given these inequalities, we can write:

\[ J'_0 - J_0 \leq \mathbb{E}^Q \left[ e^{-r\tau^*} \left( J'_{\tau^*} - J_{\tau^*} \right) \right] \]

Because the original contract was admissible, \( \lim_{\tau^* \to \infty} \mathbb{E}^Q \left[ e^{-r\tau^*} J_{\tau^*} \right] = 0 \). Since in addition the agent’s response was feasible, the new contract is also admissible, and we get \( \lim_{\tau^* \to \infty} \mathbb{E}^Q \left[ e^{-r\tau^*} J'_{\tau^*} \right] = 0 \) as well. This shows the new contract is admissible, and the cost for the principal is not greater than under the old contract. This completes the proof.

\[ \square \]

We can then simplify the contract to \( C = (c, k) \), and say an admissible contract is incentive compatible if the agent’s optimal strategy is \( (c, 0, 0, 0) \), or \( (c, 0) \) for short.
Incentive compatibility

Since the contract can depend on the history of aggregate shocks \( \tilde{Z} \), so can his continuation utility \( U_{c,0} \) and his consumption \( c \). However, because the agent is not responsible for aggregate shocks, incentive compatibility does not place any constraints on his exposure to aggregate risk. On the other hand, since the agent can invest his hidden savings, his Euler equation needs to be modified appropriately. The discounted marginal utility of a hidden dollar must be a supermartingale under any feasible hidden investment strategy, since otherwise the agent could save a dollar instead of consuming it, invest it in aggregate risk and his private technology, and consume it later when the marginal utility is expected to be higher.

**Lemma O.2.** Take the agent’s hidden investment possibility set \( H \) as given. If \( C = (c,k) \) is an incentive compatible contract, the agent’s continuation utility \( U_{c,0} \) and consumption \( c \) satisfy the laws of motion

\[
dU_{c,0} = \left( \rho U_{c,0} - \frac{c_t^{1-\gamma}}{1-\gamma} \right) dt + \Delta_t \sigma dZ_t + \tilde{\sigma}_t d\tilde{Z}_t \quad (O.3)
\]

\[
dc_t = \left( \frac{r - \rho}{\gamma} + \frac{1+\gamma}{2}(\dot{\sigma}_t^2) + \frac{1+\gamma}{2}(\tilde{\sigma}_t^2) \right) dt + \sigma_t dZ_t + \tilde{\sigma}_t d\tilde{Z}_t + dL_t \quad (O.4)
\]

for some \( \Delta, \tilde{\sigma}, \sigma^c, \tilde{\sigma}^c, \) and a weakly increasing processes \( L \), such that

\[
\Delta_t \geq c_t^{-\gamma} \phi k_t \quad (O.5)
\]

\[
z(\alpha - \sigma_t^c \sigma \gamma) \leq 0 \quad \forall z \in H \quad (O.6)
\]

\[
\tilde{\sigma}_t^c = \frac{\pi}{\gamma} \quad (O.7)
\]

**Proof.** The proof of (O.3) and (O.5) are similar to Lemma 1 and 2, where the \( d\tilde{Z} \) term appears because the contract can depend on the history of aggregate shocks. For (O.4), the proof is analogous to Lemma 2, but now we need the discounted marginal utility

\[
Y_t = e^{\int_0^t r - \rho + z_s (\alpha + \pi \tilde{\sigma} + \pi \tilde{z}_s - \frac{1}{2}(z_s \tilde{\sigma} + \tilde{z}_s)^2 - \frac{1}{2}(z_s \tilde{\sigma} + \tilde{z}_s)^2 ds + \int_0^t (z_s \sigma + \tilde{z}_s dZ_s + \int_0^t (z_s \sigma + \tilde{z}_s) d\tilde{Z}_s c_t^{-\gamma} \quad (O.8)
\]

to be a supermartingale for any investment strategy \( \tilde{z}_t \in \mathbb{R} \) and \( z_t \in H \). Using the Doob-Meyer decomposition, the Martingale Representation theorem, and Ito’s lemma, we can write

\[
dc_t = \mu_t dt + \tilde{\sigma}_t d\tilde{Z}_t + \sigma_t dZ_t + dL_t
\]

Since the drift of expression (O.8) must be weakly negative, we get

\[
\left( r - \rho - \gamma \mu_t^c + \frac{\gamma}{2} ((1+\gamma)\sigma_t^c)^2 + \frac{\gamma}{2} ((1+\gamma)\tilde{\sigma}_t^c)^2 \right) dt \quad (O.9)
\]
+ (z(\alpha + \pi \tilde{\sigma}) + \pi \tilde{z} - \gamma \tilde{\sigma}(z\tilde{\sigma} + \tilde{z})) dt - \gamma dL_t \leq 0

Taking \( z = \tilde{z} = 0 \), which are always allowed, we obtain wlog the expression for \( \mu^c \) in (O.4), and \( L \) weakly increasing. Once we plug this into (O.9), and using that \( \tilde{z}_t \) can be both positive or negative, we get (O.7). Condition (O.6) is therefore necessary to ensure (O.9) holds.

\[ \square \]

The IC constraint (O.6) depends on whether the agent is allowed to have a hidden investment in his own private technology. If hidden investment in the agent’s private technology is not allowed, \( H = \{0\} \) so condition (O.6) drops out. If instead hidden investment in the agent’s private technology is allowed, \( H = \mathbb{R}_+ \), so condition (O.6) reduces to \( \sigma^c_t \geq \frac{\alpha}{\sigma^c_t} \).

**State Space**

We can still use the the state variables \( x \) and \( \hat{c} \). Their laws of motion are

\[
\frac{dx_t}{x_t} = \left( \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x_t)^2 + \frac{\gamma}{2}(\tilde{\sigma}^x_t)^2 \right) dt + \sigma^x_t dZ_t + \tilde{\sigma}^x_t d\tilde{Z}_t \tag{O.10}
\]

\[
\frac{d\hat{c}_t}{\hat{c}_t} = \left( \frac{r - \rho}{\gamma} - \rho - \hat{c}_t^{1-\gamma} + \frac{(\sigma^x_t)^2}{2} + \gamma \sigma^x_t \tilde{\sigma}^x_t + \frac{1 + \gamma (\sigma^x_t)^2}{2} \right. \\
\left. \quad+ \frac{(\tilde{\sigma}^x_t)^2}{2} + \gamma \tilde{\sigma}^x_t \hat{c}_t \right) dt + \sigma^x_t dZ_t + \tilde{\sigma}^x_t d\tilde{Z}_t + dL_t \tag{O.11}
\]

and the incentive compatibility constraints can be written

\[
\sigma^x_t = \hat{c}_t^{1-\gamma} \hat{\phi}_t \sigma \tag{O.12}
\]

\[
z(\alpha - (\sigma^x_t + \tilde{\sigma}^x_t) \sigma) \leq 0 \quad \forall z \in H \tag{O.13}
\]

\[
\tilde{\sigma}^x_t + \sigma^x_t = \frac{\pi}{\gamma} \tag{O.14}
\]

As before, \( \hat{c} \) has an upper bound \( \hat{c}_h \), which must be modified to take into account that it is not incentive compatible to give the agent a perfectly safe consumption stream.

**Lemma O.3.** Take the agent’s hidden investment possibility set \( H \) as given. For any incentive compatible contract \( C \), for all \( t \)

\[
\hat{c}_t \leq \mathbb{E}_t^\hat{P} \left[ \int_t^\infty e^{-\int_t^u \left( \frac{e^{-(1-\gamma)} - 1}{\gamma} \sigma^c_u - \sigma^c_u \sigma^c_u \right) du} ds \right]^{-\frac{1}{1-\gamma}} \leq \hat{c}_h
\]

where \( \hat{P} \) is an equivalent measure such that \( Z_t - \int_0^t (1-\gamma) \sigma^c_s ds \) and \( \tilde{Z}_t - \int_0^t (1-\gamma) \tilde{\sigma}^c_s ds \) are
\[ \hat{c}_h \equiv \max_{\sigma^c \geq 0} \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} (\sigma^c)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \]  

\[ \text{st: } z(\alpha - \sigma^c \sigma \gamma) \leq 0 \quad \forall z \in H \]  

If ever \( \hat{c}_t = \hat{c}_h \), then the continuation contract satisfies \( \hat{c}_{t+s} = \hat{c}_h \) and \( \hat{k}_t = \frac{\sigma^x \hat{c}_h}{\sigma \sigma_h} \) for all future times \( t + s \), and \( x_t \) follows the law of motion (O.10), where \( \sigma^x \) is the optimizing choice in (O.15) and \( \hat{\sigma}^x = \frac{\pi}{\gamma} \). Let \( \hat{v}_h \) be the cost of this continuation contract:

\[ \hat{v}_h = \frac{\hat{c}_h - \frac{\alpha}{\sigma^c} \hat{c}_h^2 \sigma^x}{\rho - \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} (\sigma^x)^2 + \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2} \]

Proof. The same reasoning as in Lemma 3 yields

\[ \hat{c}_t \leq \mathbb{E}_t^\hat{P} \left[ \int_t^\infty e^{-\int_t^u \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} (\sigma^c)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) du} \right]^{-\frac{1}{1 - \gamma}} \]

where \( \hat{P} \) is an equivalent measure such that \( Z_t - \int_t^1 (1 - \gamma) \sigma_s^c ds \) and \( \bar{Z}_t - \int_0^1 (1 - \gamma) \hat{\sigma}_s^c ds \) are \( \hat{P} \)-martingales.

We can plug in the IC constraint for aggregate risk (O.7), and the maximum \( \hat{c}_t \) is \( \hat{c}_h \) as defined by (O.15), regardless of the measure. It follows that \( \hat{c}_t \leq \hat{c}_h \), and \( \hat{c}_h \) is an absorbing boundary. Using the IC constraint (O.12) and the law of motion of \( x \) we obtain the desired result. The cost of the continuation contract with \( \hat{c}_{t+s} = \hat{c}_h \) and \( \hat{k}_t = \frac{\sigma^x \hat{c}_h}{\sigma \sigma_h} \) for all future times \( t + s \) can be obtained from the HJB equation with \( \sigma^\hat{c} = \hat{\sigma}^\hat{c} = \mu^\hat{c} = 0 \), or simply applying the formula (O.29) for the cost of stationary contracts at \( \hat{c} = \hat{c}_h \). This completes the proof.

The upper bound \( \hat{c}_h \) restricts the principal’s ability to promise safety in the future. Even if the agent cannot invest his hidden savings in his private technology, \( H = \{0\} \), he can still invest in aggregate risk. In this case the maximizing choice is \( \sigma^c = 0 \) and we get

\[ \hat{c}_h = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \]. Notice that if \( \pi = 0 \) this boils down to expression (13) in the baseline setting without aggregate risk or hidden investment. If the agent can also invest his hidden savings in his own private technology, \( H = \mathbb{R}_+ \), then the maximizing choice is \( \sigma^c = \frac{\alpha}{\sigma \gamma} \), and \( \hat{c}_h \) is lower.
The HJB equation

The solution to the relaxed problem can be characterized with the same HJB equation as in the case without hidden investment, appropriately extended to incorporate aggregate risk and the new incentive compatibility constraints.

\[ 0 = \min_{\sigma^x, \sigma^x, \hat{\sigma}^x, \hat{\sigma}^x} \hat{c} - r \hat{v} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi \sigma} + \hat{v} \left( \rho - \hat{c}^{1-\gamma} \frac{1}{1-\gamma} + \frac{\sigma^x}{2} (\hat{\sigma}^x)^2 + \frac{\gamma}{2} (\hat{\sigma}^x)^2 - \pi \hat{\sigma}^x \right) \]  
\[ \text{(O.16)} \]

\[ + \hat{v}'(\hat{c}) \left( \frac{r-\rho}{\gamma} - \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{(\sigma^x)^2}{2} + (1 + \gamma) \sigma^x \hat{\sigma}^x + \frac{1 + \gamma}{2} (\hat{\sigma}^x)^2 + \frac{(\hat{\sigma}^x)^2}{2} \right) \]

\[ + (1 + \gamma) \hat{\sigma}^x \hat{\sigma}^x + \frac{1 + \gamma}{2} (\hat{\sigma}^x)^2 - \hat{\sigma}^x \pi \] + \hat{v}''(\hat{c}) \left( (\hat{\sigma}^x)^2 + (\hat{\sigma}^x)^2 \right)

subject to \( \sigma^x \geq 0 \) and (O.13) and (O.14).

Using (O.14) to eliminate \( \hat{\sigma}^x \), and taking FOC for \( \hat{\sigma}^x \), we obtain

\[ \hat{\sigma}^x = \frac{\pi}{\gamma} \quad \hat{\sigma}^x = 0 \]

This is the first best exposure to aggregate risk. The principal and the agent don’t have any conflict about aggregate risk, and the principal cannot use it to relax the moral hazard problem, so they implement the first best aggregate risk sharing.\(^{25}\)

The FOC for \( \sigma^x \) and \( \hat{\sigma}^x \) depend on whether the agent can invest his hidden savings in his private technology. Without hidden investment, the FOCs are the same as in the baseline. With hidden investment, the IC constraint (O.13) could be binding in some region of the state space. The shape of the contract, however, is the same as in the baseline without hidden investment.

**Theorem O.1.** Take the agent’s hidden investment possibility set \( H \) as given. The cost function of the relaxed problem \( \hat{v} (\hat{c}) \) has a flat portion on \([0, \hat{c}_l]\) and a strictly increasing portion on \([\hat{c}_l, \hat{c}_h]\), for some \( \hat{c}_l \in (0, \hat{c}_h) \). The HJB equation (O.16) holds with equality for \( \hat{c} \geq \hat{c}_l \). For \( \hat{c} < \hat{c}_l \), we have \( \hat{v} (\hat{c}) = \hat{v} (\hat{c}_l) \equiv \hat{v}_l \) and the HJB holds as an inequality

\[ A(\hat{c}, \hat{v}_l) \equiv \min_{\sigma^x} \hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi \sigma} - r \hat{v}_l + \hat{v}_l \left( \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma^x)^2 - \frac{\gamma}{2} (\pi)^2 \right) \quad \forall \hat{c} < \hat{c}_l. \]  
\[ \text{(O.17)} \]

At \( \hat{c}_l \) the cost function \( \hat{v} (\hat{c}) \) satisfies \( \hat{v}' (\hat{c}_l) = 0 \), \( \hat{v}'' (\hat{c}_l) > 0 \), and \( A(\hat{c}_l, \hat{v}_l) = 0 \). The cost function is below the inverse of the marginal utility of consumption, \( \hat{v} (\hat{c}) < \hat{c}^\gamma \) for all \( \hat{c} \in [\hat{c}_l, \hat{c}_h] \), with \( \hat{v} (\hat{c}_h) = \hat{v}_h \).

The state variables \( x_t \) and \( \hat{c}_t \) follow the laws of motion (O.10) and (O.11) with \( \sigma^x > 0 \),

---

\(^{25}\)If the agent didn’t have access to hidden investment in aggregate risk, and the agent’s private technology is exposed to aggregate risk \( \hat{\sigma} \neq 0 \), then the principal could potentially use the agent’s exposure to aggregate risk to relax the moral hazard problem.
\[\sigma^c_t < 0, \sigma^c_t = \frac{c}{t}, \text{ and } \sigma^c_t = 0 \text{ for all } t > 0, \text{ and } dL_t = 0 \text{ always, so the Euler equation holds as an equality. The optimal contract starts at } \hat{c}_0 = \hat{c}_t \text{ and immediately moves into the interior of the domain never reaching either boundary, that is, } \hat{c}_t \in (\hat{c}_l, \hat{c}_h) \text{ for all } t > 0.\]

At \(t = 0\) we have \(\rho^c_0 > 0\) and \(\sigma^c_0 = 0\), and \(\sigma^c_t\) is chosen myopically, without taking into account its effect on the agent’s precautionary saving motive, to maximize

\[
\sigma^c \hat{c}_t \frac{\alpha}{\hat{\sigma} c} - \hat{v}(\hat{c}_l) \frac{\gamma}{2} (\sigma^c)^2
\]

\text{(O.18)}

\text{Proof.} The proof is similar to Theorem 1, except we use the more general definition of \(A(\hat{c}, \hat{v})\)

\[
A(\hat{c}, \hat{v}) = \hat{c} - r \hat{v} - \frac{1}{2} \left( \frac{\hat{c}_t \alpha}{\hat{\sigma} c} \right) + \hat{v} \left( \frac{\rho - \hat{c}_1 - \gamma}{1 - \gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right)
\]

where \(A(\hat{c}, \hat{v}) = 0\) is the HJB equation if \(\hat{v}' = \hat{v}'' = 0\) and the IC constraint (O.13) is not binding. Notice we already know from the FOCs that \(\hat{\sigma}^c = \pi/\gamma\) and \(\hat{\sigma}^c = 0\).

\textbf{Part 1} goes through without any change. In \textbf{part 2}, to show that the region where the HJB holds cannot have flat parts, we notice that with \(\hat{v}' = \hat{v}'' = 0\), \(\sigma^c\) drops out of the HJB, and can be used to ensure that the IC constraint (O.13) holds for any choice of \(\sigma^c\). Therefore at a flat part \(A(\hat{c}, \hat{v}) = 0\) should hold, which we know cannot be the case.

In \textbf{part 3}, the proof that \(\hat{v}'(\hat{c}_t) = 0\) goes through with small modifications, as well as \(\hat{v}''(\hat{c}_t + \epsilon) \geq 0\). Indeed, if at the upper end of a flat region \(\hat{c}_2\) we have a kink, with the right derivative \(\hat{v}'(\hat{c}_2) > 0\), we must have \(\sigma^c(\hat{c}_2 + \epsilon) \to 0\) as \(\epsilon \to 0\), since otherwise we would cross into the flat region where the HJB doesn’t hold. If the drift is strictly positive there, we can start at \(\hat{c}_2 - \delta\), with the same \(\sigma^c\) that we were using at \(\hat{c}_2\). The IC constraint (O.13) does not depend on \(\hat{c}\) so it is still satisfied. Since this extends the solution and \(\hat{v}'(\hat{c}_2) > 0\), we would get a lower cost, which contradicts the flat region immediately below \(\hat{c}_2\). The proof for the case with zero drift is unchanged.

With \(\hat{v}'(\hat{c}_2) = 0\), we must have \(\hat{v}''(\hat{c}_2 + \epsilon) \geq 0\), or else \(\hat{v}'(\hat{c} + \epsilon) < 0\) for small \(\epsilon\). At \(\hat{c}_2\) we must have \(A(\hat{c}_2 - \epsilon, \hat{v}(\hat{c}_2)) \geq 0\) because otherwise we could do better by lingering below \(\hat{c}_2\) before jumping up to \(\hat{c}_2\). We must also have \(A(\hat{c}_2, \hat{v}(\hat{c}_2)) \leq 0\), because at \(\hat{c}_2\) we have \(\hat{v}' = 0\), and \(\hat{v}'' \geq 0\), the RHS of the HJB will be at least as large as \(A(\hat{c}_2, \hat{v}(\hat{c}_2))\) (it could be strictly larger if the IC constraint (O.13) is binding). Since \(A\) is continuous in \(\hat{c}\), we must have \(A(\hat{c}_2, \hat{v}(\hat{c}_2)) = 0\). Since this is true in particular with \(\hat{c}_1 = 0\) and \(\hat{c}_2 = \hat{c}_t\), we have proven \(\hat{v}'(\hat{c}_t) = 0\) and \(A(\hat{c}_t, \hat{v}_t) = 0\). The rest of part 3 goes through without changes.

\textbf{Part 4} goes through with natural modifications. In the case \(\sigma^c(\hat{c}_1, \hat{v}_1) < 0\), we consider setting \(\sigma^c = 0\), and \(\sigma^c = \frac{\alpha}{\hat{\sigma} c} \hat{c}_t^2\). This is consistent with IC constraint (O.13) because \(\hat{c}_t \geq \hat{v}_t\)
from Lemma O.13. We then obtain the first order ODE

$$A(\hat{c}, \hat{v}_{fo}) + \hat{v}_{fo}' \hat{c} \left( \frac{r - \rho}{\gamma} - \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} + \frac{\left( \frac{\alpha}{\sigma \gamma} \hat{c}^{\gamma} \right)^2}{2} + \frac{1}{2} \frac{\pi^2}{\gamma^2} \right) \mu^{\hat{c}} = 0$$

and the rest of the proof goes through to obtain $\mu^{\hat{c}} > 0$. So we get $\hat{v}''(\hat{c}_l) > 0$.

**Part 5** goes through and ignoring the IC constraint (O.13) we obtain $\sigma^{\hat{c}}(\hat{c}_l) = 0$ and the same reasoning as in Theorem 1 shows that $\hat{c}_l$ is inaccessible from $\hat{c} > \hat{c}_l$, so $\hat{c}_t > \hat{c}_l$ for all $t > 0$, $dL_t = 0$ always and the Euler equation holds as an equality.

Lemma O.15 shows that $\hat{v}(\hat{c}) < \hat{c}^{\gamma}$ for all $\hat{c} \in [\hat{c}_l, \hat{c}_h]$. It only remains to show that $\sigma^{\hat{c}}(\hat{c}) < 0$ and $\sigma^x(\hat{c}) > 0$ for all $\hat{c} \in (\hat{c}_l, \hat{c}_h)$. We already know this is the case when the IC constraint (O.13) is not binding. If it is binding, then $\sigma^{\hat{c}} = \frac{\alpha}{\sigma \gamma} - \sigma^x$, and the FOC for $\sigma^x$ yields:

$$\sigma^x = \frac{\hat{c}^{\gamma} \frac{\alpha}{\sigma} + \hat{v}' c^2 \frac{\alpha}{\sigma \gamma}}{\gamma (\hat{v} - \hat{v}' \hat{c}) + \hat{v}'' c^2} < \frac{\alpha}{\sigma \gamma}$$

which implies $\sigma^{\hat{c}} < 0$. To see this inequality, use $\hat{v} < \hat{c}^{\gamma}$ to write

$$\phi \hat{c}^{\gamma - \gamma} \hat{v} < 1$$

and use $\hat{v}' \geq 0$ to get

$$\phi \hat{c}^{\gamma} (\hat{v} - \hat{v}' \hat{c}) < 1$$

$$\frac{\alpha}{\sigma} (\hat{v} - \hat{v}' \hat{c}) < \hat{c}^{\gamma} \frac{\alpha}{\phi \sigma}$$

$$\frac{\alpha}{\gamma \sigma} (\gamma (\hat{v} - \hat{v}' \hat{c}) + \hat{v}'' c^2) < \hat{c}^{\gamma} \frac{\alpha}{\phi \sigma} + \hat{v}'' c^2 \frac{\alpha}{\gamma \sigma}$$

Finally, divide throughout by $\gamma (\hat{v} - \hat{v}' \hat{c}) + \hat{v}'' c^2$ which must be strictly positive (second order condition for optimality)

$$\frac{\alpha}{\gamma \sigma} < \frac{\hat{c}^{\gamma} \frac{\alpha}{\sigma} + \hat{v}'' c^2 \frac{\alpha}{\gamma \sigma}}{\gamma (\hat{v} - \hat{v}' \hat{c}) + \hat{v}'' c^2} = \sigma^x$$

**Part 6** Lemma O.13 implies $\mu^{\hat{c}}(\hat{c}_l) > 0$, and the same reasoning as in Theorem 1 shows that $\hat{c}_l$ is inaccessible from $\hat{c} > \hat{c}_l$, so $\hat{c}_t > \hat{c}_l$ for all $t > 0$, $dL_t = 0$ always and the Euler equation holds as an equality. For $\hat{c}_h$ we need to consider two cases. Without hidden
investment, \( H = \{0\} \), Lemma O.16 shows that

\[
\mu \hat{c} \approx (4\gamma - 6(1 + \gamma)^2)\hat{c}_h^{-\gamma} \epsilon
\]

\[
\sigma \hat{c} \approx -\sqrt{22}(1 + \gamma)\hat{c}_h^{-\gamma/2} \epsilon^{3/2}
\]

and the same analysis as in Theorem 1 shows that \( \hat{c}_h \) is inaccessible. With hidden investment, \( H = \mathbb{R}^+ \), the IC constraint will be binding near the upper boundary. Lemma O.16 shows that

\[
\mu \hat{c} \approx (\eta - 2) \frac{1}{2} \left( \frac{\alpha}{\sigma \gamma} \right)^2 \left( \frac{\gamma}{1 - \eta} \right)^2 (\hat{c}_h - \hat{c}) < 0
\]

\[
\sigma \hat{c} \approx -\left( \frac{\alpha}{\sigma \gamma} \right) \frac{\gamma}{1 - \eta} (\hat{c}_h - \hat{c})
\]

for some \( \eta \in (0, 1) \). We can compute the scale function

\[
S(\hat{c}) = \int^{\hat{c}} \exp \left( - \int^y 2\frac{\tilde{\mu}}{\sigma^2} \frac{1}{\hat{c}_h - z} dz \right) dy = -\frac{1}{2\tilde{\mu}/\tilde{\sigma}^2 + 1} (\hat{c}_h - \hat{c})^{2\tilde{\mu}^2 + 1}
\]

where \( \tilde{\mu} = (\eta - 2) \frac{1}{2} \left( \frac{\alpha}{\sigma \gamma} \right)^2 \left( \frac{\gamma}{1 - \eta} \right)^2 < 0 \) and \( \tilde{\sigma}^2 = \left( \frac{\alpha}{\sigma \gamma} \right)^2 \left( \frac{\gamma}{1 - \eta} \right)^2 \), so that \( 2\tilde{\mu}/\tilde{\sigma}^2 = \eta - 2 < -1 \). So \( S(\hat{c}_h) = \infty \), which means that \( \hat{c}_h \) is inaccessible and non-attracting (\( P\{\hat{c}_t \to \hat{c}_h\} = 0 \)). 

The long-run behavior of the contract depends on whether the agent has access to hidden investment.

**Theorem O.2.** Without hidden investment, the optimal contract in the relaxed problem does not have a stationary distribution:

\[
\frac{1}{t} \int_0^t 1\{\hat{c}_t > \hat{c}_h - \epsilon\} (\hat{c}_s) ds \to 1 \quad a.s. \quad \forall \epsilon > 0
\]

With hidden investment, the optimal contract in the relaxed problem has a stationary distribution with density proportional to \( m(\hat{c}) = \frac{1}{\sigma^2(\hat{c}_h - \hat{c})^2} \exp \left( \int^{\hat{c}} \frac{2\mu^2(z)\sigma^2}{(\sigma^2(z))^2} dz \right) \), which spikes near \( \hat{c}_h \), i.e. \( m(\hat{c}) \to \infty \) as \( \hat{c} \to \hat{c}_h \).

**Proof.** The lower boundary \( \hat{c}_l \) is an entrance boundary from the proof of Theorem O.1, so we only need to study the upper boundary \( \hat{c}_h \). We use the approximations in Lemma O.16. For the case without hidden investment, the proof is analogous to Theorem 2. For the case with hidden investment, we must compute the speed measure

\[
m(\hat{c}) = \frac{1}{\sigma^2(\hat{c}_h - \hat{c})^2} \exp \left( \int^{\hat{c}} \frac{2\tilde{\mu}^2(z)}{\sigma^2(z)} \frac{1}{\hat{c}_h - z} dz \right)
\]
Using the approximation,

\[ \mu \hat{c} \approx (\eta - 2) \left( \frac{\alpha}{\sigma \gamma} \right)^2 \left( \frac{\gamma}{1 - \eta} \right)^2 (\hat{c}_h - \hat{c}) < 0 \]

\[ \sigma \hat{c} \approx - \left( \frac{\alpha}{\sigma \gamma} \right) \frac{\gamma}{1 - \eta} (\hat{c}_h - \hat{c}) \]

we get that near \( \hat{c}_h \)

\[ m(\hat{c}) \approx \frac{1}{\sigma^2} (\hat{c}_h - \hat{c})^{-\frac{2}{\sigma^2} - 2} \]

where \(-\frac{2}{\sigma^2} - 2 = 2 - \eta - 2 = -\eta < 0\). This means that \( m(\hat{c}) \rightarrow \infty \) as \( \hat{c} \rightarrow \hat{c}_h \). But the integral of \( m(\hat{c}) \) is

\[ M(\hat{c}) = \int_{\hat{c}_l}^{\hat{c}} \frac{1}{\sigma^2} (\hat{c}_h - z)^{-\eta} dz = \frac{1}{1 - \eta} \frac{1}{\sigma^2} (\hat{c}_h - \hat{c})^{1 - \eta} \]

which is finite as \( \hat{c} \rightarrow \hat{c}_h \). Since \( \hat{c}_l \) is an entrance boundary, we have a stationary distribution,

\[ \psi(\hat{c}) = \frac{m(\hat{c})}{\int_{\hat{c}_l}^{\hat{c}_h} m(z) dz} \]

with a spike near \( \hat{c}_h \).

We also have a verification theorem for the HJB equation.

**Theorem O.3.** Take the agent’s hidden investment possibility set \( H \) as given. Let \( \hat{v}(\hat{c}) : [\hat{c}_l, \hat{c}_h] \rightarrow [\hat{v}_1, \hat{v}_h] \) be a strictly increasing \( C^2 \) solution to the HJB equation (O.16) for some \( \hat{c}_l \in (0, \hat{c}_h) \), such that \( \hat{v}_l \equiv \hat{v}(\hat{c}_l) \in (0, \hat{v}_h] \), \( \hat{v}'(\hat{c}_l) = 0 \), \( \hat{v}''(\hat{c}_l) > 0 \) and \( \hat{v}(\hat{c}_h) = \hat{v}_h \). Assume also that \( \hat{v}(\hat{c}) \leq \hat{c}_l^\gamma \) for \( \hat{c} \in [\hat{c}_l, \hat{c}_h] \), and, if \( \gamma < \frac{1}{2} \) that

\[ 1 - \hat{v}_l \left( \hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma - 1} \alpha^2 (\phi \sigma)^{-2} \hat{c}_l^{-2} \right) \leq 0 \]

Then,

1) For any incentive compatible contract \( C = (c, k) \) that delivers at least utility \( u_0 \) to the agent, we have \( \hat{v}(\hat{c}_l)((1 - \gamma)u_0)^{\frac{1}{1 - \gamma}} \leq J_0(\hat{C}) \).

2) Let \( C^* \) be a contract generated by the policy functions of the HJB. Specifically, the state variables \( x^* \) and \( \hat{c}^* \) are solutions to (O.10) and (O.11) with \( dL_t = 0 \) (and potential absorption at \( \hat{c}_h \)), with initial values \( x_0^* = ((1 - \gamma)u_0)^{\frac{1}{1 - \gamma}} \) and \( \hat{c}_0^* = \hat{c}_l \). If \( C^* \) is admissible and \( \sigma \hat{c}^* \) bounded, then \( C^* \) is an optimal contract, with cost \( J_0(C^*) = \hat{v}(\hat{c}_l)((1 - \gamma)u_0)^{\frac{1}{1 - \gamma}} \).

**Proof.** The proof is very similar to Theorem 3, but we use the definition

\[ A(\hat{c}, \hat{v}) = \hat{c} - r \hat{v} - \frac{1}{2} \left( \frac{\hat{c} \alpha}{\sigma \gamma} \right) + \hat{v} \left( \frac{\rho - \hat{c}^{1 - \gamma}}{1 - \gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right) \]
and use it to show that the HJB holds as an inequality below $\hat{c}_t$. Notice that the optimal policies in the HJB imply $\hat{\sigma}^x = \pi/\gamma$ and $\hat{\sigma}^\pi = 0$ throughout. With $\hat{v}'(\hat{c}_t) = 0$ and $\hat{v}''(\hat{c}_t) > 0$, the IC constraint (O.13) is not binding at $\hat{c}_t$, because $\hat{v}(\hat{c}_t) \leq \hat{c}^\gamma$, so we get $\sigma^x(\hat{c}_t) = \frac{\alpha}{\sigma^\gamma \phi \gamma} \hat{c}_t^\gamma$ and $\sigma^\pi(\hat{c}_t) = 0$ and therefore $A(\hat{c}_t, \hat{c}_t) = 0$. Lemma O.13 then shows that $\mu^\hat{c}(\hat{c}_t) > 0$, and the same reasoning as in Theorem 3 shows that $\hat{c}_t$ is inaccessible from $\hat{c} > \hat{c}_t$, so $\hat{c}_t \in [\hat{c}_l, \hat{c}_h]$. We also want to show that $\sigma^x \geq 0$ and $\sigma^\pi \leq 0$ for all $\hat{c} \in [\hat{c}_l, \hat{c}_h]$. When know this is true when the IC constraint (O.13) is not binding. Using $\hat{v}(\hat{c}) \leq \hat{c}^\gamma$ we can show this is also the case if the constraint is binding, as in Part 5 of Theorem O.1. We can then use Theorem O.4 to establish the global incentive compatibility of the candidate optimal contract. The rest of the proof is unchanged.

\[ \square \]

The following result is useful to verify admissibility.

**Lemma O.4.** If the candidate contract $C^*$ constructed in Theorem O.3 has $\mu^{x^*} < r + \pi^2/\gamma$, then $C^*$ is admissible and delivers utility $u_0$ to the agent.

**Proof.** We know that $\hat{c}_l^* \in [\hat{c}_l, \hat{c}_h]$ and recall that $\hat{c}_l > 0$. Then an upper bounded $\mu^{x^*} < r + \pi^2/\gamma$ implies a bounded $0 \leq \sigma^{x^*} \leq \bar{\sigma}_X$, and we also know that $\hat{\sigma}^x = \pi/\gamma$. Then

$$
E^Q \left[ \int_0^\infty e^{-rt}(|c^*_t| + |k^*_t\alpha|)dt \right] \leq 2 \max \left\{ \hat{c}_h, \frac{\bar{\sigma}_X \hat{c}_h^\gamma}{\phi \gamma} \alpha \right\} E^Q \left[ \int_0^\infty e^{-rt}x^*_t dt \right] < \infty
$$

where the last inequality follows from $\mu^{x^*} < r + \pi^2/\gamma$ (notice the expectations is taken under $Q$). Let $U^* = \frac{(x^*)^{1-\gamma}}{1-\gamma}$, so using the law of motion of $x^*$, (14), we get

$$
U^*_0 = E \left[ \int_0^{\tau^n} e^{-\rho t} \frac{c^1_{1-\gamma}}{1-\gamma} dt + e^{-\rho \tau^n} U^*_n \right]
$$

with $\tau^n \to \infty$ a.s. Use the monotone convergence theorem and notice that

$$
\lim_{n \to \infty} E \left[ e^{-\rho \tau^n} U^*_n \right] = 0
$$

because $\rho - (1-\gamma)(\mu^{x^*} - \frac{\gamma}{2}(\sigma^{x^*})^2 - \frac{\gamma}{2}(\pi^2)^2) = \hat{c}^{1-\gamma} \geq \min\{\hat{c}_l^{1-\gamma}, \hat{c}_h^{1-\gamma}\} > 0$. We then get that $U^*_{t=0} = U^*_0 = u_0$. We conclude that the contract is indeed admissible.

\[ \square \]

**Verifying global incentive compatibility**

We can extend Theorem 4 to verify global incentive compatibility.

**Theorem O.4.** Let $C = (c, k)$ be an admissible contract with associated processes $x$ and $\hat{c}$ satisfying (O.10) and (O.11), and (O.12), (O.14), and in the case of hidden investment
Following the proof of Theorem 4, we can write the integrand as the sum of four parts:

\[
\frac{\tilde{c}^{1 - \gamma}}{1 - \gamma} x_t \mu_x \left( \frac{\rho - \tilde{c}^{1 - \gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma^x_t)^2 + \frac{\gamma}{2} (\sigma^\delta_t)^2 - \frac{\gamma}{2} \sigma^x_t \sigma^\delta_t \right) + \dot{c} \gamma (r \gamma_t + z_i \gamma_t (\alpha + \pi \tilde{\sigma}) + \tilde{z}_i \gamma_t \pi + c_t - \ddot{c} + \phi \gamma_k a_t) + h_i \dot{c} \gamma (\rho - r - \dot{c}^{1 - \gamma} - \frac{\gamma}{1 - \gamma} (\sigma^x_t)^2 - \frac{\gamma}{2} (\sigma^x_t)^2 - \frac{\gamma}{2} (\sigma^\delta_t)^2 - \frac{\gamma}{2} (\sigma^x_t \sigma^\delta_t + \gamma \sigma^x_t a_t)
\]

\[
= \tilde{\sigma}^x_t x_t \mu_x \left( \frac{\rho - \tilde{c}^{1 - \gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma^x_t)^2 + \frac{\gamma}{2} (\sigma^\delta_t)^2 - \frac{\gamma}{2} \sigma^x_t \sigma^\delta_t \right)
\]

Following the proof of Theorem 4, we can write the integrand as the sum of four parts

\[
\frac{\tilde{c}_u^{1 - \gamma}}{1 - \gamma} x_u + x_u \mu_u - x_u \frac{\gamma}{2} (\sigma_u^x)^2 - x_u \frac{\gamma}{2} (\sigma_u^\delta)^2 = A_u a_u + B_u + C_u + \tilde{\gamma} u
\]
$A_t$ and $B_t$ are unchanged and we know they are non-positive:

$$A_t = \hat{c}_t^{-\gamma} \phi k_t - x_t \frac{\sigma^x_t}{\sigma} + h_t \hat{c}_t^{-\gamma} \gamma \sigma_t^c \leq 0$$

$$B_t = x_t \rho - c_1 t^{-\gamma} + \hat{c}_t^\gamma (rh_t + \hat{c}_t x_t - \hat{c}_t x_t) + h_t \hat{c}_t^{-\gamma} (\rho - r - \gamma \frac{c_1 t^{-\gamma} - \rho}{1 - \gamma}) + \bar{x}_t \frac{c_1 t^{-\gamma} - \rho}{1 - \gamma} \leq 0$$

$C_t$ needs to be modified to account for hidden investment, and the new term $\tilde{C}_t$ collects the terms dealing with aggregate risk.

$$C_t = x_t \frac{\gamma}{2} (\sigma_t^x)^2 + h_t \hat{c}_t^{-\gamma} \left( z_t \alpha - \gamma (\sigma_t^c z_t \sigma - \frac{\gamma}{2} (\sigma_t^x)^2 - \gamma^2 \sigma_t^x \sigma_t^c)\right) - \bar{x}_t \frac{\gamma}{2} (\sigma_t^x)^2$$

$$C_t = \frac{\gamma}{2} \left( x_t (\sigma_t^x)^2 + 2h_t \hat{c}_t^\gamma z_t (\frac{\alpha}{\gamma} - \sigma_t^c \sigma) - h_t \hat{c}_t^{-\gamma} (\sigma_t^x)^2 - h_t \hat{c}_t^{-\gamma} 2\gamma \sigma_t^x \sigma_t^c - \frac{1}{\bar{x}_t} (\bar{x}_t \sigma_t^x)^2 \right)$$

Notice that we included the $z_t h_t \alpha$ term here. We will include $\pi(z_t h_t \sigma + \bar{z}_t h_t)$ in $\tilde{C}_t$.

Expand $(\bar{x}_t \sigma_t^x)^2$:

$$C_t = \frac{\gamma}{2} \left( x_t (\sigma_t^x)^2 + 2h_t \hat{c}_t^\gamma z_t \left( \frac{\alpha}{\gamma} - \sigma_t^c \sigma \right) - h_t \hat{c}_t^{-\gamma} \left( \sigma_t^x \right)^2 - h_t \hat{c}_t^{-\gamma} 2\gamma \sigma_t^x \sigma_t^c - \frac{1}{\bar{x}_t} \left( \bar{x}_t \sigma_t^x \right)^2 \right)$$

Take the $1/\bar{x}_t$ out of the parenthesis:

$$C_t = \frac{\gamma}{2} \left( x_t (\sigma_t^x)^2 + h_t \hat{c}_t^{-\gamma} \left( \frac{\alpha}{\gamma} - \sigma_t^c \sigma \right) + 2 \left( h_t \hat{c}_t^{-\gamma} \right)^2 z_t \left( \frac{\alpha}{\gamma} - \sigma_t^c \sigma \right) \right)$$

$$- x_t h_t \hat{c}_t^{-\gamma} \left( \sigma_t^x \right)^2 - \left( h_t \hat{c}_t^{-\gamma} \right)^2 \left( \sigma_t^x \right)^2 - x_t h_t \hat{c}_t^{-\gamma} 2\gamma \sigma_t^x \sigma_t^c - \left( h_t \hat{c}_t^{-\gamma} \right)^2 2\gamma \sigma_t^x \sigma_t^c$$

$$- \left( x_t^2 (\sigma_t^x)^2 + h_t \hat{c}_t^{-\gamma} \left( z_t \sigma - \gamma \sigma_t^c \right) \right) + 2 x_t \sigma_t^x h_t \hat{c}_t^{-\gamma} \left( z_t \sigma - \gamma \sigma_t^c \right)$$

Cancel some terms:

$$C_t = \frac{\gamma}{2} \left( x_t h_t \hat{c}_t^{-\gamma} z_t \left( \frac{\alpha}{\gamma} - \sigma_t^c \sigma \right) + 2 \left( h_t \hat{c}_t^{-\gamma} \right)^2 z_t \left( \frac{\alpha}{\gamma} - \sigma_t^c \sigma \right) - \left( h_t \hat{c}_t^{-\gamma} \right)^2 \left( \sigma_t^x \right)^2$$

$$- \left( h_t \hat{c}_t^{-\gamma} \right)^2 2\gamma \sigma_t^x \sigma_t^c - \left( h_t \hat{c}_t^{-\gamma} \left( z_t \sigma - \gamma \sigma_t^c \right) \right)$$

And group the remaining ones to form a square:
When \( \gamma \geq 1/2 \), if \( \alpha \leq \frac{\phi \sigma \sqrt{2(1-\gamma)}}{\sqrt{2}} \left( \frac{\rho - r (1-\gamma)}{\gamma} - \frac{1-\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{2}} \) then \( \hat{v} \geq \hat{c}_h^\gamma / 2 > 0 \). When \( \gamma \leq 1/2 \), if \( \alpha \leq \frac{\phi \sigma \sqrt{2(1-\gamma)}}{\sqrt{2}} \left( \frac{\rho - r (1-\gamma)}{\gamma} - \frac{1-\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{2}} \) then \( \hat{v} \geq (1-\gamma)(2\gamma)^{\frac{1}{2}} \hat{c}_h^\gamma > 0 \).
function. To do this, it’s sufficient to show that \( A(\hat{c}, \hat{v}_l) \geq 0 \) for any \( \hat{c} \in (0, \hat{c}_h) \).

\[
A(\hat{c}, \hat{v}_l) = \hat{c} + \hat{v}_l \frac{\hat{c}_h^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2} \frac{\hat{c}^2 (\frac{\alpha}{\theta})^2}{\hat{v}_l \gamma^2}
\]

\[
\geq \hat{c} + \frac{\hat{c}_h^{\gamma} \hat{c}_h^{1-\gamma} - \hat{c}^{1-\gamma}}{2} - \frac{1}{2} \frac{\gamma}{\hat{v}_l \gamma^2} \frac{\hat{c}_h^{1+\gamma} \hat{c}^{-\gamma}}{2} \left( 2 y^{1+\gamma} + \frac{\gamma y^{\gamma} - y - y^{3\gamma}}{1-\gamma} \right)
\]

where \( y = \hat{c}/\hat{c}_h \in (0, 1) \). Since \( y^{3\gamma} < y^{1+\gamma} \) the expression in parenthesis is greater or equal to

\[
y^{1+\gamma} + \frac{\gamma y^{\gamma} - y}{1-\gamma}
\]

Here we have three powers of \( y \), and the middle coefficient is always negative, while the outside coefficients are positive (this is true both if \( \gamma < 1 \) and \( \gamma > 1 \)). Moreover, the sum of the coefficients is 0 and the weighted sum (with weights equal to the powers) is

\[
1 + \gamma + \frac{\gamma^2 - 1}{1-\gamma} = 0
\]

Hence, by Jensen’s inequality, the expression is positive.

For the case \( \gamma \leq 1/2 \). We will show that \( \hat{v}_l = (1-\gamma)\hat{c}_m^{\gamma} \) is a lower bound, where \( \hat{c}_m \) is defined by \( 2\gamma\hat{c}_h^{1-\gamma} = \hat{c}_m^{1-\gamma} \). We have

\[
A(\hat{c}, \hat{v}_l) = \hat{c} + \hat{v}_l \frac{\hat{c}_h^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2} \frac{\hat{c}^2 (\frac{\alpha}{\theta})^2}{\hat{v}_l \gamma^2}
\]

\[
\geq \hat{c} + \hat{c}_m^{\gamma} (\hat{c}_m^{1-\gamma} / 2 - \hat{c}^{1-\gamma}) - \frac{1}{2} \frac{\gamma}{\hat{v}_l \gamma^2} \frac{\hat{c}_h^{1+\gamma} \hat{c}_m^{-\gamma}}{\hat{c}_m^{1-\gamma}}
\]

\[
= \hat{c} + \hat{c}_m/2 - \hat{c}^{1-\gamma} \hat{c}_m^{-\gamma} - \frac{1}{2} \frac{\gamma}{\hat{v}_l \gamma^2} \frac{\hat{c}_h^{1-\gamma}}{\hat{c}_m^{1-\gamma}}
\]

\[
= (\hat{c}_m/2) (1 + (\hat{c}/\hat{c}_m)^\gamma) - \hat{c}^{1-\gamma} \hat{c}_m^{\gamma} (1 - (\hat{c}/\hat{c}_m)^\gamma)
\]

If \( \hat{c}/\hat{c}_m < 1 \) then \( \hat{c}_m^{1-\gamma} \hat{c}_m^{\gamma} < \hat{c}^{1-\gamma} \hat{c}_m^{\gamma} < \hat{c}_m \) and therefore the expression is positive. If \( \hat{c}/\hat{c}_m > 1 \), then \( \hat{c}_m^{1-\gamma} \hat{c}_m^{\gamma} > \hat{c}^{1-\gamma} \hat{c}_m^{\gamma} > \hat{c}_m \) and also the expression is positive. This completes the proof.

**Benchmark contracts and autarky limit**

We can extend the benchmark contracts in Section 3 to incorporate aggregate risk. In addition, we can find conditions under which the gains from trade are exhausted and the optimal contract coincides with autarky, as mentioned in Section 4 in the paper.
The optimal contract without hidden savings is characterized by the HJB equation:
\begin{equation}
rvn = \min_{\sigma^x, \hat{c}, \tilde{\sigma}^x} \hat{c} - \sigma^x \hat{c}^\gamma + \hat{v}_n \left( \frac{\hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x)^2 + \frac{\gamma}{2}(\tilde{\sigma}^x)^2 - \tilde{\sigma}^x \pi \right)
\tag{O.20}
\end{equation}
where \(v_n(x) = \hat{v}_nx\) is the principal’s cost function. The FOC are:
\begin{align}
\sigma^x &= \frac{\alpha}{\gamma(\hat{v}_n \hat{c}^\gamma \phi) \sigma} \tag{O.21} \\
1 &= \hat{v}_n \hat{c}^{-\gamma} + \hat{v}_n \gamma^2 (\sigma^x)^2 \hat{c}^{-1} \tag{O.22} \\
\tilde{\sigma}^x &= \frac{\pi}{\gamma} \tag{O.23}
\end{align}

The optimal contract exists only if \(\gamma \leq 1/2\) and only if \(\alpha\) is sufficiently low; otherwise the principal’s value function becomes infinite.

The Inverse Euler equation says that \(e^{(r-\rho)t \hat{c}^\gamma}\) is a \(Q\)-martingale. If the contract has constant \(\hat{c}\), it requires
\begin{equation}
\sigma^x = \sqrt{\left(\hat{c}^*\right)^{1-\gamma} - \hat{c}^{1-\gamma} \frac{2}{1-\gamma} \frac{1}{1-2\gamma}} \tag{O.24}
\end{equation}
where \(\hat{c}^* = \left(\frac{\rho-r}{\gamma} - (1-\gamma)\frac{1}{2}(\pi)^2\right)^{1-\gamma}\) coincides with \(\hat{c}_h\) without hidden investment, so this expression is analogous to (25) in the paper. We can compute the cost of contracts consistent with the Inverse Euler equation by plugging (O.24) into (O.20). To ensure admissibility, we must restrict \(\hat{c} > \hat{c}_h(2\gamma)^{1-\gamma}\) (assuming \(\gamma < 1/2\)), or else the No-Ponzi condition is violated.

**Lemma O.6.** The optimal contract without hidden savings satisfies the Inverse Euler equation, i.e. \(e^{(r-\rho)t \hat{c}^\gamma}\) is a \(Q\)-martingale, and myopic optimization over \(\sigma^x\), i.e. (O.21). The marginal cost of utility is lower than the inverse of the marginal utility of consumption, \(\hat{v}_n < \hat{c}^\gamma_n\).

**Proof.** Myopic optimization follows from the FOC (O.21), and \(\hat{v}_n < \hat{c}^\gamma_n\) from FOC (O.22). Given stationarity, the Inverse Euler equation is equivalent to:
\begin{equation}
\mu^x = \frac{r-\rho}{\gamma} + (1-\gamma)\frac{1}{2}(\sigma^x)^2 + (1+\gamma)\frac{1}{2}(\tilde{\sigma}^x)^2
\end{equation}
Using the FOC for \(\hat{c}\) we can write
\begin{equation}
\Rightarrow \hat{v}\hat{c}^{1-\gamma} = \hat{c} - \hat{v}\gamma^2 (\sigma^x)^2
\tag{O.25}
\end{equation}
Plug into the HJB (O.20) along with the FOC for \(\sigma^x\), (O.21), to obtain
\begin{equation}
rv = \hat{c} - (\sigma^x)^2 \hat{v}\gamma + \frac{1}{2}(\sigma^x)^2 \hat{v}\gamma + \hat{v}\rho - \hat{c}^{1-\gamma} - \hat{v}\gamma (\tilde{\sigma}^x)^2
\end{equation}
Divide by $\hat{v}$ and use (O.25), to obtain

$$\hat{c}^{1-\gamma} = \left( \frac{\rho - r(1-\gamma)}{\gamma} - (1-\gamma)\left(\hat{\sigma}^x\right)^2 \right) - \frac{(\sigma^x)^2}{2}(1-2\gamma)(1-\gamma) \quad (O.26)$$

And now compute $\mu^x$:

$$\mu^x = \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x)^2 + \frac{\gamma}{2}(\hat{\sigma}^x)^2$$

After some algebra we obtain the Inverse Euler equation

$$\mu^x = \frac{r - \rho}{\gamma} + (1-\gamma)\frac{1}{2}(\sigma^x)^2 + (1+\gamma)\frac{1}{2}(\hat{\sigma}^x)^2$$

Stationary contracts have a constant $\hat{c}$ and are obtained by setting $\sigma^c = \hat{\sigma}^c = 0$, $\hat{\sigma}^x = \pi/\gamma$, and $\sigma^x$ to satisfy

$$\sigma^x = \sigma^x(\hat{c}) \equiv \sqrt{2\left(\hat{c}^\gamma - \hat{c}^{1-\gamma} \frac{1}{1-\gamma}\right)}$$

so that $\mu^c = 0$ in (15). To ensure admissibility, we must restrict

$$\hat{c} > \hat{c}_* \equiv \frac{2\gamma}{1+\gamma} \frac{1}{1-\gamma} < \hat{c}^* \quad (O.28)$$

so that $\mu^x < r$ and the No-Ponzi condition (O.2) is satisfied. Theorem O.4 is general enough to ensure that stationary contracts are globally incentive compatible. The HJB equation (18) yields the cost of the stationary contract,

$$\hat{v}_r(\hat{c}) = \frac{\hat{c} - \frac{\alpha}{\gamma \phi} \hat{c}^\gamma \sigma^x(\hat{c})}{2r - \rho - (1+\gamma)\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \gamma(\pi/\gamma)^2}. \quad (O.29)$$

A special stationary contract corresponds to myopic optimization,

$$\sigma^x(\hat{c}_p) = \frac{\alpha}{\gamma(\hat{v}_r(\hat{c}_p) \hat{c}_p^\gamma \phi)\sigma}$$

which yields

$$\hat{c}_p = \left(\hat{c}^{1-\gamma}_h - (1-\gamma)\frac{1}{2}\left(\frac{\alpha}{\gamma \phi \sigma}\right)^2 - (1-\gamma)\frac{1}{2}(\pi/\gamma)^2\right)^{\frac{1}{1-\gamma}}, \quad \sigma^x_p = \frac{\alpha}{\gamma \phi \sigma}, \quad \hat{v}_p = \hat{c}_p^\gamma \quad (O.30)$$

The best stationary contract minimizes the cost, $\hat{c}_{r\min} \equiv \arg\max_{\hat{c} \in [\hat{c}_*, \hat{c}_h]} \hat{v}_r(\hat{c})$ and $\hat{v}_{r\min} \equiv \hat{v}_r(\hat{c}_{r\min})$. 

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Lemma O.7. For any \( \hat{c} \in (\hat{c}_*, \hat{c}_h] \), the corresponding stationary contract is globally incentive compatible and has cost \( \hat{v}_r(\hat{c}) \) given by (O.29). Since stationary contracts are incentive compatible, we have \( \hat{v}(\hat{c}) \leq \hat{v}_r(\hat{c}) \).

The myopic stationary contract is an incentive compatible stationary contract corresponding to \( \hat{c}_p \), and the marginal cost of utility is equal to the inverse of the marginal utility of consumption, \( \hat{v}_r(\hat{c}_p) = \hat{c}^\gamma_p \). The best stationary contract is less risky for the agent, i.e. we have \( \hat{c}_s < \hat{c}_p < \hat{c}^\min_r \) and \( \sigma^x_r(\hat{c}^\min_r) < \sigma^x_p \). For all \( \hat{c} \in (\hat{c}_p, \hat{c}_h) \) the marginal cost of utility is below the inverse of the marginal utility of consumption \( \hat{v}_r(\hat{c}) < \hat{c}^\gamma \), and we depart from myopic optimization, \( \sigma^x_r(\hat{c}) < \frac{\alpha}{\gamma(\hat{v}_r(\hat{c})(\bar{c}^{-\gamma}\phi)\sigma)} \).

Proof. First, using \( \alpha < \bar{\alpha} \), we can verify that \( 0 \leq \hat{c}_h \leq \hat{c}^* \), regardless of whether the agent can invest in his hidden savings. Second, \( \hat{v}_r(\hat{c}) > 0 \) for all \( \hat{c} \in (\hat{c}_s, \hat{c}_h) \) from Lemma O.8. The same argument as in Theorem O.3 shows that \( \hat{v}_r(\hat{c}) \) from (O.29) is the cost corresponding to the stationary contract with \( \hat{c} \) and \( \sigma^x \) given by (O.27), as long as the contract is indeed admissible and delivers utility \( u_0 \) to the agent. We can check that \( \mu^x < r + \frac{\pi^2}{\gamma} \) for the stationary contract if and only if \( \hat{c} > \hat{c}_s \). In this case, since \( \mu^x < r + \frac{\pi^2}{\gamma} \) arguing as in the proof of Lemma O.4 we can show that the stationary contract is admissible and delivers utility \( u_0 \) to the agent if and only if \( \hat{c} > \hat{c}_s \). Since the contract satisfies (O.10), (O.11), and (O.12), and (O.14) by construction, we only need to check that (O.13) holds too. It’s east to see this is the case because \( \hat{c} \leq \hat{c}_h \). Theorem O.4 then ensures that it is incentive compatible.

The myopic stationary contract has \( \hat{c} = \hat{c}_p \) given by (O.30). Lemma O.14 ensures that \( \hat{c}_p \in (\hat{c}_s, \hat{c}_h] \) and therefore by the argument above, it is an incentive compatible contract. The best stationary contract has \( \hat{c}^\min_r > \hat{c}_p > \hat{c}_s \) from part 1) of Lemma O.14. From (O.27) it follows that \( \sigma^x_p > \sigma^x_r(\hat{c}^\min_r) \). Part 2) of Lemma O.14 shows that \( \hat{v}_r(\hat{c}) < \hat{c}^\gamma \) for all \( \hat{c} \in (\hat{c}_p, \hat{c}_h) \), with equality at \( \hat{c}_p \) and \( \hat{c}_h \). Therefore,

\[
\sigma^x_r(\hat{c}) < \sigma^x_p = \frac{\alpha}{\gamma\phi\sigma} < \frac{\alpha}{\gamma(\hat{v}_r(\hat{c})(\bar{c}^{-\gamma}\phi)\sigma)}
\]

Lemma O.8. The cost function of stationary contracts \( \hat{v}_r(\hat{c}) \) defined by (O.29) is strictly positive for all \( \hat{c} \in (\hat{c}_*, \hat{c}_h] \) if and only if \( \alpha < \bar{\alpha} \).

Proof. We need to check the numerator in (O.29), since the denominator is positive for all \( \hat{c} \geq \hat{c}_s \):

\[
\hat{c} \left(1 - \frac{\alpha}{\phi\sigma} \sqrt{2} \sqrt{\hat{c}^{\gamma-1} \left( \frac{\hat{c}^{\gamma}(1-\gamma)}{\gamma} - \frac{1-\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \hat{c}^{\gamma-1} - 1 \right)} \right)
\]
The rest of the proof consists of evaluating this expression at \( \hat{c} = \hat{c}_* \) and showing it is non-positive iff the bound is violated, since the expression is increasing in \( \hat{c} \). We get \( \hat{c} \) times

\[
1 - \frac{\alpha}{\phi \sigma} \sqrt{2} \sqrt{1 + \gamma} \frac{1}{2} \gamma \left[ \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{-1} \right]
\]

So if \( \alpha \geq \bar{\alpha} \) the numerator is non-positive, and if \( \alpha < \bar{\alpha} \) then it’s strictly positive. This completes the proof. \( \square \)

**Lemma O.9.** If the agent has access to hidden investment, \( H = \mathbb{R}_+ \) and \( \phi = 1 \), the optimal contract is the myopic stationary contract characterized in (O.30).

**Proof.** The myopic stationary contract is both admissible and incentive compatible by lemma O.7. Since in this case

\[
\hat{c}_h = \hat{c}_p = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\gamma \sigma} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right),
\]

we can use the same verification argument as in Theorem O.3, using the flat value function \( \hat{v}(\hat{c}) = \hat{v}_p \) for all \( \hat{c} \in (0, \hat{c}_h) \). For the argument to go through, it must be the case that the HJB holds as an inequality for all \( \hat{c} < \hat{c}_p \):

\[
A(\hat{c}, \hat{v}_p) = \hat{c} - r \hat{v}_p - \frac{1}{2} \frac{v_0}{\hat{v}_p \gamma} + \hat{v}_p \left( \rho - \hat{c}^{1-\gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right) > 0
\]

This is true because \( \hat{v}_p = \hat{c}_p^\gamma \), and from lemma O.14 we know that \( \partial_1 A(\hat{c}_p, \hat{c}_p^\gamma) < 0 \). From Lemma O.11 we know that \( A(\hat{c}, \hat{v}) \) is positive near 0 and either has one root in \( \hat{c} \) if \( \gamma \geq 1/2 \), or is convex with at most two roots if \( \hat{c} \leq 1/2 \). This means that \( A(\hat{c}, \hat{c}_p^\gamma) > 0 \) for all \( \hat{c} \in (0, \hat{c}_p) \).

\( \square \)

**Renegotiation**

Here we provide technical details for Section 5 of the paper. This section is consistent with the presence of aggregate risk and hidden investment introduced in Section 4 and the Online Appendix.

We say that an incentive compatible contract \( C = (c, k) \) is renegotiation-proof (RP) if

\[
\infty = \arg \min_{\tau} \mathbb{E}_Q \left[ \int_0^\tau e^{-\tau(c_k \alpha)} dt + e^{-\tau x} \inf \hat{v} \right]
\]

The optimal contract with hidden savings is not renegotiation proof, because after any
history \( \hat{v}_t > \hat{v}_t = \inf \hat{v}(\omega, s) \), so the principal is always tempted to “start over”. In fact, it is easy to see that RP contracts must have a constant \( \hat{v}_t \). The converse it also true.

**Lemma O.10.** An incentive compatible contract \( C \) is renegotiation proof if and only if the continuation cost \( \hat{v} \) is constant.

**Proof.** If ever \( \hat{v}_t > \inf \hat{v}(\omega, s) \), then renegotiating at that point is better than never renegotiating and obtaining \( \hat{v}_0 \). In the other direction, if \( \hat{v} \) is constant, any stopping time \( \tau \) yields the same value to the principal, so \( \tau = \infty \) is an optimal choice.

Stationary contracts have a constant \( \hat{v} \), because \( \hat{c} \) is constant. However, those contracts were built using \( dL_t = 0 \). There are other contracts with a constant \( \hat{c} \) that use \( dL_t > 0 \), i.e. the drift of \( \hat{c} \) would be negative without \( dL_t \). In addition, there could be non-stationary contracts with a constant cost \( \hat{v}(\hat{c}) \) for all \( \hat{c} \) in the domain. The next Lemma shows they are all worse than the best stationary contract \( C_r^{\min} \), with cost \( \hat{v}_r^{\min} = \min \{ \hat{c} \in (\hat{c}_L, \hat{c}_U) \mid \hat{v}(\hat{c}) \} \).

**Theorem O.5.** The optimal renegotiation-proof contract is the optimal stationary contract \( C_r^{\min} \) with cost \( \hat{v}_r^{\min} \).

**Proof.** Since the optimal stationary contract is incentive compatible and has a constant \( \hat{v} \), we only need to show that any incentive compatible contract with constant \( \hat{v} \) has \( \hat{v} \geq \hat{v}_r^{\min} \). This is clearly true for all stationary contracts as defined in Lemma O.7 with aggregate risk.

There could also be stationary contracts with a constant \( \hat{c} \), but \( dL_t > 0 \). For these contracts the drift \( \mu^\hat{c} < 0 \) in the absence of \( dL_t \). Consider the optimization problem

\[
0 = \min_{\sigma^x} \hat{c} - r\hat{v} - \sigma^x \hat{c} \gamma \frac{\alpha}{\phi \sigma} + \hat{v} \left( \frac{\rho - \hat{c}^{1 - \gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma^x)^2 - \frac{\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)
\]

subject to:

\[
\frac{r - \rho}{\gamma} - \frac{\rho - \hat{c}^{1 - \gamma}}{1 - \gamma} + \frac{1}{2} (\sigma^x)^2 + \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 \leq 0
\]

If the constraint is binding, we get the stationary contracts with \( dL_t = 0 \), so \( \hat{v} = \hat{v}_r \). We want to show that it must be binding. Towards contradiction, if the constraint is not binding we have \( \sigma^x = \frac{\alpha}{\sigma \gamma} \frac{\hat{c}^{1 - \gamma}}{\hat{c}} \) and therefore we have \( A(\hat{c}, \hat{v}) = 0 \), where \( A \) is defined as in Lemma O.11. If \( \hat{v} \leq \hat{v}_r^{\min} \) then \( \hat{v} \leq \hat{v}_p \), because the myopic stationary contract is incentive compatible (\( \hat{c}_p \leq \hat{c}_h \) for any valid hidden investment). Then Lemma O.13 ensures that \( \frac{r - \rho}{\gamma} - \frac{\rho - \hat{c}^{1 - \gamma}}{1 - \gamma} + \frac{1}{2} (\sigma^x)^2 + \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 > 0 \), which violates the constraint. This means that \( \hat{v} \geq \hat{v}_r^{\min} \).

Finally, if we have a non-stationary contract with a constant \( \hat{v} < \hat{v}_r^{\min} \), the domain of \( \hat{c} \) must have an upper bound \( \hat{c}^* \leq \hat{c}_h \) because otherwise they would have a lower cost than the optimal contract near \( \hat{c}_h \), and this cannot be for an IC contract. For the upper bound \( \hat{c}^* \)
we must have $\sigma^\dot = 0$ and $\mu^\dot \leq 0$. But this is the same situation with stationary contracts with $dL_t > 0$, and we know their cost is above $c_r^{\min}$.

Remark. It is possible that $c_r^{\min} = \hat{c}_h$ if the agent can invest his hidden savings and $\phi$ is close enough to 1. In the special case with hidden investment and $\phi = 1$, we have $c_r^{\min} = \hat{c}_p = \hat{c}_h$, as shown in Lemma O.9.

Intermediate results

Lemma O.11. Define the function

$$A(\hat{c}, \hat{v}) \equiv \hat{c} - r\hat{v} - \frac{1}{2} \left( \frac{\hat{c} \alpha}{\phi \sigma} \right)^2 + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right)$$

For any $\hat{v} \in (0, (\hat{c}^*)^\gamma)$, we have $A(\hat{c}, \hat{v}) > 0$ for $\hat{c}$ near 0, where $\hat{c}^* = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}}$. In addition, if $\gamma \geq \frac{1}{2}$ then $A(\hat{c}, \hat{v})$ has at most one root in $[0, \hat{c}^*]$. If instead $\gamma < \frac{1}{2}$, $A(\hat{c}, \hat{v})$ is convex and has at most two roots.

Proof. First, for $\gamma < 1$ $\lim_{\hat{c} \to 0} A(\hat{c}; \hat{v}) = \hat{v} \frac{\gamma}{1 - \gamma} \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) > 0$. For $\gamma > 1$, $\lim_{\hat{c} \to 0} A(\hat{c}; \hat{v}) = \infty$.

For $\gamma \geq 1/2$, to show that $A(\hat{c}; \hat{v})$ has at most one root in $[0, \hat{c}^*]$ for any $\hat{v} \in (0, \hat{v}_h)$, we will show that $A'(\hat{c}; \hat{v}) = 0 \implies A(\hat{c}; \hat{v}) > 0$ for all $\hat{c} < \hat{c}^*$. Compute the derivative (dropping the arguments to avoid clutter)

$$A' = 1 - \hat{v} \hat{c}^{-\gamma} - \hat{c}^{2\gamma - 1} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{v}}$$

So

$$A' = 0 \implies \hat{c} - \hat{v} \hat{c}^{-\gamma} = \hat{c}^{2\gamma} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{v}}$$

Plug this into the formula for $A$ to get

$$A = \hat{c} - r\hat{v} + \hat{v} \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} - \frac{\hat{c}^{2\gamma}}{2 \hat{v} \gamma} \left( \frac{\alpha}{\phi \sigma} \right)^2 - \hat{v} \frac{\pi^2}{2 \gamma}$$

$$= \frac{2\gamma - 1}{2\gamma} \hat{c} + \frac{1 - 3\gamma}{2\gamma} \hat{v} \frac{\hat{c}^{1-\gamma}}{1 - \gamma} + \hat{v} \frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{\hat{v} \pi^2}{2 \gamma} \equiv B(\hat{c}, \hat{v})$$
\( B(\hat{c}, \hat{v}) \) is convex in \( \hat{c} \) because \( 1 - 3\gamma < 0 \) for \( \gamma \geq \frac{1}{2} \), so it’s minimized in \( \hat{c} \) when \( B'_c = 0 \):

\[
\frac{2\gamma - 1}{3\gamma - 1} = \hat{v} \hat{c}^{-\gamma}
\]  

(0.31)

and it is strictly decreasing before this point. Now we have two possible cases:

**CASE 1:** The minimum of \( B \) is achieved for \( \hat{c} \geq \hat{c}^* \), so in the relevant range, it is minimized at \( \hat{c}_0 \). Let’s plug \( \hat{c}^* \) into \( B(\hat{c}, \hat{v}) \):

\[
2\gamma B(\hat{c}^*, \hat{v}) = (2\gamma - 1) \hat{c}^* + \frac{\hat{v}}{1 - \gamma} ((\rho - r(1 - \gamma)) 2\gamma + (1 - 3\gamma)(\hat{c}^*)^{1-\gamma}) - \hat{v} \pi^2
\]

\[
= (2\gamma - 1) \hat{c}^* + \hat{v} \left( \frac{\rho - r(1 - \gamma)}{\gamma} (1 - 2\gamma) - \frac{1}{2} (1 - \gamma) \left( \frac{\pi}{\gamma} \pi \right)^2 \hat{v} (1 - 2\gamma) \right)
\]

\[
(2\gamma - 1) \left( \hat{c}^* - \hat{v} \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1}{2} (1 - \gamma) \left( \frac{\pi}{\gamma} \pi \right)^2 \right) \right) \geq 0
\]

and the inequality is strict if \( \hat{v} < (\hat{c}^*)^{\gamma} \). So \( A(\hat{c}, \hat{v}) = B(\hat{c}, \hat{v}) > B(\hat{c}^*, \hat{v}) \geq 0 \) for any \( \hat{c} < \hat{c}^* \).

**CASE 2:** If the minimum is achieved for \( \hat{c}_m \in [0, \hat{c}^*] \) it must be that \( \gamma > 1/2 \). Then plugging in (0.31) into \( B \):

\[
B(\hat{c}, \hat{v}) \geq \frac{2\gamma - 1}{2\gamma} \hat{c}_m - \frac{2\gamma - 1}{2\gamma} \hat{c}_m + \hat{v} \rho - r(1 - \gamma) - \hat{v} \pi^2 \frac{2}{2 \gamma}
\]

\[
= \frac{1 - 2\gamma}{2} \hat{c}_m + \hat{v} \rho - r(1 - \gamma) - \hat{v} \pi^2 \frac{2}{2 \gamma}
\]

\[
= \frac{1 - 2\gamma}{2} \hat{c}_m + \frac{2\gamma - 1}{3\gamma - 1} \hat{c}_m \left( \rho - r(1 - \gamma) - \frac{1}{2} \pi^2 \frac{1}{2 \gamma} \right)
\]

and dividing throughout by \( 2\gamma - 1 > 0 \):

\[
= \frac{1}{2} \hat{c}_m + \frac{\hat{c}_m}{3\gamma - 1} \left( \rho - r(1 - \gamma) - \frac{1}{2} \pi^2 \frac{1}{2 \gamma} \right)
\]

and multiplying by \( \hat{c}_m^{\gamma} > 0 \) and using \( \frac{\hat{c}_m^{1-\gamma}}{1-\gamma} < \frac{(\hat{c}^*)^{1-\gamma}}{1-\gamma} \):

\[
> - \frac{1}{2} (\hat{c}^*)^{1-\gamma} + \frac{1}{3\gamma - 1} \left( \rho - r(1 - \gamma) - \frac{1}{2} \pi^2 \frac{1}{2 \gamma} \right)
\]

\[
= \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1}{2} (1 - \gamma) \left( \frac{\pi}{\gamma} \pi \right)^2 \right) \left( - \frac{1}{2} \frac{1}{2 \gamma} + \frac{\gamma}{(3\gamma - 1)(1 - \gamma)} \right)
\]
\[
\left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1}{\gamma} (1 - \gamma) \left( \frac{\pi}{\gamma} \right)^2 \right) \frac{1 - 3\gamma + 2\gamma}{(3\gamma - 1)^2} = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1}{2} (1 - \gamma) \left( \frac{\pi}{\gamma} \right)^2 \right) \frac{1}{(3\gamma - 1)^2} > 0
\]

So \(A(\hat{c}; \hat{v}) \geq B(\hat{c}, \hat{v}) > 0\) for all \(\hat{c} \in [0, \hat{c}^*]\).

For the case with \(\gamma < \frac{1}{2}\), the second derivative of \(A\) is
\[
A''(\hat{c}) = \gamma \hat{v} \hat{c}^{-\gamma - 1} - (2\gamma - 1) \hat{c}^{2\gamma - 2} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{v}} > 0
\]

So \(A(\hat{c}; \hat{v})\) is strictly convex and so can have at most two roots. \(\square\)

**Lemma O.12.** Assume there are some constants \(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\), and \(\lambda_4 > 0\) such that for any feasible strategy \((\hat{c}, a, z, \tilde{z})\) there is a non-negative process \(N\) with
\[
dN_t \leq ((\lambda_1 + \lambda_2 \hat{c}_t^N + \lambda_3 \hat{a}_t^N)N_t - \lambda_4 \hat{c}_t)dt + \sigma_t^N N_t dZ_t^a + \tilde{\sigma}_t^N N_t d\tilde{Z}_t
\]
for some processes \(\sigma^N\) and \(\tilde{\sigma}^N\), which can depend on the strategy. Then there is a constant \(\lambda_5 > 0\) such that for any \(T > 0\) and any feasible strategy \((\hat{c}, a, z, \tilde{z})\)
\[
\mathbb{E}^a \left[ \int_0^T e^{-\rho t} \hat{c}_t^{1-\gamma} dt \right] \leq \lambda_5 \frac{N_0^{1-\gamma}}{1-\gamma}
\]

**Proof.** First define \(n_t\) as the solution to the SDE
\[
dn_t = ((\lambda_1 + \lambda_2 \hat{c}_t^N + \lambda_3 \hat{a}_t^N)n_t - \hat{c}_t)dt + \sigma_t^n n_t dZ_t^a + \tilde{\sigma}_t^N n_t d\tilde{Z}_t
\]
and \(n_0 = N_0^\lambda\). It follows that \(n_t \geq \frac{N_0}{\lambda_4} \geq 0\). Now define \(\zeta\) as
\[
\frac{d\zeta_t}{\zeta_t} = -\lambda_1 dt - \lambda_2 dZ_t^a - \lambda_3 d\tilde{Z}_t, \quad \zeta_0 = 1
\]
and
\[
\tilde{n}_t = \int_0^t \zeta_s \tilde{c}_s ds + \zeta_t n_t
\]
We can check that \(\tilde{n}_t\) is a local martingale under \(P^a\). Since \(\zeta_t > 0\) and \(n_t \geq 0\) it follows that
\[
\mathbb{E}^a \left[ \int_0^{\tau^m \wedge T} \zeta_s \tilde{c}_s ds \right] \leq \mathbb{E}^a \left[ \int_0^{\tau^m \wedge T} \zeta_s \tilde{c}_s ds + \zeta_{\tau^m \wedge T} n_{\tau^m \wedge T} \right] = n_0
\]
where \(\{\tau^m\}\) reduces the stochastic integral and has \(\lim_{m \to \infty} \tau^m = \infty\) a.s. Taking \(m \to \infty\) and using the monotone convergence theorem we obtain
\[
\mathbb{E}^a \left[ \int_0^T \zeta_s \tilde{c}_s ds \right] \leq n_0
\]
Now we want to maximize \(\mathbb{E}^a \left[ \int_0^T e^{-\rho t} \hat{c}_t^{1-\gamma} dt \right]\) subject to this budget constraint. Notice
that $a$ appears both in the budget constraint and objective function, but does not affect the law of motion of $\zeta$ under $P^n$, so we can ignore it since we are choosing $\bar{c}$. The candidate solution $c$ has

$$e^{-\rho t} c_t^{-\gamma} = \zeta_t \mu$$

where $\mu > 0$ is the Lagrange multiplier and is chosen so that the budget constraint holds with equality. For any $\bar{c}$ that satisfies the budget constraint we have

$$E^a \left[ \int_0^T e^{-\rho t} \frac{\bar{c}_t^{1-\gamma}}{1-\gamma} dt \right] \leq E^a \left[ \int_0^T e^{-\rho t} \left( \frac{c_t^{1-\gamma}}{1-\gamma} + c_t^{-\gamma} (\bar{c}_t - c_t) \right) dt \right]$$

$$= E^a \left[ \int_0^T e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] + \mu E^a \left[ \int_0^T \zeta_t (\bar{c}_t - c_t) dt \right] \leq E^a \left[ \int_0^T e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]$$

Now since $c_t = (\zeta_t \mu)^{-\frac{1}{\gamma}} e^{-\frac{\gamma}{\gamma} t}$ it follows a geometric Brownian motion so $E^a \left[ \int_0^T e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]$ is finite. Because of homothetic preferences, we know that $E^a \left[ \int_0^T e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] = \lambda_5 \frac{n_1^{1-\gamma}}{1-\gamma} = \lambda_5 \frac{N_1^{1-\gamma}}{1-\gamma}$ for some $\lambda_5 > 0$.

**Corollary.** For $\gamma > 1$, $\lim_{n \to \infty} E^a \left[ e^{-\rho T} N_1^{1-\gamma} \right] = 0$ for any feasible strategy $(\bar{c}, a, z, \bar{z})$.

**Proof.** The continuation utility at any stopping time $\tau < \infty$ has

$$U^\tau_{\tau} = E^a \left[ \int_{\tau}^{\tau + T} e^{-\rho (t - \tau)} \frac{\bar{c}_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho (T - \tau)} U^\tau_{\tau + T} \right]$$

$$\leq E^a \left[ \int_{\tau}^{\tau + T} e^{-\rho (t - \tau)} \frac{\bar{c}_t^{1-\gamma}}{1-\gamma} dt \right] \leq \lambda_5 \frac{N_1^{1-\gamma}}{1-\gamma}$$

So at $t = 0$ we get

$$U^\tau_{\tau} = E^a \left[ \int_0^{\tau} e^{-\rho t} \frac{\bar{c}_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho \tau} U^\tau_{\tau + T} \right] \leq E^a \left[ \int_0^{\tau} e^{-\rho t} \frac{\bar{c}_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho \tau} \lambda_5 \frac{N_1^{1-\gamma}}{1-\gamma} \right]$$

Take limits $n \to \infty$ and use the monotone convergence theorem on the first term on the right hand side to get $0 \geq \lim_{n \to \infty} E^a \left[ e^{-\rho T} N_1^{1-\gamma} \right] \geq 0$. 

**Lemma O.13.** Let $\hat{c}_t \in (0, \hat{c}_h)$ and $\hat{v}_t \leq \hat{v}_p$. If $\sigma^x = \bar{\sigma} = 0$, $\sigma^x = \frac{\alpha}{\sigma^y \bar{v} \bar{\phi}}$, and $\bar{\sigma} = \pi / \gamma$, and $A(\hat{c}_t, \hat{v}_t) = 0$, where

$$A(\hat{c}, \hat{v}) = \hat{c} - \hat{r} \hat{v} - \frac{1}{2} \left( \frac{\bar{v} \sigma^y}{\bar{\phi} \gamma} \right)^2 + \hat{v} \left( \frac{\mu - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right)$$

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then \( \hat{v}_t < \hat{c}_t^\gamma \) and
\[
\mu^\hat{c} = \frac{r - \rho}{\gamma} - \frac{\rho - \hat{c}_t^{1 - \gamma}}{1 - \gamma} + \frac{1}{2} (\sigma^x)^2 + \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 > 0
\]

Proof. Looking at (O.11), with \( \sigma^\hat{c} = \tilde{\sigma}^\hat{c} = 0 \) we get for the drift
\[
\mu^\hat{c} = \frac{r - \rho}{\gamma} + \frac{1}{2} (\sigma^x)^2 + \frac{1}{2} (\tilde{\sigma}^x)^2 - \frac{\rho - \hat{c}_t^{1 - \gamma}}{1 - \gamma}
\]
So \( \mu^\hat{c} > 0 \) implies
\[
\frac{1}{2} (\sigma^x)^2 + \frac{1}{2} (\tilde{\sigma}^x)^2 > \frac{\rho - r}{\gamma} + \frac{\rho - \hat{c}_t^{1 - \gamma}}{1 - \gamma}
\]
Since we also want \( A(\hat{c}; \hat{v}) = 0 \), we get
\[
0 = \hat{c} - \hat{r} \hat{v} + \hat{v} \left( \frac{\rho - \hat{c}_t^{1 - \gamma}}{1 - \gamma} - \frac{\gamma}{2} (\sigma^x)^2 - \frac{\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)
\]
\[
< \hat{c} - \hat{v} \hat{c}_t^{1 - \gamma} \equiv M
\]
Notice that if \( \hat{v} = \hat{c}_t^\gamma \) we have \( M = 0 \). If \( \hat{v} > \hat{c}_t^\gamma \) we have \( M < 0 \) and if \( \hat{v} < \hat{c}_t^\gamma \) we have \( M > 0 \). So for \( A(\hat{c}; \hat{v}) = 0 \) and \( \mu^\hat{c} > 0 \) we need \( \hat{v} < \hat{c}_t^\gamma \). In fact, if \( \hat{v} = \hat{c}_t^\gamma \) and in addition
\[
\frac{1}{2} \left( \frac{\alpha}{\phi \sigma \gamma} \right)^2 + \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 = \frac{\rho - \hat{c}_t^{1 - \gamma}}{1 - \gamma} + \frac{\rho - r}{\gamma}
\]
then we have \( A = 0 \) and \( \mu^\hat{c} = 0 \). In this case, because we have \( \mu^\hat{c} = 0 \) we therefore have the value of a stationary contract, i.e. \( \hat{v} = \hat{v}_r(\hat{c}) \) given by (O.29). This point corresponds to the myopic stationary contract with \( (\hat{c}_p, \hat{v}_p) \). We know from Lemma O.14 that \( \hat{c}_p \in [\hat{c}_s, \hat{c}_h] \).

By assumption, \( \hat{v}_t \leq \hat{v}_p \).

First we will show that \( \mu^\hat{c} \geq 0 \), and then make the inequality strict. Towards contradiction, suppose \( \mu^\hat{c} < 0 \) at \( \hat{c}_t \). Then it must be the case that \( \hat{v}_t > \hat{c}_t^\gamma \) because we have \( A(\hat{c}_t, \hat{v}_t) = 0 \). We will show that \( A(\hat{c}_t, \hat{v}_t) > 0 \) and get a contradiction. First take the derivative of \( A \):
\[
A'_\hat{c}(\hat{c}_t, \hat{v}_t) = 1 + \hat{v}_t \left( \hat{c}_t^{-\gamma} + \hat{c}_t^{2\gamma - 1} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{v}_t^2} \right) < 0
\]
where the inequality holds for all \( \hat{c} < \hat{c}_t^{\frac{1}{\gamma}} \). So \( A(\hat{c}_t, \hat{v}_t) > A(\hat{c}_t^{\frac{1}{\gamma}}, \hat{v}_t) \). Letting \( \hat{c}_m = \hat{c}_t^{\frac{1}{\gamma}} \) we get
\[
A(\hat{c}_t, \hat{v}_t) > \hat{c}_m - r \hat{v}_t + \hat{v}_t \left( \frac{\rho - \hat{c}_m^{1 - \gamma}}{1 - \gamma} - \frac{1}{2} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\gamma} - \frac{\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)
\]
\[
= \hat{c}_m - r \hat{v}_t + \hat{v}_t \left( \frac{\rho - \hat{c}_m^{1 - \gamma}}{1 - \gamma} - \gamma \frac{\rho - \hat{c}_p^{1 - \gamma}}{1 - \gamma} - (\rho - r) \right)
\]
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\[ A(\hat{c}_l, \hat{v}_l) > \hat{c}_m + \hat{v}_l \gamma \frac{c_p^{1-\gamma} - c_m^{1-\gamma}}{1 - \gamma} = \hat{c}_m^{\gamma} \frac{c_p^{1-\gamma} - c_m^{1-\gamma}}{1 - \gamma} \geq 0 \]

where the last equality uses \( \hat{v}_l = \hat{c}_m^{\gamma} \) and the last inequality uses \( \hat{c}_m = \hat{v}_l^{\frac{1}{\gamma}} \leq \hat{v}_p^{\frac{1}{\gamma}} = \hat{c}_p \). This is a contradiction, and therefore it must be the case that \( \mu^c \geq 0 \) at \( \hat{c}_l \).

It’s clear from the previous argument that \( \mu^c(\hat{c}_l) = 0 \) only if \( (\hat{c}_l, \hat{v}_l) = (\hat{c}_p, \hat{v}_p) \). We will show this cannot be the case because \( \alpha > 0 \). First, note that \( (\hat{c}_p, \hat{v}_p) \) is a tangency point where \( \hat{v}_r(\hat{c}) \) touches the locus \( \hat{v}_b(\hat{c}) \) defined by \( A(\hat{c}; \hat{v}_b(\hat{c})) = 0 \). If \( (\hat{c}_l, \hat{v}_l) = (\hat{c}_p, \hat{v}_p) \) then this must be the minimum point for \( \hat{v}_r(\hat{c}) \), so the derivative of both \( \hat{v}_r(\hat{c}) \) and \( \hat{v}_b(\hat{c}) \) must be zero. This means that \( A^c_l(\hat{c}_l, \hat{v}_l) = 0 \). However,

\[ 1 - \hat{v}_l \left( \hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma-1} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{v}_l^{\gamma}} \right) < 0 \]

where the inequality follows from \( \hat{v}_l = \hat{v}_p = \hat{c}_p^{\gamma} \) (note that \( \hat{c}_l > 0 \) because as Lemma O.11 shows \( A(\hat{c}, \hat{v}_l) \) is strictly positive for \( \hat{c} \) near 0). This can’t be a minimum of \( \hat{v}_r(\hat{c}) \). Therefore \( (\hat{c}_l, \hat{v}_l) \neq (\hat{c}_p, \hat{v}_p) \) and \( \mu^c(\hat{c}_l) > 0 \). This completes the proof.

**Lemma O.14.** Let

\[ \hat{c}_p = \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\phi \sigma \gamma} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \]

\[ \hat{v}_p = \hat{c}_p^{\gamma} \]

be the \( \hat{c} \) and \( \hat{v} \) corresponding to the myopic stationary contract. We have the following properties

1) \( \hat{c}_s < \hat{c}_p < \hat{c}_r^{\min} \leq \hat{c}_h \), for any valid hidden investment setting

2) \( \hat{c}_r \) intersects \( \hat{v}_r(\hat{c}) \) only at \( \hat{c}_p \) and \( \hat{c}_s = \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \) in \([0, \hat{c}_s]\). Furthermore, \( \hat{c}_r \geq \hat{v}_r(\hat{c}) \) for all \( \hat{c} \in [\hat{c}_p, \hat{c}_s] \), and \( \hat{c}_s \geq \hat{v}_r(\hat{c}) \) for all \( \hat{c} \in [\hat{c}_s, \hat{c}_p] \), with strict inequality in the interior of each region.

3) \( A(\hat{c}, \hat{c}_r) = 0 \) only at \( \hat{c} = 0 \) and \( \hat{c}_p \). Furthermore, \( A(\hat{c}, \hat{c}_r) \leq 0 \) for all \( \hat{c} \in [\hat{c}_p, \hat{c}_h] \) and \( A(\hat{c}, \hat{c}_r) \geq 0 \) for all \( \hat{c} \in [0, \hat{c}_p] \), and \( \partial_1 A(\hat{c}, \hat{c}_r) < 0 \) for all \( \hat{c} \in (0, \hat{c}_h] \).

**Proof.** First let’s show that \( \hat{c}_p \in (\hat{c}_s, \hat{c}_h) \). Clearly, \( \hat{c}_p < \hat{c}_h \) for any type of valid hidden investment, because \( \phi < 1 \). Now write \( \hat{c}_p \)

\[ \hat{c}_p = \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\phi \sigma \gamma} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \]
\[
> \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{\gamma'}} \left( 1 - \frac{1 - \gamma}{1 + \gamma} \right)^{\frac{1}{\gamma}}
\]

where the inequality comes from \( \alpha < \hat{\alpha} = \frac{\phi \sigma \sqrt{2}}{\gamma \sqrt{1 + \gamma}} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2} \). Notice \( 1 - \frac{1 - \gamma}{1 + \gamma} = \frac{2 \gamma}{1 + \gamma} \) and use the definition of \( \hat{c}_* \):

\[
\hat{c}_* = \left( \frac{2\gamma}{1 + \gamma} \right)^{\frac{1}{\gamma'}} \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{\gamma'}}
\]

to conclude that \( \hat{c}_* < \hat{c}_p \). The cost of this contract is \( \hat{v}_p = \hat{c}_p \).

Now go to 2). We are looking for roots of \( \hat{v}_r(\hat{c}) = \hat{c}^{\gamma} \):

\[
\hat{c} - \frac{\alpha}{\phi \sigma} \sqrt{2} \gamma \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 - \hat{c}^{\gamma} - \hat{c}^{1 - \gamma} (1 + \gamma)}
\]

Divide throughout by \( \hat{c}^{\gamma} > 0 \) and reorganize the right hand side

\[
\frac{\hat{c}^{1 - \gamma}}{1 - \gamma} - \frac{\alpha}{\phi \sigma} \sqrt{2} \gamma \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 - \hat{c}^{1 - \gamma}} = -2\gamma \frac{\rho - r(1 - \gamma)}{1 - \gamma} + \hat{c}^{1 - \gamma} (1 + \gamma)
\]

\[
\frac{\alpha}{\phi \sigma} \sqrt{2} \gamma \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 - \hat{c}^{1 - \gamma}} = -2\gamma \frac{\rho - r(1 - \gamma)}{1 - \gamma} + \hat{c}^{1 - \gamma} (1 + \gamma)
\]

If \( \hat{c} = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{\gamma'}} \) we have a root. If not, then we can write

\[
\frac{\alpha}{\phi \sigma} = \sqrt{2} \gamma \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 - \hat{c}^{1 - \gamma}}
\]

\[
\hat{c} = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\phi \sigma} \right) - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{\gamma'}} = \hat{c}_p < \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{\gamma'}}
\]

We know that at \( \hat{c} = 0, \hat{c}^{\gamma} = 0 \), while \( \hat{v}_r(\hat{c}) \) is always positive above \( \hat{c}_* \) and diverges to infinity as \( \hat{c} \searrow \hat{c}_* \). So we know that \( \hat{c}_p \) is the first time they intersect and therefore \( \hat{c}^{\gamma} \)
intersects $\hat{v}_r(\hat{c})$ from below. Since they won’t intersect again until $\hat{c}^*$, we get the other inequality.

Back to 1), consider the locus $\hat{v}_b(\hat{c})$ defined by $A(\hat{c}, \hat{v}_b(\hat{c})) = 0$. Since $A(\hat{c}, \hat{v})$ minimizes over $\sigma^x$, it is always below $\hat{v}_r(\hat{c})$. At $(\hat{c}_p, \hat{v}_p)$ we have $\hat{v}_b(\hat{c}) = \hat{v}_r(\hat{c})$ by part 3) below, which means this is a tangency point of $\hat{v}_b$ and $\hat{v}_r$. We can now show that $A'_b(\hat{c}_p, \hat{v}_p) < 0$ and $A'_c(\hat{c}_p, \hat{v}_p) < 0$, so that $\hat{v}_b(\hat{c}_p) = \hat{v}_r(\hat{c}_p) < 0$ which means that the $C_p$ is not the optimal stationary contract, since $\hat{c}_p < \hat{c}_h$. Write

$$A'_c(\hat{c}_p, \hat{v}_p) = 1 - \hat{v}_p \left( \hat{c}_p^{-\gamma} + \hat{c}_p^{2\gamma-1} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{v_p^2} \right) = -\hat{c}_p^{-\gamma-1} \left( \frac{\alpha}{\phi \sigma} \right) < 0$$

$$A'_b(\hat{c}_p, \hat{v}_p) = \frac{1}{1-\gamma} \left( \gamma \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) - \frac{1-\gamma}{2} \left( \frac{\alpha}{\phi \sigma \gamma} \right)^2 \right) + \frac{1}{2} \left( \frac{\hat{v}_p}{\hat{c}_p} \right)^2$$

$$= \frac{1}{1-\gamma} \left( (\gamma - 1) \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) \right) + \frac{1 + \gamma}{2} \left( \frac{\alpha}{\phi \sigma \gamma} \right)^2 < 0$$

where the last inequalities follows from the bound on $\alpha < \hat{\alpha} \equiv \frac{\phi \sigma \sqrt{2}}{\sqrt{1+\gamma}} \sqrt{\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2}$.

To find the best stationary contract, use the HJB

$$rr_r^{\min} = \min_{\hat{c}} \hat{c} - \sigma^x_r(\hat{c}) \hat{c}^{\gamma} \frac{\alpha}{\phi \sigma} + \hat{v}_r^{\min} \left( \frac{\rho - \hat{c}^{1-\gamma}_{\hat{c}}}{1-\gamma} + \frac{\gamma}{2} (\sigma^x_r(\hat{c}))^2 + \frac{\gamma}{2} (\pi/\gamma)^2 \right)$$

with FOC for $\hat{c}$:

$$1 - \gamma \hat{c}^{\gamma-1} \frac{\alpha}{\phi \sigma} \sigma^x_r(\hat{c}) - \hat{v}_r^{\min} \hat{c}^{-\gamma} + (\hat{v}_r^{\min}) \gamma \sigma^x_r(\hat{c}) - \hat{c}^{\gamma} \frac{\alpha}{\phi \sigma}) \partial_c \sigma^x_r(\hat{c}) = 0$$

We already know that for $\hat{c} \leq \hat{c}_p$ we have $\hat{v}_r(\hat{c}) \geq \hat{c}^\gamma$ and $\sigma^x_r(\hat{c}) \geq \frac{\alpha}{\phi \sigma}$. We can then show that for $\hat{c} < \hat{c}_p$ the left hand side of the FOC is strictly negative:

$$lhs = 1 - \gamma \hat{c}^{\gamma-1} \frac{\alpha}{\phi \sigma} \sigma^x_r(\hat{c}) - \hat{v}_r(\hat{c}) \hat{c}^{-\gamma} + (\hat{v}_r(\hat{c}) ) \gamma \sigma^x_r(\hat{c}) - \hat{c}^{\gamma} \frac{\alpha}{\phi \sigma}) \partial_c \sigma^x_r(\hat{c})$$

Use $\hat{v}_r(\hat{c}) \geq \hat{c}^\gamma$ and $\partial_c \sigma^x_r(\hat{c}) < 0$ to obtain:

$$lhs \leq -\gamma \hat{c}^{\gamma-1} \frac{\alpha}{\phi \sigma} \sigma^x_r(\hat{c}) + (\hat{v}_r(\hat{c}) \gamma \sigma^x_r(\hat{c}) - \hat{c}^{\gamma} \frac{\alpha}{\phi \sigma}) \partial_c \sigma^x_r(\hat{c})$$

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and
\[ \text{lhs} \leq -\gamma \hat{c}^{\gamma-1} \frac{\alpha}{\phi \sigma} \sigma^\gamma_r(\hat{c}) + \hat{c}^\gamma (\gamma \sigma^\gamma_r(\hat{c}) - \frac{\alpha}{\phi \sigma}) \partial_t \sigma^\gamma_r(\hat{c}) \]

Finally, \( \sigma^\gamma_r(\hat{c}) \geq \frac{\alpha}{\phi \sigma} \) yields \( \text{lhs} < 0 \). This means the best stationary contract must have \( \hat{c}_r \min > \hat{c}_p \). We know \( \hat{c}_r \min \leq \hat{c}_h \) from the definition of \( \hat{c}_r \min \).

For 3), we are looking for roots of
\[ \hat{c} - r \hat{c}^\gamma - \frac{1}{2} \left( \frac{\alpha \hat{c}}{\phi \sigma} \right)^2 + \hat{c}^\gamma \left( \rho - \hat{c}^{1-\gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right) = 0 \]

This works for \( \hat{c} = 0 \). Otherwise, divide by \( \hat{c}^\gamma \)
\[ \frac{\hat{c}^{1-\gamma}}{1-\gamma} (1-\gamma) - r - \frac{1}{2} \left( \frac{\alpha}{\phi \sigma} \right)^2 + \rho - \hat{c}^{1-\gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} = 0 \]
\[ \frac{\rho - r (1-\gamma)}{1-\gamma} - \frac{\gamma}{2} \left( \frac{\alpha}{\phi \sigma \gamma} \right)^2 - \frac{\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 = \hat{c}^{1-\gamma} \]
\[ \frac{\rho - r (1-\gamma)}{1-\gamma} - \frac{1-\gamma}{2} \left( \frac{\alpha}{\phi \sigma \gamma} \right)^2 - \frac{1-\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 = \hat{c}^{1-\gamma} \]
\[ \hat{c} = \left( \frac{\rho - r (1-\gamma)}{1-\gamma} - \frac{1-\gamma}{2} \left( \frac{\alpha}{\phi \sigma \gamma} \right)^2 - \frac{1-\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{1\gamma} = \hat{c}_p \]

So we have only \( \hat{c}_p \) and \( \hat{c} = 0 \) as roots. This argument also shows that \( A(\hat{c}, \hat{c}^\gamma) \leq 0 \) for \( \hat{c} \in [\hat{c}_p, \hat{c}_h] \), and \( A(\hat{c}, \hat{c}^\gamma) \geq 0 \) for \( \hat{c} \in [0, \hat{c}_p] \). Also, evaluating the derivative \( \partial_t A(\hat{c}, \hat{c}^\gamma) \)
\[ \partial_t A(\hat{c}, \hat{c}^\gamma) = 1 - \hat{c}^\gamma \hat{c}^{-\gamma} - \hat{c}^{2\gamma-1} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{c}^\gamma} \]
\[ \partial_t A(\hat{c}, \hat{c}^\gamma) = 1 - 1 - \hat{c}^{\gamma-1} \left( \frac{\alpha}{\phi \sigma} \right)^2 = -\hat{c}^{\gamma-1} \left( \frac{\alpha}{\phi \sigma} \right)^2 < 0 \]
for all \( \hat{c} \in (0, \hat{c}_h] \).

\[ \square \]

**Lemma O.15.** For all \( \hat{c} \in [\hat{c}_l, \hat{c}_h] \)
\[ \hat{v}(\hat{c}) < \hat{c}^\gamma \]
If \( \hat{c}_h = \left( \frac{\rho - r (1-\gamma)}{\gamma} \right)^{\frac{1}{\gamma^\gamma}} \) then \( \hat{v}(\hat{c}_h) = \hat{c}_h^\gamma \). If \( \hat{c}_h < \left( \frac{\rho - r (1-\gamma)}{\gamma} \right)^{\frac{1}{\gamma^\gamma}} \) then \( \hat{v}(\hat{c}_h) < \hat{c}^\gamma \).

**Proof.** We already know from Lemma O.13 that at \( \hat{c}_l \) we have \( \hat{v}(\hat{c}_l) < \hat{c}_l^\gamma \). We also know that \( \hat{v}(\hat{c}) \leq \hat{v}_r(\hat{c}) \), so from Lemma O.14 if ever \( \hat{v}(\hat{c}) = \hat{c}^\gamma \) for some \( \hat{c} > \hat{c}_l \), it must be
either that \( \hat{c} = \hat{c}^* \approx \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2} \left( \frac{\eta}{\gamma} \right)^2 \right) \frac{1}{\gamma} \geq \hat{c}_h \); or that \( \hat{c} \leq \hat{c}_p \) and therefore \( A(\hat{c}, \hat{c}^\gamma) = A(\hat{c}, \hat{v}(\hat{c})) \geq 0 \) and \( \partial_1 A(\hat{c}, \hat{v}(\hat{c})) < 0 \) (because \( \hat{c} \geq \hat{c}_l > 0 \)). From Lemma O.11 we know that \( A(\hat{c}, \hat{v}) \) is positive near 0 and either has one root in \( \hat{c} \) if \( \gamma \geq 1/2 \), or is convex with at most two roots if \( \hat{c} \leq 1/2 \). This means that \( A(\hat{c} - \delta, \hat{c}^\gamma) > 0 \) for all \( \delta \in (0, \hat{c}] \).

Now we’ll use the same reasoning as in Theorem 1. We can pick an \( \alpha' < \alpha \) so that \( \hat{v}_{\alpha'}(\hat{c}^{\alpha'}) = \hat{v}(\hat{c}) \), and \( \hat{c}^{\alpha'} < 1 \), because \( \hat{v}_{\alpha'}(\hat{c}) \) is decreasing in \( \alpha' \). However, \( A_{\alpha'}(\hat{c}, \hat{c}^\gamma) \) is decreasing in \( \alpha \), so we get \( A_{\alpha'}(\hat{c}^{\alpha'}, \hat{c}^\gamma) > A(\hat{c}^{\alpha'}, \hat{c}^\gamma) > 0 \), which contradicts \( A_{\alpha'}(\hat{c}^{\alpha'}, \hat{v}_{\alpha'}(\hat{c}^{\alpha'})) = 0 \).

So we conclude that \( \hat{v}(\hat{c}) < \hat{c}^\gamma \) for all \( \hat{c} \in [\hat{c}_l, \hat{c}_h] \). If \( \hat{c}_h = \hat{c}^* \) then we have \( \hat{v}(\hat{c}_h) = \hat{c}_h^\gamma \), but if \( \hat{c}_h < \hat{c}^* \) then we must also have \( \hat{v}(\hat{c}_h) < \hat{c}_h^\gamma \).

Lemma O.16. Without hidden investment, \( H = \{0\} \), the drift and volatility of \( \hat{c} \) near \( \hat{c}_h \) are approximately,

\[
\mu^\hat{c} \hat{c} \approx (4\gamma - 6(1+\gamma)^2) \hat{c}^{-\gamma} \epsilon \\
\sigma^\hat{c} \hat{c} \approx -\sqrt{22}(1+\gamma) \hat{c}^{-\gamma/2} \epsilon^{3/2}
\]

where \( \epsilon = \hat{c}_h - \hat{c} \). With hidden investment, \( H = \mathbb{R}^+ \), we have

\[
\mu^\hat{c} \hat{c} \approx (\eta - 2) \frac{1}{2} \left( \frac{\alpha}{\sigma \gamma} \right)^2 \left( \frac{\gamma}{1-\eta} \right)^2 \epsilon < 0 \\
\sigma^\hat{c} \hat{c} \approx - \left( \frac{\alpha}{\sigma \gamma} \right) \frac{\gamma}{1-\eta} \epsilon
\]

with \( \eta \in (0, 1) \).

Proof. WITHOUT HIDDEN INVESTMENT. First we derive the drift of \( \hat{v} \) using the HJB equation (18). Differentiating with respect to \( \hat{c} \) and using the envelope theorem, we obtain

\[
rv' = 1 - \gamma \sigma^x \hat{c}^{-1} \frac{\alpha}{\phi \sigma} + \hat{v}' \left( \rho - \hat{c}^{1-\gamma} \frac{2 \gamma}{2 \gamma} - \frac{\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) - \hat{v}^{-\gamma} \hat{c}^{\gamma} \\
+ \hat{v}'' \hat{c} \left( \frac{\hat{c}^{1-\gamma} - \hat{c}_h^{1-\gamma}}{1-\gamma} \right) \left( \frac{(\sigma^x)^2}{2} + (1+\gamma) \sigma^x \hat{c}^{\gamma} + \frac{1+\gamma}{2} (\hat{c}^{\gamma})^2 \right) \frac{\hat{v}''}{2} \hat{c}^{2(\sigma^x)^2} \\
+ \hat{v}' \hat{c}^{1-\gamma} + \hat{v}' \left( \frac{\hat{c}^{1-\gamma} - \hat{c}_h^{1-\gamma}}{1-\gamma} \right) \left( \frac{(\sigma^x)^2}{2} + (1+\gamma) \sigma^x \hat{c}^{\gamma} + \frac{1+\gamma}{2} (\hat{c}^{\gamma})^2 \right) \hat{v}'' \hat{c}^{(\sigma^x)^2} \\
= -\hat{v}' (1+\gamma) (\sigma^x + \sigma^x)
\]

The middle line has the drift of \( \hat{v} \) plus an extra term \( \hat{v}'' \hat{c}^{(\sigma^x)} \), which we can combine with the term containing \( \hat{v}'' \) in the third line. We also know that \( \hat{v}'' \hat{c} \hat{c} = -\hat{v}' (1+\gamma) (\sigma^x + \sigma^x) \)
from the FOC for \( \sigma^\hat{c} \). Using this we find the drift of \( \hat{v}' \) to be

\[
+ \hat{v}'' \hat{c} \left( \hat{c}^{1-\gamma} \frac{1}{1-\gamma} - \hat{c}_h^{1-\gamma} \right) + \frac{1+\gamma}{2} (\sigma^x)^2 + \frac{1+\gamma}{2} (\sigma^\hat{c})^2 \right) + \frac{1}{\hat{v}''} \hat{c}^2 (\sigma^\hat{c})^2
\]

\[
= \hat{v}' \left( \frac{1+\gamma}{2} (\sigma^x + \sigma^\hat{c})^2 + \frac{1+\gamma}{2} (\sigma^x - \sigma^\hat{c})^2 \right) + \gamma \sigma^x \hat{c}^{\gamma-1} \frac{\alpha}{\phi \sigma} + \hat{v} \hat{c}^{-\gamma} - 1
\]

Now we approximate the cost function near \( \hat{c}_h \). Conjecture, and later verify, that \( \hat{v}(\hat{c}) = \hat{c}_h^{\gamma} - K \sqrt{\epsilon} \). Then

\[
\hat{v}' = K \epsilon^{-1/2}, \quad \hat{v}'' = K \epsilon^{-3/2}, \quad \hat{v}''' = \frac{3K}{8} \epsilon^{-5/2}
\]

plus smaller order terms.

Now conjecture that \( \sigma^\hat{c} \) is of smaller order than \( \sigma^x \) (also verified later) and use the FOC for \( \sigma^x \) to obtain

\[
\hat{c}^{\gamma} \frac{\alpha}{\phi \sigma} = \frac{K}{2} \epsilon^{-1/2} \hat{c}^x \implies \sigma^x = \frac{2K}{\hat{c}^{\gamma-1} \frac{\alpha}{\phi \sigma} \sqrt{\epsilon}}
\]

plus smaller order terms. Now plug into the FOC for \( \sigma^\hat{c} \):

\[
\frac{K}{2} \epsilon^{-1/2} (1+\gamma)(\sigma^x + \sigma^\hat{c}) + \frac{K}{4} \epsilon^{-3/2} \sigma^\hat{c} = 0 \implies \hat{c} \sigma^\hat{c} = -2(1+\gamma) \sigma^x \epsilon
\]

This verifies that indeed \( \sigma^\hat{c} \) is of smaller order than \( \sigma^x \).

Now we plug everything into the HJB equation and collect terms of order \( \sqrt{\epsilon} \) (the constant order terms match because the cost function we specified works at \( \hat{c}_h \)). The only terms of order \( \sqrt{\epsilon} \) are

\[
-\sigma^x \hat{c}^{\gamma} \frac{\alpha}{\phi \sigma} + \left( \hat{c} + \hat{v} \gamma \hat{c}_h^{1-\gamma} - \hat{c}_h^{1-\gamma} \right) \hat{v}' \hat{c} + \hat{v}' \left( \frac{(\sigma^x)^2}{2} - \frac{\hat{c}_h^{1-\gamma} - \hat{c}_h^{1-\gamma}}{1-\gamma} \right) = 0
\]

\[
-2 \hat{c}_h^{2 \gamma - 1} \frac{\alpha}{\phi \sigma}^2 - K^2 \gamma \hat{c}_h^{1-\gamma} - \hat{c}_h^{1-\gamma} \frac{1}{1-\gamma} + \frac{K}{2} \hat{c}_h \left( \frac{4}{2} K^2 \hat{c}_h^{2 \gamma - 2} - \hat{c}_h^{1-\gamma} \right) = 0
\]

We can solve for

\[
K = \sqrt{2} \hat{c}_h^{1.5 \gamma - 1} \frac{\alpha}{\phi \sigma}
\]

Now we plug into our expression for \( \sigma^x \) and \( \sigma^\hat{c} \):

\[
\sigma^x = \sqrt{2} \hat{c}_h^{-\gamma/2} \sqrt{\epsilon}
\]

\[
\sigma^\hat{c} \hat{c} = -\sqrt{2} (1+\gamma) \hat{c}_h^{-\gamma/2} \epsilon^{3/2}
\]

as desired.
For the drift, evaluate the drift of \( \dot{v}' \) using the formula above

\[
\frac{K}{2} \epsilon^{-1/2} \left( \frac{1 + \gamma \hat{c}_h \epsilon - (1 - \gamma) \hat{c}_h \epsilon}{2} \right) + \gamma (\sqrt{2} \hat{c}_h / \sqrt{\epsilon}) \hat{c}_h \epsilon^{-1} + (\hat{c}_h - K \sqrt{\epsilon}) \hat{c}_h \epsilon^{-1} - 1
\]

\[
\gamma K \hat{c}_h \epsilon^{-1/2} \left( 1 + \gamma (2 \hat{c}_h / \sqrt{\epsilon}) \hat{c}_h \epsilon^{-1} \right) + (\hat{c}_h - K \sqrt{\epsilon}) \hat{c}_h \epsilon^{-1} - 1
\]

But we can also use Ito’s lemma to obtain the drift of \( \dot{v}' \)

\[
\dot{v}'' \hat{c}_h \hat{c} + \frac{1}{2} \dot{v}'' (\hat{c}_h \epsilon^2) = \frac{K}{4} \epsilon^{-3/2} \hat{c}_h \hat{c} + \frac{1}{2} \frac{3K}{8} \epsilon^{-5/2} (1 + \gamma)^2 \hat{c}_h \epsilon^{-1} = \gamma K \hat{c}_h \epsilon^{-1/2}
\]

Solve for \( \hat{c}_h \)

\[
\hat{c}_h = (4 \gamma - 6(1 + \gamma)^2) \hat{c}_h \epsilon^{-1/2}
\]

which completes the proof.

**WITH HIDDEN INVESTMENT.** The IC constraints for hidden investment will be binding near \( \hat{c}_h \), so we have

\[
\sigma^x = \frac{\hat{c}_h \alpha + \dot{v}'' \hat{c}_h \epsilon^2}{\gamma (\hat{v} - \dot{v}' \hat{c}) + \dot{v}'' \hat{c}^2}
\]

\[
\sigma^x = \frac{\alpha}{\gamma \sigma} - \sigma^x
\]

In this case we use the approximation

\[
\hat{v} = \hat{v}_h - K \epsilon \hat{c}
\]

\[
\dot{v}' = K \eta \epsilon \gamma^{-1}
\]

\[
\dot{v}'' = -K \eta (\eta - 1) \epsilon \gamma^{-2}
\]

Divide the FOC for \( \sigma^x \) by \( \dot{v}'' \hat{c} \) on both sides (\( \dot{v}'' \neq 0 \), or we would have \( \sigma^x > \alpha / (\gamma \sigma) \) and the IC wouldn’t be binding):

\[
\sigma^x = \frac{\frac{\alpha}{\sigma \gamma} + \frac{\epsilon^{-1} K \eta \epsilon / (\eta - 1)}{K \eta (\eta - 1) \epsilon^2} \epsilon^{-1/2}}{1 + \frac{\gamma (\hat{v}_h - K \epsilon \hat{c} \eta^{-1})}{\gamma (\eta - 1) \epsilon^2} \epsilon^{-1/2}}
\]

The largest terms are of order \( \epsilon \) because \( \eta \in (0, 1) \), so we get:

\[
\sigma^x \approx \frac{\alpha}{\sigma \gamma} (1 + A \epsilon)
\]

where \( A = \gamma \hat{c}_h^{-1} \frac{1}{1 - \eta} > 0 \), and therefore

\[
\sigma^x \approx -\frac{\alpha}{\sigma \gamma} A \epsilon
\]
We need to make sure the HJB holds up to terms of order $\epsilon^\eta$. Plug into the HJB to obtain

\begin{align*}
0 &= (\hat{c}_h - \epsilon) - \left(\frac{\alpha}{\sigma^\gamma}\right) (1 + Ae) (\hat{c}_h^\eta - \gamma \hat{c}_h^{\gamma - 1} \epsilon) \frac{\alpha}{\phi^\gamma} \\
+ (\hat{v}_h - K \epsilon^\eta) & \left(\frac{\gamma (\hat{c}_h^{\gamma - 1})}{\gamma} - \frac{1 - \gamma (\hat{c}_h^{\gamma - 1})}{2} - \hat{c}_h^\gamma\right) + \epsilon \hat{c}_h^{\gamma - 1} \epsilon + \frac{\gamma}{2} \left(\frac{\alpha}{\sigma^\gamma}\right)^2 (1 + Ae)^2 \\
+ K \eta^\eta - 1 & (\hat{c}_h - \epsilon) + \left(\frac{\alpha}{\gamma^\sigma}\right)^2 \left(\frac{(1 + Ae)^2}{2} - (1 + \gamma)(1 + Ae)\epsilon + \frac{1 + \gamma^2}{2} A^2 \epsilon^2\right) \\
- \left(\frac{\rho r (1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} (\hat{c}_h^{\gamma - 1})^2 - \hat{c}_h^{\gamma - 1}\right) & - K \eta (\eta - 1) e^\eta - 2 (\hat{c}_h^2 - 2 \hat{c}_h \epsilon) \left(\frac{\alpha}{\sigma^\gamma}\right)^2 A^2 \epsilon^2
\end{align*}

The constant terms match. Then there are terms of order $\epsilon^\eta - 1$:

\begin{align*}
K \eta \hat{c}_h \left(\frac{\alpha}{\gamma^\sigma}\right)^2 \frac{1}{2} - \left(\frac{\rho r (1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} (\hat{c}_h^{\gamma - 1})^2 - \hat{c}_h^{\gamma - 1}\right) & = K \eta \hat{c}_h \left(\frac{\alpha}{\gamma^\sigma}\right)^2 \frac{1}{2} - \frac{1 - \gamma}{1 - \gamma} \left(\frac{\alpha}{\gamma^\sigma}\right)^2 = 0
\end{align*}

Then there are terms of order $\epsilon^\eta$:

\begin{align*}
-K \left(\frac{\gamma (\hat{c}_h^{\gamma - 1})}{\gamma} - \frac{1 - \gamma}{2} (\hat{c}_h^{\gamma - 1})^2 - \hat{c}_h^{\gamma - 1}\right) & + \frac{\gamma}{2} \left(\frac{\alpha}{\sigma^\gamma}\right)^2 \\
+ K \eta \hat{c}_h \left(\frac{\alpha}{\gamma^\sigma}\right)^2 (A - (1 + \gamma)A - \hat{c}_h^{\gamma - 1}) & - \frac{1}{2} K \eta (\eta - 1) c_h^2 \left(\frac{\alpha}{\gamma^\sigma}\right)^2 A^2
\end{align*}

We want this to be zero. $K$ factors out, and there is a unique $\eta \in (0, 1)$ that makes this expression zero. After some algebra we obtain

\begin{equation}
\hat{c}_h^{1 - \gamma}(1 - \eta)^2 + \eta (1 - \gamma) \left(\frac{\alpha}{\sigma^\gamma}\right)^2 - \gamma \left(\frac{\alpha}{\sigma^\gamma}\right)^2 = 0 \tag{O.33}
\end{equation}

The bound $\alpha < \hat{a}$ implies $\hat{c}_h^{1 - \gamma} > \gamma \left(\frac{\alpha}{\sigma^\gamma}\right)^2 > 0$, so at $\eta = 0$ the rhs is strictly positive. At $\eta = 1$ we have $\gamma \left(\frac{\alpha}{\sigma^\gamma}\right)^2 - (1 - \gamma) \left(\frac{\alpha}{\sigma^\gamma}\right)^2 = \frac{\gamma}{2} \left(\frac{\alpha}{\sigma^\gamma}\right)^2 > 0$, so the rhs is negative. And because the vertex of the quadratic term is $\eta = 1$ there is a unique $\eta$ that satisfies the expression. So we have

\begin{align*}
\sigma^x &= \left(\frac{\alpha}{\sigma^\gamma}\right) \left(1 + \gamma \hat{c}_h^{-\gamma} \frac{1}{1 - \eta}\right)
\end{align*}

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\[ \sigma \hat{c} \approx -\left( \frac{\alpha}{\sigma \gamma} \right) \frac{\gamma}{1 - \eta} \epsilon \]

Now let’s find the drift of \( \hat{c} \). Using (15) and plugging in the expression for \( \sigma^x \) and \( \sigma^\hat{c} \), the constant terms cancel, and we get terms of order \( \epsilon \) (plus smaller terms)

\[ \mu^\hat{c} = \left( -\hat{c}_1^{1-\gamma} + \left( \frac{\alpha}{\sigma \gamma} \right)^2 \frac{\gamma}{1 - \eta} (1 - \gamma) \right) \epsilon \]

Now use (O.33) to replace \( \hat{c}_1^{1-\gamma} \) and obtain

\[ \mu^\hat{c} = \left( \frac{\alpha}{\sigma \gamma} \right)^2 \left( \frac{\gamma}{1 - \eta} (1 - \gamma) + \eta \frac{2 - \gamma}{2} \frac{\gamma}{(1 - \eta)^2} - \frac{\gamma}{(1 - \eta)^2} \right) \]

After some algebra we get

\[ \mu^\hat{c} \approx \frac{\eta - 2}{2} \left( \frac{\alpha}{\sigma \gamma} \right)^2 \left( \frac{\gamma}{1 - \eta} \right)^2 \epsilon < 0 \]

As desired. \( \square \)