Traditional Real Options Models

- Important and influential
- Based on a simple analogy
- Focus on uncertainty of future shocks
- Option to wait to achieve a higher cash flow level
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- Option to wait to achieve a higher cash flow level

**Implications:**

- Fixed upper trigger
- Record-setting news principle
- Maturity structure of project cash flows is irrelevant, for any given PV
Build a real options model that combines two sources of uncertainty:

- Uncertainty of **future shocks**
- Uncertainty regarding fundamental nature of **past shocks**
Build a real options model that combines two sources of uncertainty:
- Uncertainty of **future shocks**
- Uncertainty regarding fundamental nature of **past shocks**

Consider a specific application:
- Cash flow shocks can be fundamental (**permanent**) and non-fundamental (**temporary**)
- Firm is unable to distinguish the nature of past shocks
- As time passes, the firm updates its prior about past shocks: the longer a shock persists, the more likely it was permanent.
An additional real option:

- Option to learn about the nature of past shocks

Implications for investment dynamics:

- Investment trigger depends on timing of past shocks
- Investment not only in booms, but also at times of stable or decreasing cash flows
- Sluggish response to cash flow shocks
- Maturity structure of cash flows matters even for projects with the same PVs
Results

- **An additional real option:**
  - Option to learn about the nature of past shocks

- **Implications for investment dynamics:**
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  - Maturity structure of cash flows matters even for projects with the same PVs
1. **Both permanent and temporary shocks are important:**
   - Cash flows are more volatile than asset values
   - Correlation between cash flows and asset values is far from being perfect
   - Many shocks revert to mean
2. **Distinguishing between permanent and temporary shocks is often difficult:**

   “Most important, distinguishing between temporary and permanent shocks to commodity prices can be extraordinary difficult. The swings in commodity prices can be too large and uncertain to ascertain their causes and nature. The degree of uncertainty about duration of a price shock varies. For example, market participants could see that the sharp jump in coffee prices caused by the Brazilian frost of 1994 was likely to be reversed, assuming a return to more normal weather. By contrast, most analysts assumed that the high oil prices during the mid-1970s and early 1980s would last indefinitely.”

   (World Bank’s Global Economic Prospects annual report)
Office space in the oil-patch cities experienced incredible growth during the early 1980’s.

For example, over the thirty-year period from 1960 through 1990, over half of all office construction in Denver and Houston was completed over the four-year interval: 1982 – 1985.

Construction likely initiated over the period 1979-1983.

During this period, office vacancies were at record levels (around 30%).

What were developers thinking?
Domestic Crude Oil Prices
(Inflation-adjusted)
What were developers thinking?

- Quote from a Dallas office developer: “We thought we were in the real estate business, but we were really in the oil business.”
- The belief that high oil prices were the result of a permanent shock (OPEC) likely spurred on office development in oil-producing cities.
- Of course, this proved incorrect, but that is with hindsight.
Research Idea: Illustration
Research Idea: Illustration

Traditional Models

![Graph showing Traditional Models with Cash flow X(t) over Time t, indicating Trigger and Shock events.](image-url)
Research Idea: Illustration

Traditional Models

Time $t$  
Cash flow $X(t)$

Trigger

Shock

$X(t)$
Research Idea: Illustration

Bayesian Approach

![Graph showing Cash flow X(t) over time t with a shock and trigger event.](image-url)
A firm has an irreversible investment opportunity:

- At any time $\tau \geq 0$ pay an investment cost $I > 0$ and receive a perpetual cash flow $X(t)$, $t \geq \tau$
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- A **permanent** shock occurs with intensity $\lambda_1$ and increases $X(t) \rightarrow (1 + \varphi)X(t)$ forever
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- $dX(t)$ can also have drift $(\alpha X dt)$ and diffusion $(\sigma X dB_t)$ components
Model: Versions

1. **Simple Model**
   - only one shock which can be either permanent or temporary
   - no drift or diffusion
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2. Model with interaction between Bayesian and Brownian uncertainties
   - only one shock
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   - unlimited number of shocks
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3. **Model with an unlimited number of shocks**
   - unlimited number of shocks
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Let \( H(X) \) be the investment option value after the shock reverses.

Before \( X(t) \) reaches the investment threshold \( X^* \):

\[
rH(X) = \alpha XH'(X) + \frac{\sigma^2}{2} X^2 H''(X)
\]

This is solved subject to the appropriate boundary conditions:

- Value-matching:
  \[
  H(X^*) = \frac{X^*}{r - \alpha} - 1
  \]

- Smooth-pasting
  \[
  H'(X^*) = \frac{1}{r - \alpha}
  \]
The solution is standard (e.g., Dixit and Pindyck (1994))
After a Shock Reverses

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- Investment trigger:

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where \( \beta = \frac{1}{\sigma^2} \left[ - \left( \alpha - \frac{\sigma^2}{2} \right) + \sqrt{\left( \alpha - \frac{\sigma^2}{2} \right)^2 + 2r\sigma^2} \right] > 1 \]
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- Option value:
  \[ H(X) = \begin{cases} 
    \left( \frac{X}{X^*} \right)^\beta \left( \frac{X^*}{r - \alpha} - I \right), & \text{if } X < X^* \\
    \frac{X}{r - \alpha} - I, & \text{if } X \geq X^* 
  \end{cases} \]
While the Shock Persists: Bayesian Learning Process

- \( p(t) \): conditional probability of the past shock being temporary
- Using Bayes rule:

\[
dp(t) = -\lambda_3 p(t)(1 - p(t))\,dt \quad \text{with} \quad p(t_0) = \frac{\lambda_2}{\lambda_1 + \lambda_2}
\]

- Hence,

\[
p(t) = \frac{\lambda_2}{\lambda_1 e^{\lambda_3 (t-t_0)} + \lambda_2}
\]
Let $G(X, p)$ be the investment option value while the shock persists before $X(t)$ reaches the investment threshold $\bar{X}(p)$:

$$rG(X, p) = \alpha XG_X(X, p) + \frac{\sigma^2}{2} X^2 G_{XX}(X, p)$$

$$-\lambda_3 p(1 - p) G_p(X, p) + p \lambda_3 \left( H \left( \frac{X}{1 + \varphi} \right) - G(X, p) \right),$$

where $H(\cdot)$ is the option value after the shock reverts.
While The Shock Persists: Option Value

- This is solved subject to the appropriate boundary conditions:
  - Value-matching:
    \[
    G(\bar{X}(p), p) = \left[ (1 - p) \frac{1}{r - \alpha} + p \frac{1 + \frac{\lambda_3}{(r - \alpha)(1 + \phi)}}{r - \alpha + \lambda_3} \right] \bar{X}(p) - I
    \]
  - Smooth-pasting with respect to $X$:
    \[
    G_X(\bar{X}(p), p) = (1 - p) \frac{1}{r - \alpha} + p \frac{1 + \frac{\lambda_3}{(r - \alpha)(1 + \phi)}}{r - \alpha + \lambda_3}
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  - Smooth-pasting with respect to $p$:
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    G_p(\bar{X}(p), p) = \left[ -\frac{1}{r - \alpha} + \frac{1 + \frac{\lambda_3}{(r - \alpha)(1 + \phi)}}{r - \alpha + \lambda_3} \right] \bar{X}(p)
    \]
Investment Trigger

\[ X(p) = rl + \frac{\sigma^2}{2} X(p)^2 G_{XX}(X(p), p) \]

\[ + \rho_3 \left[ H \left( \frac{X(p)}{1 + \phi} \right) - \left( \frac{X(p)}{(1 + \phi)(r - \alpha)} - 1 \right) \right] \]

Brownian effect

Bayesian effect
Investment Trigger

Investment trigger $\bar{X}(p)$

Firm's belief $p$

Bayesian effect

Brownian effect ($\sigma = 0.05$)

Brownian effect ($\sigma = 0.10$)

Bayesian effect
Let $F(X)$ be the investment option value before the shock arrives.

Before $X(t)$ reaches the investment threshold $\hat{X}$:

$$rF(X) = \alphaXF'(X) + \frac{\sigma^2}{2}X^2F''(X)$$

$$+ (\lambda_1 + \lambda_2) \left( G(X(1 + \varphi), p_0) - F(X) \right).$$

This is solved subject to the usual boundary conditions.
Graphical Illustration: A Simulated Sample Path

Cash flow $X$ as a function of time $t$, showing a shock at time $0.5$. The path fluctuates significantly, with peaks and troughs indicating varying cash flows over time.
Graphical Illustration: A Simulated Sample Path

![Graphical Illustration](image-url)
Graphical Illustration: A Simulated Sample Path
Model Implications

- Two forces that lead to later investment:
  - the standard option to wait for realizations of future shocks
  - option to learn more about past shocks
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Sluggish response of investment to past shocks:
- Consistent with slow response of aggregate investment and labor demand to shocks (e.g., Caballero and Engel (2004))
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- Investment in the face of stable or declining cash flows:
  - A potential explanation for the period of rapid construction in Denver and Houston during 1982-1985 (Grenadier (1996))
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Investment in the face of stable or declining cash flows:
- A potential explanation for the period of rapid construction in Denver and Houston during 1982-1985 (Grenadier (1996))

Importance of the timing of project cash flows
Consider the simple model (one shock, no drift or diffusion) with one alteration

Upon exercise at time $\tau$, the firm gets a perpetual stream of payments

$$
\left(1 + \frac{k}{r}\right) e^{-k(t-\tau)} X(t), \ t \geq \tau
$$

$k$ : measure of how "front-loaded" the project is
Investment in a more "front-loaded" project occurs earlier even when the projects are identical in other dimensions.

Example: immediate sale of the asset vs development of an oil well.
Now, the belief process $p(t)$ is a vector $p(t) = (p_1(t), p_2(t), \ldots)'$:

- $p_k(t)$ is the probability at time $t$ that there are $k$ outstanding temporary shocks
- the dynamics of $p_k(t)$:

$$\frac{dp_k(t)}{dt} = -\lambda_3 p_k(t) \left( k - \sum_{i=1}^{\infty} p_i(t) i \right)$$
The investment threshold $\bar{X}(p)$ satisfies

$$
\bar{X}(p) = \lambda_3 \sum_{i=1}^{\infty} p_i i \left[ G \left( \frac{\bar{X}(p)}{1+\varphi}, \tilde{\rho}(p) \right) + 1 - S \left( \frac{\bar{X}(p)}{1+\varphi}, \tilde{\rho}(p) \right) \right] \\
+ rl + \frac{\sigma^2}{2} \bar{X}(p)^2 G_{XX} (\bar{X}(p), p).
$$

where $\tilde{\rho}(p)$ is the updated vector of beliefs if the shock reverts immediately and $S(X, p)$ is the present value of project cash flows.
Unlimited Number of Shocks

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- Solve numerically for $\bar{X}(p)$ when $p = (p_1 \ p_2 \ 0 \ldots 0\ldots)'$
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- Intuition and main results do not change
Introduce a novel kind of real options problem

Bayesian uncertainty about past shocks leads to
- valuable option to learn and, hence, sluggish response to shocks
- investment at times of stable or decreasing cash flows
- importance of maturity structure of project cash flows

Uncertainty about past shocks may be as important as uncertainty about future shocks
The paper presented studies learning of a decision-maker about the nature of the project.

However, there is an alternative type of learning:

- The decision-maker and outsiders are asymmetrically informed about the nature ("type") of the project.
- The exercise decision reveals information to the outsiders, because different "types" exercise the option at different thresholds.
- The payoff of the decision-maker depends on the perception of his "type" by the outsiders.
The option payoff function from exercise at time $\tau$ is

$$\alpha (P(\tau) - \theta) + G(P(\tau)) \, W(\tilde{\theta}, \theta),$$

where $W(\tilde{\theta}, \theta)$ is the function that captures the effect of the outsiders’ incorrect beliefs. We normalize $W(\theta, \theta) = 0$. 
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Then, for a given vector of inference function of the outsiders $\tilde{\theta}(\hat{P})$ the informed agent’s maximization problem is

$$\max_{\hat{P}} \left\{ \hat{P}^{-\beta} (\alpha (\hat{P} - \theta) + G(\hat{P}) W(\tilde{\theta} (\hat{P}), \theta)) \right\}.$$
Mathematical Formulation

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$$

- This implies the first-order condition

$$
\frac{\hat{P}}{\beta} \left( \alpha + G' (\hat{P}) W (\tilde{\theta} (\hat{P}), \theta) + G (\hat{P}) W_{\tilde{\theta}} (\tilde{\theta} (\hat{P}), \theta) \tilde{\theta}' (\hat{P}) \right).
$$
In equilibrium, the inference function must be consistent with the agent’s investment strategy:

$$\tilde{\theta}(\bar{P}(\theta)) = \theta \ \forall \theta \in [\theta, \theta].$$
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Hence, the first-order condition yields the equilibrium differential equation for \( \bar{P}(\theta) \):

\[
\frac{d\bar{P}(\theta)}{d\theta} = \frac{\bar{P}(\theta) G(\bar{P}(\theta)) W_{\tilde{\theta}}(\theta, \theta) / \alpha}{(\beta - 1) \bar{P}(\theta) - \beta \theta}.
\]

The differential equation is solved subject to the appropriate boundary condition:

- If \( W_{\tilde{\theta}}(\tilde{\theta}, \theta) < 0 \), then \( \bar{P}(\tilde{\theta}) = P^*(\tilde{\theta}) = \tilde{\theta} \beta / (\beta - 1) \).
- If \( W_{\tilde{\theta}}(\tilde{\theta}, \theta) > 0 \), then \( \bar{P}(\tilde{\theta}) = P^*(\tilde{\theta}) = \tilde{\theta} \beta / (\beta - 1) \).
The agent invests a suboptimal threshold in order to shift the beliefs of the outsiders.

The direction of the effect depends on whether the agent benefits more when the outsiders believe that the project is more or less profitable:

- The agent benefits from “better” beliefs $\Rightarrow$ exercise is earlier than in the symmetric information case.
- The agent benefits from “worse” beliefs $\Rightarrow$ exercise is later than in the symmetric information case.

Signal-jamming occurs:

- The outsiders infer the “type” correctly anticipating the suboptimal exercise by the agent.
Venture Capital Grandstanding:
- Perceived type of the general partner affects the amount of new investment in his fund.
- Better projects are taken public earlier.
- Hence, an inexperienced venture capital firm has incentives to take the project public earlier in order to signal a better “type”.

Managerial Myopia:
- Managerial compensation depends on the current stock price (e.g., because of incentives to exercise executive options).
- A manager is better informed about quality of the investment project than shareholders.
- Hence, a manager has incentives to invest at a lower threshold in order to signal higher quality.
Applications: Earlier Exercise

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Applications: Later Exercise

- **Cash Flow Diversion:**
  - A portion of project value is observable only by the manager.
  - The manager can divert it for his own consumption.
  - Hence, a manager has incentives to invest later in order to signal worse quality of the project and be able to divert more.

- Sequential Investment in Product Markets:
  - Two firms are asymmetrically informed about the value of a new product.
  - Investment by the informed firm reveals information about the value of the product to the uninformed firm.
  - Hence, the informed firm has incentives to invest later in order to understate profitability of the project and thereby deter entry of the uninformed firm.
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