Option Exercise Games: An Application to the Equilibrium Investment Strategies of Firms

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Abstract

Under the standard real options approach to investment under uncertainty, agents formulate optimal exercise strategies in isolation and ignore competitive interactions. However, in many real-world asset markets exercise strategies cannot be determined separately, but must be formed as part of a strategic equilibrium. This article provides a very general and tractable solution approach for deriving the equilibrium investment strategies of firms in a continuous-time Cournot-Nash framework. The impact of competition on exercise strategies is dramatic. For example, while standard real options models emphasize that a valuable "option to wait" leads firms to invest only at large positive net present values, the impact of competition drastically erodes the value of the option to wait and leads to investment at very near the zero net present value threshold. The Nash equilibrium exercise strategies are shown to display the useful property that they are equivalent to those derived in an "artificial" perfectly competitive industry under a modified demand curve. This transformation permits a simplified solution approach for the inclusion of various realistic features into the model, such as time-to-build.

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1 Introduction

The real options approach to analyzing investment under uncertainty has become part of the mainstream literature of financial economics. The real options approach to investment now typically comprises an entire chapter in corporate finance textbooks.[1] Essentially, the real options approach posits that the opportunity to invest in a project is analogous to an American call option on the investment opportunity. Once that analogy is made, the vast and rigorous machinery of financial options theory is at the disposal of real investment analysis. The real options approach is well summarized in Dixit and Pindyck (1994) and Trigeorgis (1996).[2]

A feature that the vast majority of real options articles have in common is the lack of strategic interaction across option holders. Investment (exercise) strategies are formulated in isolation, without regard to the potential impact of other firms' exercise strategies. The standard starting point for such models is an exogenous process for underlying asset values (e.g., the stock price in the Black-Scholes framework) or cash flows [e.g., Brennan and Schwartz (1985) or McDonald and Siegel (1986)]. This assumption is crucial to the implications of the approach. For example, perhaps the most well-known result of the real options literature is the invalidation of the standard net present value rule of investing in any project with a non-negative net present value. Because the future value of the asset is uncertain, there is an opportunity cost to investing today. This is often referred to as the "option to wait." Thus, the optimal investment rule, as described in the real options literature, is to invest when the asset value exceeds the investment cost by a potentially large option premium. As stated in Dixit and Pindyck (1994), "Hence the simple NPV rule is not just wrong; it is often very wrong." However, clearly this result depends on the lack of competitive access to this project. If firms fear preemption, then the option to wait becomes less valuable. To understand investment in industries with competitive pressure, a game-theoretic analysis of equilibrium exercise strategies is essential.

A strategic exercise game can be applied to a wide array of real-world investment contexts, where the payoffs from a firm's investment are fundamentally affected by the investment strategies of its competitors. For example, expenditures on research and development are analogous to exercising the option to invest in future growth opportunities. However, firms contemplating such investments are keenly aware of the potential for its competitors to do likewise. The payoff from such research efforts may be substantial if the firm has monopolistic access to the research technology,

[1] For example, see Brealey and Myers (2000) and Van Horne (1998).

[2] The application of the real option approach to investment is increasingly broad. Brennan and Schwartz (1985) use an option pricing approach to analyze investment in natural resources. McDonald and Siegel (1986) provided the standard continuous-time framework for analysis of a firm's investment in a single project. Majd and Pindyck (1987) enrich the analysis with a time-to-build feature. Dixit (1989a) uses the real option approach to examining entry and exit from a productive activity. Triantis and Hodder (1990) analyze manufacturing flexibility as an option. Titman (1985) and Williams (1991) use the real options approach to analyze real estate development.

but can become significantly muted if competitors have similar access. Similarly, a developer considering the construction of an office tower realizes that the economic prospects of the completed building will critically depend on the actions of other developers. There is substantial empirical evidence that rents and occupancy rates can plummet in the presence of construction booms.

This article provides a very general and tractable solution approach for deriving the equilibrium investment strategies of firms in a Cournot-Nash framework. Each firm faces a sequence of investment opportunities and must determine an exercise strategy for its path of investment. The cash flows from investment are determined by a continuous-time stochastic shock process as well as the investment strategies of all firms in the industry. A symmetric Nash equilibrium in exercise policies is determined such that each firm's equilibrium exercise strategy is optimal, conditional on its competitors following their equilibrium exercise strategies. The resulting equilibrium is analytically quite simple and has potentially wide applications.

There have been relatively few real options applications concerned with strategic equilibrium in exercise policies.[3] This is partially a result of the fact that such issues are typically ignored in the financial options literature, as most financial options are widely held side-bets between agents external to the underlying firm, and therefore their exercise does not influence the characteristics of the underlying security or the options themselves.[4] In addition, the application of game theory to continuous-time models is not well-developed, and often quite tricky. Nevertheless, some progress has already been made. Williams (1993) and Grenadier (1996) use option exercise games to understand real estate development. Smets (1993) uses an option exercise game for an application in international finance. Applications related to real option investments in strategic settings include Trigeorgis (1996, Chapter 9) and Kulatilaka and Perotti (1998). Grenadier (1999) analyzes the case of a strategic equilibrium in exercise strategies under asymmetric information over the underlying option parameters. The closest applications of option exercise games to this paper are in Lambrecht and Perraudin (1999) and Baldursson (1997). Lambrecht and Perraudin (1999) provide an example of an exercise game in which firms compete over the exercise of a real option. Their model uses a different solution approach than the present model, and deals with firms competing over a single investment opportunity. In contrast, the present model describes an industry equilibrium with multiple active firms. The equilibrium framework of Baldursson (1997) is very similar to the present model. However, the solution approach offered by Baldursson (1997) is only applicable to a very specific setting in which demand is linear; his methodology will not work for more general specifications.

Following the determination of the equilibrium exercise strategies, the implications of the equilibrium are analyzed. As mentioned earlier, for a firm with monopolistic

[3]For a summary of the existing literature game-theoretic option theory, see Grenadier (2000b).

[4]A notable exception is the case of warrants and convertible securities [Emanuel (1983), Constantinides (1984), Spatt and Sterbenz (1988)].

access to a project, the option to wait can be quite valuable, leading to investment at triggers well above the standard $NPV = 0$ threshold. However, the model demonstrates that the presence of competition quickly erodes the value of the option to wait. While for reasonable parameter values a monopolist may not invest until the net present value is double the cost of investment, with competition the traditional net present value rule becomes approximately correct, even for industries with only a few competitors. The fact that the option to wait can be very valuable for a monopolist helps explain the large resources that firms expend in order to protect their position in markets.

A related implication of the model is that as the number of industry competitors increases, the likelihood of existing asset values falling below their investment cost increases. We know that in many real world asset markets, investments that initially appeared promising turned out poorly as the market quickly turned downward. However, in the traditional real options framework, such ex-post losses are highly infrequent. Since a monopolist invests at a substantial option premium, the likelihood for large asset value reversals is remote. Thus, it is very difficult for the standard real options framework to explain boom-and-bust markets such as real estate, where periodic bouts of overbuilding result in waves of high vacancy and foreclosure rates. It is found, however, that given the much smaller option premiums in competitive environments, such reversals become much more likely.

A useful and surprising feature of the equilibrium set of exercise strategies is that they can be determined in an "artificial" perfectly competitive industry. That is, if the industry demand curve is transformed in a particular manner, the equilibrium set of exercise strategies in an oligopolistic framework can be determined *as if* the industry was perfectly competitive. Since it is well-known [Lucas and Prescott (1971), Dixit (1989b)] that such perfectly competitive strategies can be determined by a central planner maximizing the path of societal surplus, then the exercise game can be solved as a single agent's optimization problem. A number of articles have addressed the problem of linking a strategic equilibrium with the solution of a single optimization problem, including Spence (1976a, 1976b), Bergstrom and Varian (1985), and Slade (1994).

The fact that the equilibrium exercise strategies can be obtained from an artificial competitive equilibrium (or from the solution of a single optimization problem) permits the extension of the results to more complex environments. One extension that is presented in this article is the addition of a time-to-build feature. While the addition of a time-to-build component is not particularly difficult in the context of a monopolist, it is difficult in the context of an industry equilibrium. In an industry setting, the state of the market at any point in time is path-dependent, as the relevant state space depends not only on current demand and supply, but also on all units currently in the pipeline (as well as the instants at which they were initiated). However, since Grenadier (2000a) was able to solve this problem for a perfectly competitive industry, we can transform the oligopolistic equilibrium into an "equivalent" competitive equilibrium, and obtain simple closed-form solutions for the equilibrium investment strategies.

The article is organized as follows. Section 2 presents the basic model for the option exercise game. Section 3 provides a simplified approach to equilibrium determination as well as closed-form solutions for a particular base-case example. Section 4 examines the impact of strategic competition on the value of the option to wait. Section 5 demonstrates that the strategic equilibrium is equivalent to that achieved in an artificial competitive equilibrium. Section 6 examines the impact of time-to-build on the strategic equilibrium exercise strategies. Section 7 provides solutions for a variety of alternative specifications. Section 8 concludes.

2 Model and Assumptions

The general framework of the model is the real options approach to investment under uncertainty. Each firm holds a sequence of investment opportunities that are analogous to the exercise of a call option on an asset. In the vast majority of option pricing models (both real and financial), the starting point is an exogenous process for the underlying asset value (e.g., the stock price in the Black-Scholes framework) or cash flows [e.g., Brennan and Schwartz (1985) or McDonald and Siegel (1986)]. However, in this case, the value of the payoff from investment (exercise) is endogenous as it depends on the exercise strategies of all option holders. In such a strategic environment, optimal exercise strategies cannot be derived in isolation, but must be calculated as part of a game-theoretic equilibrium.

Consider an oligopolistic industry comprised of n identical firms producing a single, homogeneous good.[5] At time t, firm i produces $q_i(t)$ units of output. Output is infinitely divisible. The price of a unit of output, $P(t)$, fluctuates stochastically over time so as to clear the market:

$$P(t) = D[X(t), Q(t)], \tag{1}$$

where D is the inverse demand function, $X(t)$ is an exogenous shock process to demand, and $Q(t) = \sum_{j=1}^{n} q_j(t)$ is the industry supply process. I assume that the function D is twice continuously differentiable, strictly increasing in X, and strictly decreasing in Q.[6] For simplicity, I assume no variable costs of production, and thus $P(t)$ represents the cash flow per unit. Denote the output process of all firms except firm i by $Q_{-i}(t) = \sum_{j=1, j\neq i}^{n} q_j(t)$. Let $\pi_i\left[X(t), q_i(t), Q_{-i}(t)\right] \equiv q_i(t) \cdot D\left[X(t), q_i(t) + Q_{-i}(t)\right]$

[5] The assumption that the number of firms in the industry is exogenous can be relaxed. For fixed n, let $F(n)$ denote the equilibrium value of each firm (to be derived below), where $F'(n) < 0$. Suppose that there are set-up costs of Ω that must be paid to enter the industry. Then, the equilibrium number of firms in the industry is the value of n that satisfies the conditions $F(n) \geq \Omega$, and $F(n+1) < \Omega$.

[6] In addition, in order to assume finite asset values, $E\left\{\int_0^\infty e^{-rt} D\left[X(t), \hat{q}\right] dt\right\} < \infty$ for all fixed $\hat{q} \in \mathbf{R}_+$.

denote the profit flow for firm i, given the current state of the industry. I assume that the function D is such that the conditions $\frac{\partial^2 \pi_i}{\partial X \partial q_i} > 0$ and $\frac{\partial^2 \pi_i}{\partial q_i^2} < 0$ holds for all i.

At any point in time, each firm can invest in additional capacity to increase its output by an infinitesimal increment $dq_i \equiv \frac{dQ}{n}$. Thus, the path of output is continuous, and if all firms increase capacity simultaneously, $Q(t)$ increases by the increment dQ. The cost of increasing output is linear: increases in a firm's output involve a cost of K per unit of output.[7]

The shock, $X(t)$, is the ultimate source of uncertainty in the model. At each point in time, even demand at the next instant is uncertain. I assume that $X(t)$ follows a time-homogeneous diffusion process of the form,

$$dX = \mu(X)dt + \sigma(X)dz, \tag{2}$$

where $z(t)$ is a Wiener process.[8] This class of processes embeds several special cases that are often used in the real options literature. If $\mu(X) = \mu X$, and $\sigma(X) = \sigma X$, then $X(t)$ follows a geometric Brownian motion and is log-normally distributed. If $\mu(X) = \mu$, and $\sigma(X) = \sigma$, then $X(t)$ follows a Brownian motion, and is normally distributed. If $\mu(X) = \kappa(\theta - X)$, and $\sigma(X) = \sigma\sqrt{X}$, then $X(t)$ follows a square-root process. I assume that cash flows are valued in a risk-neutral framework, where r is the risk-free rate.

The optimal exercise decision must be part of an endogenous, Nash equilibrium solution in exercise strategies. Each firm chooses its output process $q_i(t)$ so as to maximize its value, conditional on the assumed exercise strategies of its competitors.[9] The n-tuple of strategies $[q_1^*(t), ..., q_n^*(t)]$ constitutes a Nash equilibrium if $q_i^*(t)$ is the optimal strategy for each firm i when it takes the strategies of its competitors, $Q_{-i}^*(t)$, as given. Mathematically, let $V^i\left[X, q_i, Q_{-i}; q_i(t), Q_{-i}(t)\right]$ denote the value of firm i,

[7] Note that the assumptions that the profit function is concave in q_i and that investment costs are proportional ensure that optimal capacity expansion will indeed be of the "optimal instantaneous control" variety. See Harrison and Taksar (1983).

[8] It is assumed that $\mu(X)$ and $\sigma(X)$ satisfy the standard conditions for a solution to (2) to exist [see Karatzas and Shreve (1988)]. In addition, I restrict $X(t)$ to the class of processes that, when combined with the earlier restriction that $\frac{\partial^2 \pi_i}{\partial X \partial q_i} > 0$, ensures that the optimal exercise strategy is a "trigger" strategy where output is increased at the moment $X(t)$ rises to a specific upper threshold. The precise condition is discussed in Appendix B in Chapter 4 of Dixit and Pindyk (1994). As stated on page 104 of Dixit and Pindyk (1994), (the restriction of the process for $X(t)$) "is true for random walks, Brownian motion, mean-reverting autoregressive processes, and indeed in almost all economic applications we can think of."

[9] The current assumption of a constant returns to scale production technology can be generalized. We can allow for more general production by allowing output to be determined by a production function with invested capital as an argument. Thus, firm's choose investment in capital (which is transformed into output by a production function) in their dynamic optimization problems. Provided that each firm's profit function is concave in capital, and that the partial derivative of each firm's profits with respect to capital is increasing in X, the basic structure of the model continues to hold. Note however that these conditions imply restrictions on both the demand and production functions, while the current model places restrictions only on the demand function.

for given strategies $q_i(t)$ and $Q_{-i}(t)$, where (X, q_i, Q_{-i}) are the initial values of the state variables. V^i can then be written as the discounted expectation of future cash flows:

$$V^i\left[X, q_i, Q_{-i}; q_i(t), Q_{-i}(t)\right] = E\left\{\int_0^\infty e^{-rt}\pi_i\left[X(t), q_i(t), Q_{-i}(t)\right]dt - \int_0^\infty e^{-rt}K dq_i(t)\right\}, \tag{3}$$

where the expectation operator is conditional on the current state $[X, q_i, Q_{-i}]$. Thus, the strategies $[q_1^*(t), ..., q_n^*(t)]$ constitute a Nash equilibrium if

$$V^i\left[X, q_i, Q_{-i}; q_i^*(t), Q_{-i}^*(t)\right] = \sup_{\{q_i(t):t>0\}} V^i\left[X, q_i, Q_{-i}; q_i(t), Q_{-i}^*(t)\right], \forall i. \tag{4}$$

I will focus on the case of a symmetric Nash equilibrium. For the case of a symmetric equilibrium, $q_i^*(t) = q_j^*(t)$ for all i, j, and thus $q_i^*(t) = Q^*(t)/n$, $\forall i$.[10]

Each firm faces a dynamic programming problem of determining its optimal investment strategy, while taking into account its competitors' investment strategies. The state of the economy at any instant is fully characterized by the vector $[X(t), q_1(t), ..., q_n(t)]$ where the initial values of the variables are denoted by $[X, q_1, ..., q_n]$, where $X(0) = X$, and $q_i(0) = q_i$, $\forall i$. The solution to such a dynamic programming problem can be derived using the methods of option pricing, using the following analogy. Each individual firm holds a sequence of options to increase its output, fully recognizing that the exercise of investment options by its competitors will impact its own payoffs from exercise. Firm i's decision problem can be envisioned as a sequence of options on marginal investments. For each output level of q_i, it holds an option (of the perpetual, American variety) on the marginal flow of profits through increasing its output to $q_i + dq$, with an exercise price of $K \cdot dq$. Similarly, the other $(n-1)$ firms hold options to increase their output, which directly impact the profits of firm i. Thus, while firm i controls the evolution of the process $q_i(t)$, it recognizes that the evolution of the process $Q_{-i}(t)$, which is beyond firm i's control, helps determine the payoff from the exercise of its investment options. Given the assumed properties of $X(t)$ and $D(X, Q)$, the optimal exercise policies of all of these options will take the form of trigger policies: options will be exercised the first moment that $X(t)$ rises to a threshold that is a function of the current state of the industry, (q_i, Q_{-i}).

I now apply this option pricing approach to solve for the equilibrium exercise strategies and firm values. Let us begin by considering firm i's optimal investment strategy, where firm i takes all competitors' strategies as given. Suppose that firm i's competitors are assumed to incrementally increase capacity at each moment when

[10] As in most models of oligopolistic equilibrium, the assumption of identical firms and the resulting focus on symmetric equilibrium substantially simplifies the derivation of equilibrium. Most importantly, while the current state space involves two variables, $X(t)$ and $Q(t)$, if there are n different firms, the state space would grow to encompass $n+1$ variables, $X(t)$, $q_1(t)$, ... , $q_n(t)$. Baldursson (1997) provides a numerical solution for an equilibrium with n firms, which differ in their initial capacity (but are identical otherwise). He finds that there will be convergence to the symmetric equilibrium after a sufficient period of time.

$X(t)$ rises to the given trigger function $X^{-i}(q_i, Q_{-i})$, which for simplicity we write as X^{-i}. Let $F^i[X, q_i, Q_{-i}; X^{-i}]$ denote the value of firm i, contingent on this assumed investment strategy of its competitors. Consider the instantaneous return on $F^i[X, q_i, Q_{-i}; X^{-i}]$ over a region in which no new investment occurs. By Itô's Lemma, the instantaneous change in F^i is:

$$dF^i = \left[\frac{1}{2}\sigma(X)^2 F^i_{XX} + \mu(X) F^i_X\right] dt + \sigma(X) F^i_X dz. \tag{5}$$

In addition to the capital gain, firm i's current output level yields the dividend flow rate of $\pi_i(X, q_i, Q_{-i})$. Therefore, the total expected return on F^i per unit time, α_i, is:

$$\begin{aligned} \alpha_i &= E\left[\frac{dF^i + \pi_i(X, q_i, Q_{-i})dt}{F^i} \cdot \frac{1}{dt}\right] \\ &= \frac{1}{F^i} \cdot \left[\frac{1}{2}\sigma(X)^2 F^i_{XX} + \mu(X) F^i_X + \pi_i(X, q_i, Q_{-i})\right]. \end{aligned} \tag{6}$$

Since the equilibrium expected return on this asset must equal r, setting α_i equal to r and simplifying yields the following equilibrium differential equation:

$$0 = \frac{1}{2}\sigma(X)^2 F^i_{XX} + \mu(X) F^i_X - rF^i + \pi_i(X, q_i, Q_{-i}). \tag{7}$$

The solution $F^i[X, q_i, Q_{-i}; X^{-i}]$ to differential equation (7) must then satisfy appropriate boundary conditions.

The first boundary condition that F^i must satisfy is a "value-matching" condition. Suppose firm i exercises its expansion option at the trigger $X^i(q_i, Q_{-i})$. At the moment of exercise, q_i increases by the infinitesimal increment dq, and the firm pays the exercise price $K \cdot dq$. Thus, at the moment of exercise, $F^i[X^i(q_i, Q_{-i}), q_i, Q_{-i}; X^{-i}] = F^i[X^i(q_i, Q_{-i}), q_i + dq, Q_{-i}; X^{-i}] - K \cdot dq$. Dividing by the incremental increment dq, the value-matching condition can be written in derivative form as:

$$\frac{\partial F^i}{\partial q_i}\left[X^i(q_i, Q_{-i}), q_i, Q_{-i}; X^{-i}\right] = K. \tag{8}$$

The second boundary condition that F^i must satisfy is an optimality condition that ensures that the trigger $X^i(q_i, Q_{-i})$ is determined optimally. The "smooth-pasting" condition is that $F^i_X[X^i(q_i, Q_{-i}), q_i, Q_{-i}; X^{-i}] = F^i_X[X^i(q_i, Q_{-i}), q_i + dq, Q_{-i}; X^{-i}]$.[11] Writing this in derivative form, we have:

$$\frac{\partial^2 F^i}{\partial q_i \partial X}\left[X^i(q_i, Q_{-i}), q_i, Q_{-i}; X^{-i}\right] = 0. \tag{9}$$

[11] See Merton (1973) for a derivation of the "smooth-pasting" or "high-contact" condition.

In the theory of optimal instantaneous control, Dumas (1991) calls this the "super-contact" condition. This is analogous to the more commonly known "high-contact" condition in the standard option pricing literature.

The final boundary condition that F^i must satisfy is a value-matching condition at the competitors' trigger function, $X^{-i}(q_i, Q_{-i})$. At the moment firm i's competitors exercise, Q_{-i} increases by the infinitesimal increment dQ_{-i}. Thus, at the moment of competitive exercise, $F^i\left[X^{-i}(q_i, Q_{-i}), q_i, Q_{-i}; X^{-i}\right] = F^i\left[X^{-i}(q_i, Q_{-i}), q_i, Q_{-i} + dQ_{-i}; X^{-i}\right]$. Dividing by the incremental increment dQ_{-i}, this condition can be written in derivative form as:

$$\frac{\partial F^i}{\partial Q_{-i}}\left[X^{-i}(q_i, Q_{-i}), q_i, Q_{-i}; X^{-i}\right] = 0. \tag{10}$$

In addition to satisfying boundary conditions (8), (9), and (10), F^i must satisfy regularity conditions that depend on the precise specifications of $X(t)$ and $D(X, Q)$. These will be considered when specific examples are worked out.

We can now fully characterize a symmetric Nash equilibrium in exercise strategies. Since the equilibrium is symmetric, $X^i(q_i, Q_{-i}) = X^{-i}(q_i, Q_{-i})$ for all i, and denote this common equilibrium trigger by $\bar{X}(q_i, Q_{-i})$. Therefore, the determination of the equilibrium trigger function, $\bar{X}(q_i, Q_{-i})$, is the solution to a rather complicated fixed point problem in function space. For an assumed competitive trigger $\bar{X}(q_i, Q_{-i})$, the optimal trigger strategy for firm i must also be $\bar{X}(q_i, Q_{-i})$. Therefore, for $\bar{X}(q_i, Q_{-i})$ to be the symmetric equilibrium trigger, $F^i\left[X, q_i, Q_{-i}; \bar{X}\right]$ must satisfy differential equation (7), subject to boundary conditions (8), (9), and (10), where the optimal trigger $X^i(q_i, Q_{-i})$ equals $\bar{X}(q_i, Q_{-i})$. Write the equilibrium value of firm i as $V^i(X, q_i, Q_{-i}) \equiv F^i\left[X, q_i, Q_{-i}; \bar{X}\right]$. The equilibrium is summarized in the following proposition.

Proposition 1 *Let the equilibrium value of each firm i be denoted by $V^i(X, q_i, Q_{-i})$. Each firm's equilibrium investment strategy is characterized by increasing output incrementally whenever $X(t)$ rises to the trigger function $\bar{X}(q_i, Q_{-i})$. The functions $V^i(X, q_i, Q_{-i})$ and $\bar{X}(q_i, Q_{-i})$ are solutions to the following differential equation:*

$$0 = \frac{1}{2}\sigma(X)^2 V^i_{XX} + \mu(X) V^i_X - rV^i + \pi_i(X, q_i, Q_{-i}),$$

subject to:

$$\begin{aligned}
\frac{\partial V^i}{\partial q_i}\left[\bar{X}(q_i, Q_{-i}), q_i, Q_{-i}\right] &= K, \\
\frac{\partial^2 V^i}{\partial q_i \partial X}\left[\bar{X}(q_i, Q_{-i}), q_i, Q_{-i}\right] &= 0, \\
\frac{\partial V^i}{\partial Q_{-i}}\left[\bar{X}(q_i, Q_{-i}), q_i, Q_{-i}\right] &= 0.
\end{aligned}$$

In the following section I demonstrate a much simpler approach to determining the equilibrium exercises strategies, $\bar{X}(q_{i,}Q_{-i})$. In particular, I demonstrate that $\bar{X}(q_{i,}Q_{-i})$ is the optimal trigger for a "myopic" option problem in which competitors' exercise is ignored. The solution to this myopic problem does not necessitate the solution to a fixed-point problem, and can be solved by traditional option pricing methods.[12]

3 A Simplified Approach to Equilibrium Derivation

The determination of a Nash equilibrium in exercise strategies is seemingly a most complex problem. In equilibrium it must be the case that conditional on firm i believing that its competitors will exercise at the trigger function $\bar{X}(q_{i,}Q_{-i})$, its own optimal exercise trigger must also be $\bar{X}(q_{i,}Q_{-i})$. The determination of equilibrium seems to require an optimal exercise strategy to represent the solution to a fixed point problem in function space. Fortunately, the problem of equilibrium exercise strategy determination can be vastly simplified into a form that is no more complicated than the standard real option problem. After presenting the simplified equilibrium solution approach, a closed-form solution is derived for an illustrative example.

In a continuous-time, perfectly competitive equilibrium setting, Leahy (1993) demonstrates that the equilibrium investment policy of an individual firm is identical to a myopic strategy in which a firm ignores the effect that other firms exert on the price process. In this section, I demonstrate that the Nash equilibrium exercise strategy also evolves *as if* it were determined by a firm pursuing a form of myopic exercise strategy.[13]

Consider a myopic firm i that ignores all potential competitive exercise. That is, given a current level of competitive supply, Q_{-i}, the myopic firm assumes that Q_{-i} will remain fixed forever. Let the value of the myopic firm be denoted by $M^i(X, q_i, Q_{-i})$ and its optimal exercise trigger be denoted by $X^m(q_i, Q_{-i})$. $M^i(X, q_i, Q_{-i})$ and $X^m(q_i, Q_{-i})$ are simultaneously determined as solutions to the following differential

[12] The model can be generalized to allow for firms to hold options to both increase and decrease output. In such a case, there would be an upper threshold function that triggers output increases and a lower threshold function that triggers output decreases. The result that a Nash equilibrium can be constructed using a simple myopic approach would continue to hold. Note that the option to decrease output has value when profits can become negative, as would be the case with variable costs.

[13] Just as in Leahy (1993), in order to obtain the result that the equilibrium outcomes can be solved via a myopic strategy one must assume that investment is infinitely divisible. This is essential to a limiting argument that is made in the proof of Proposition 2. An equilibrium determination with discrete investment can be solved using the approach of Smets (1993) or Grenadier (1996). While discrete investment is likely to be a more accurate description of reality, incremental investment may be a reasonable approximation, particularly in the case in which new investment is a small fraction of current industry capacity.

equation:

$$0 = \frac{1}{2}\sigma(X)^2 M^i_{XX} + \mu(X)M^i_X - rM^i + \pi_i(X, q_i, Q_{-i}), \tag{11}$$

subject to:

$$\begin{aligned} \frac{\partial M^i}{\partial q_i}\left[X^m(q_{i,}Q_{-i}), q_i, Q_{-i}\right] &= K, \\ \frac{\partial^2 M^i}{\partial q_i \partial X}\left[X^m(q_{i,}Q_{-i}), q_i, Q_{-i}\right] &= 0. \end{aligned} \tag{12}$$

Note that this is identical to the system for F^i in the previous section, with the exception that the value-matching condition for the competitive exercise strategies is ignored.

Note that the solution for $M^i(X, q_i, Q_{-i})$ and its associated trigger $X^m(q_i, Q_{-i})$ is a very standard problem in the real options literature. There is no fixed-point problem that needs to be solved, but only a differential equation subject to a value-matching and a smooth-pasting condition. We shall later find it useful to consider the value of a myopic firm's marginal output. Denote the value of a myopic firm's marginal output by $m^i(X, q_i, Q_{-i})$, with $m^i(X, q_i, Q_{-i}) = \frac{\partial M^i}{\partial q_i}(X, q_i, Q_{-i})$.

The following proposition states that the myopic exercise trigger is identical to the Nash equilibrium exercise trigger.

Proposition 2 *The symmetric Nash equilibrium exercise strategy described in Proposition 1 is characterized by each firm increasing output whenever $X(t)$ rises to the myopic trigger function $X^m(q_{i,}Q_{-i})$. That is, $\bar{X}(q_{i,}Q_{-i}) = X^m(q_{i,}Q_{-i})$.*

Proof: See Appendix.

The dimensionality of the equilibrium can be further reduced by recognizing the fact that in a symmetric equilibrium, $q_i = \frac{1}{n}Q$, and $Q_{-i} = \frac{n-1}{n}Q$. We can then write the equilibrium exercise strategy as solely a function of Q. Let the equilibrium exercise trigger be denoted by $X^*(Q)$, with $X^*(Q) \equiv X^m(\frac{1}{n}Q, \frac{n-1}{n}Q)$. The following proposition presents a simple means of deriving the optimal exercise trigger, $X^*(Q)$.

Proposition 3 *In a symmetric Nash equilibrium, each firm will exercise its investment option whenever $X(t)$ rises to the trigger $X^*(Q)$. Let $m(X, Q)$ denote the value of a myopic firm's marginal investment, with $m(X, Q) \equiv m^i(X, \frac{1}{n}Q, \frac{n-1}{n}Q)$. $X^*(Q)$ and $m(X, Q)$ are jointly determined by the following differential equation:*

$$0 = \frac{1}{2}\sigma(X)^2 m_{XX} + \mu(X)m_X - r \cdot m + D(X, Q) + \frac{Q}{n}D_Q(X, Q), \tag{13}$$

subject to:

$$\begin{aligned} m\left[X^*(Q), Q\right] &= K, \\ \frac{\partial m}{\partial X}\left[X^*(Q), Q\right] &= 0. \end{aligned} \tag{14}$$

Proof: See Appendix.

Note that we have now reduced the search for an equilibrium exercise strategy to the solution of a standard problem from the real-options literature: $m(X,Q)$ represents the value of a perpetual American call option, where the option pays a cash flow of $D(X,Q) + \frac{Q}{n}D_Q(X,Q)$, an exercise payoff of K, and a zero exercise price.

Given the equilibrium exercise strategy $X^*(Q)$, it is then a simple matter of calculating the value of each firm in equilibrium. Let $F(X,Q)$ denote the equilibrium value of any firm. $F(X,Q)$ must satisfy the following equilibrium differential equation:

$$0 = \frac{1}{2}\sigma(X)^2 F_{XX} + \mu(X)F_X - rF + \frac{Q}{n}D(X,Q), \tag{15}$$

subject to:

$$\frac{\partial F}{\partial Q}\left[X^*(Q), Q\right] = K/n. \tag{16}$$

Boundary condition (16) ensures that when the trigger $X^*(Q)$ is reached, Q increases by the infinitesimal increment dQ, and the firm incurs a proportional exercise cost of $K/n \cdot dQ$.

Finally, it shall prove useful to also determine the value of a unit of investment. Let $G(X,Q)$ denote the value of the Qth unit of investment in production capacity, which simply provides the cash flow stream of $P(t) = D[X(t), Q(t)]$ in perpetuity. $G(X,Q)$ must satisfy the following equilibrium differential equation:

$$0 = \frac{1}{2}\sigma(X)^2 G_{XX} + \mu(X)G_X - rG + D(X,Q), \tag{17}$$

subject to:

$$\frac{\partial G}{\partial Q}\left[X^*(Q), Q\right] = 0. \tag{18}$$

Boundary condition (18) ensures that when the trigger $X^*(Q)$ is reached, Q increases by the infinitesimal increment dQ, and the cash flow increases from $D(X,Q)$ to $D(X, Q + dQ)$.

I will now illustrate the option exercise game equilibrium through a specific example. Closed-form solutions for the equilibrium processes are presented. This base-case equilibrium will be analyzed in the future sections. Additional solutions for alternative specifications will be presented in Section 7.

In this base-case, I assume prototypical specifications for both the shock process $X(t)$ and the inverse demand curve $D(X,Q)$. These specifications are commonplace in the real options literature. Assume that the market inverse demand function is of a constant-elasticity form:

$$P(t) = X(t) \cdot Q(t)^{-\frac{1}{\gamma}}. \tag{19}$$

We assume that $\gamma > 1/n$ to ensure that marginal profits (for any assumed value of n) are increasing in X.[14] $X(t)$ represents a multiplicative demand shock, and evolves

[14] Concavity of the profit function with respect to each firm's output is guaranteed for all $n > 1$.

as a geometric Brownian motion:

$$dX = \mu X dt + \sigma X dz. \tag{20}$$

Assume that $r > \mu$ in order to ensure convergence.[15]

By solving differential equation (13) subject to boundary condition (14), the equilibrium value of a myopic firm's marginal investment, $m(X, Q)$, and its equilibrium investment trigger, $X^*(Q)$, can be expressed as:

$$\begin{aligned} m(X,Q) &= -\frac{n\gamma - 1}{n\gamma} \frac{v_n^{1-\beta}}{\beta(r-\mu)} Q^{-\frac{\beta}{\gamma}} X^{\beta} + \frac{n\gamma - 1}{n\gamma} \frac{Q^{-\frac{1}{\gamma}}}{r-\mu} X, \\ X^*(Q) &= v_n \cdot Q^{\frac{1}{\gamma}}, \end{aligned} \tag{21}$$

where:

$$\begin{aligned} v_n &= \left(\frac{\beta}{\beta - 1}\right)\left(\frac{n\gamma}{n\gamma - 1}\right)(r-\mu) \cdot K, \\ \beta &= \frac{-\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2}}{\sigma^2} > 1. \end{aligned} \tag{22}$$

The equilibrium trigger, $X^*(Q)$, is an increasing, linear function of $Q^{\frac{1}{\gamma}}$. The explicit dependence of the equilibrium on the degree of competition is through the function v_n. The equilibrium trigger is a decreasing function of n:

$$\frac{\partial X^*(Q)}{\partial n} = -\frac{X^*(Q)}{n\,(n\gamma - 1)} < 0. \tag{23}$$

Increasing competition leads firms to exercise their options sooner, as the fear of preemption diminishes the value of their "options to wait."

We can verify that the equilibrium exercise strategy coincides with two special cases derived in the real options literature: the monopolist solution ($n = 1$), and the perfectly competitive solution ($n \to \infty$). For the case of the monopolist, Chapter 11 of Dixit and Pindyck (1994) derives the optimal exercise strategy for a monopolist choosing incremental investment in capacity. The monopolist faces a profit function of $\pi(X, Q) = X \cdot H(Q)$, with a multiplicative log-normal shock X. The optimal exercise strategy for the monopolist derived by Dixit and Pindyck (with slightly altered notation) is $X^*(Q) = \frac{\beta}{\beta-1}\frac{(r-\mu)\cdot K}{H'(Q)}$. In the present example, the profit function is $\pi(X, Q) = Q \cdot (X \cdot Q^{-\frac{1}{\gamma}})$, and thus $H(Q) = Q^{\frac{\gamma-1}{\gamma}}$. Therefore, their solution coincides

[15] The solutions to the partial differential equations require the imposition of a boundary condition at the lower limit of the space for the process of $X(t)$. For the Geometric Brownian Motion, zero is an absorbing barrier. Thus, the appropriate boundary condition is that the value of the option at $X = 0$ is the present value of all future cash flows with $X(t)$ remaining at zero, and $Q(t)$ remaining at its current level forever.

with $X^*(Q) = v_1 Q^{\frac{1}{\gamma}}$. For the case of perfect competition, Chapter 9 of Dixit and Pindyck (1994) derives the optimal equilibrium trigger for entry into an industry with inverse demand curve: $P = X \cdot D(Q)$, with X being log-normal. They derive an equilibrium exercise trigger of $X^*(Q) = \frac{\beta}{\beta-1}\frac{(r-\mu)\cdot K}{D(Q)}$. Once again, in this example with $D(Q) = Q^{-\frac{1}{\gamma}}$, their solution coincides with $X^*(Q) = \lim_{n\to\infty} v_n Q^{\frac{1}{\gamma}}$.

Figure 1 plots a simulated equilibrium for a hypothetical 10-firm industry. A simulated path of $X(t)$ is plotted, along with the trigger function $X^*(Q_t)$. At any instant at which $X(t)$ rises to the trigger, each firm exercises its option. The units of the $X(t)$ and $X^*(Q_t)$ curves are displayed along the left axis. The impact of the moments of option exercise is displayed by the path of $Q(t)$, whose units are displayed along the right axis. Each moment of option exercise translates into an incremental increase in firm output, $Q(t)$.

A closed-form solution for the conditional distribution of the future path of supply is easily derived. The optimal trigger strategy in (21) translates into an expression for the process of equilibrium supply, $Q(t)$:

$$Q(t) = \max\left[Q(0), \left(\frac{M(t)}{v_n}\right)^{\gamma}\right], \tag{24}$$

where $M(t) \equiv \sup\left[X(s) : 0 \le s \le t\right]$. The distribution function for equilibrium supply at time t, conditional on $X(0) = X$ and $Q(0) = Q$, can be written as:

$$\begin{aligned} J_t(q; X, Q) &\equiv \Pr\left[Q(t) \le q\right] \qquad (25) \\ &= \Pr\left[\left(\frac{M(t)}{v_n}\right)^{\gamma} \le q\right] \\ &= \Pr\left[\sup\left(\ln\left[\frac{X(s)}{X}\right] : 0 \le s \le t\right) \le d(q)\right] \\ &= \Phi\left[\frac{d(q) - (\mu - \sigma^2/2)\cdot t}{\sigma\sqrt{t}}\right] - \left(\frac{Xq^{-\frac{1}{\gamma}}}{v_n}\right)^{-\frac{2(\mu-\sigma^2/2)}{\sigma^2}} \cdot \Phi\left[\frac{-d(q) - (\mu - \sigma^2/2)\cdot t}{\sigma\sqrt{t}}\right], \end{aligned}$$

for $q \ge Q$, and 0 otherwise, where:

$$d(q) = -\ln\left(\frac{Xq^{-\frac{1}{\gamma}}}{v_n}\right),$$

"Pr" represents probabilities conditional on $X(0) = X$ and $Q(0) = Q$, and where $\Phi(\cdot)$ is the standard normal cumulative distribution function. Since $\ln\left[\frac{X(s)}{X}\right]$ is a Brownian motion with drift parameter $(\mu - 1/2\sigma^2)$ and variance parameter σ^2, we can use the distribution function provided in Harrison (1985, Chapter 1, Equation 9.4) to calculate the result on the fourth line of the derivation.

Similarly, a closed-form solution for the conditional distribution of the future path of prices is obtainable. In this case, the equilibrium output price, $P(t) = X(t) \cdot Q^{-\frac{1}{\gamma}}$, takes on a particularly simple form. Since $Q(t)$ increases incrementally only when $X(t) = v_n \cdot Q^{\frac{1}{\gamma}}$, $P(t)$ will follow geometric Brownian motion with an upper reflecting barrier at v_n. That is, whenever $P(t) < v_n$, $dP(t) = dX(t)$, but whenever $P(t)$ rises to v_n, $Q(t)$ increases just enough so as to reflect $P(t)$ off the barrier v_n. Note that for more general demand specifications, $P(t)$ will have a reflecting barrier that itself is a function of $Q(t)$. However, for demand specifications in which $X(t)$ appears as a multiplicative shock (as in the present example), the reflecting barrier will be a constant.[16] The distribution function for the equilibrium output price at time t, conditional on $P(0) = P$, can be written as:

$$\begin{aligned} L_t(p;P) &\equiv \Pr\left[P(t) \le p\right] \qquad (26) \\ &= \Pr\left[\ln\left[\frac{v_n}{P(t)}\right] \ge \ln\left[\frac{v_n}{p}\right]\right] \\ &= \Phi\left[\frac{\ln(p/P) - (\mu - \sigma^2/2)\cdot t}{\sigma\sqrt{t}}\right] \\ &\quad + (p/v_n)^{\frac{2(\mu-\sigma^2/2)}{\sigma^2}} \cdot \Phi\left[\frac{\ln(p/v_n) + \ln(P/v_n) + (\mu - \sigma^2/2)\cdot t}{\sigma\sqrt{t}}\right], \end{aligned}$$

for $0 \le p \le v_n$, where "Pr" represents probabilities conditional on $P(0) = P$. Note that $\ln\left[\frac{v_n}{P(t)}\right]$ is a Brownian motion with a lower reflecting barrier at zero, a drift parameter $(\mu - 1/2\sigma^2)$ and a variance parameter σ^2. Thus, we can use the distribution function provided in Harrison (1985, Chapter 3, Equation 6.1) to calculate the result on the third line of the derivation.

Provided $\mu > \sigma^2/2$, $P(t)$ has a long-run stationary distribution.[17] That is,

$$\lim_{t\to\infty} L_t(p;P) = (p/v_n)^{\frac{2(\mu-\sigma^2/2)}{\sigma^2}}, \text{ for } \mu > \sigma^2/2. \qquad (27)$$

By solving differential equation (15) subject to boundary condition (16), the equilibrium value of each firm, $F(X,Q)$, can be expressed as:

$$F(X,Q) = C_1(Q)\cdot X^\beta + \frac{X}{n(r-\mu)} Q^{\frac{\gamma-1}{\gamma}}, \qquad (28)$$

where:

$$C_1(Q) = \left(\frac{v_n^{-\beta}}{n}\right)\left(\frac{\gamma}{\gamma-\beta}\right)\left[K - \left(\frac{v_n}{r-\mu}\right)\left(\frac{\gamma-1}{\gamma}\right)\right]\cdot Q^{\frac{\gamma-\beta}{\gamma}}. \qquad (29)$$

[16] In a more general setting, such as Grenadier (1995) with the investment cost, K, following a geometric Brownian motion, it is the ratio $\frac{P(t)}{K(t)}$ that follows a reflected geometric Brownian motion.

[17] As in the previous footnote, with K following a geometric Brownian motion process, it will be the distribution of $\frac{P(t)}{K(t)}$ that will have a stationary distribution.

Intuition for this solution is easily obtained. The second term on the right-hand side of (28) is equal to the present value of expected future stream of profits from selling Q/n units of output forever, while other firms produce $\frac{(n-1)}{n}Q$ units of output forever. The first term, therefore, represents the impact of future increases in output on a firm's future cash flow stream. The net impact of future increases in supply on a firm's value is negative.[18]

By solving differential equation (17) subject to boundary condition (18), the equilibrium value of a unit of investment, $G(X,Q)$, can be expressed as:

$$G(X,Q) = -\frac{v^{1-\beta}}{\beta \cdot (r-\mu)} \cdot \left(XQ^{-\frac{1}{\gamma}}\right)^{\beta} + \frac{XQ^{-\frac{1}{\gamma}}}{(r-\mu)}. \tag{30}$$

The intuition is similar to that for the value of the firm. The second term on the right-hand side of (30) is equal to the present value of expected future stream of profits from selling one unit of output forever, assuming that Q is fixed. The first term, therefore, represents the impact of future changes in the industry supply, Q, on the asset's value. Clearly, the impact of future supply increases results in a negative impact on the future cash flow stream from the asset.

4 Competition and the Erosion of the Investment Option Value

The most often cited result from the literature on real options is the invalidation of the simple net present value rule that one should invest as long as the value of the investment is greater than or equal to the cost of investment. Because the future value of the asset is uncertain, there is an opportunity cost to investing today. This is often referred to as the "option to wait." Thus, the optimal investment rule, as described in the real options literature, is to invest when the asset value exceeds the investment cost by a potentially large option premium. As stated in Dixit and Pindyck (1994), "Hence the simple NPV rule is not just wrong; it is often very wrong." This principle is emphasized in Pindyck (1988),

> "This aspect of investment has been explored in an emerging literature, and most notably by Robert McDonald and Daniel Siegel (1986). They

[18] In order to ensure convergence, we assume that $\gamma < \beta$. To see why this is necessary, recall that the term $C_1(Q) \cdot X^{\beta}$ represents the impact of all future increases in output on the firm's cash flows. We can decompose this into increments by noting that $-C_1'(Q) \cdot X^{\beta} dQ$ represents the impact of industry output increasing from Q to $Q + dQ$. Thus, the present value of the impact of all future supply increases can be written as the integral $\int_Q^{\infty} -C_1'(\omega) \cdot X^{\beta} d\omega$. From (29), this can be written as $-\left(\frac{v_n^{-\beta}}{n}\right)\left[K - \left(\frac{v_n}{r-\mu}\right)\left(\frac{\gamma-1}{\gamma}\right)\right] X^{\beta} \int_Q^{\infty} \omega^{-\frac{\beta}{\gamma}} d\omega$. If the condition $\gamma < \beta$ did not hold, than the impact of future supply increases would translate into an infinitely negative value, making the existence of the industry unviable.

> show that with even moderate levels of uncertainty, the value of this opportunity cost (the cost of extinguishing the option to wait) can be large, and investment rules that ignore it will be grossly in error. Their calculations, and those in related papers by Michael Brennan and Eduardo Schwartz (1985) and Saman Majd and Robert Pindyck (1987), show that in many cases projects should be undertaken only when their present value is at least double their direct cost."

Although this phenomenon may be consistent with what firms would *want* to do, it is not consistent with what firms *can* do when facing competitive pressure. In this model of strategic equilibrium, a firm that waits too long faces the consequences of competitors preempting them. We shall see that in a world in which firms face competitive pressure, the standard net present value rule reemerges and becomes increasingly descriptive of industries with more firms. Firms can no longer realize the full, unhindered option premium of waiting to invest, but must exercise earlier or face the prospect of losing out on an investment opportunity. In the limit of a perfectly competitive industry (as $n \to \infty$), individual optimizing behavior of competitive firms leads to entry precisely at a point of zero net present value. In this sense, the urgency of potential entrants results in an option to wait with zero value; the fear of competitors usurping one's investment opportunity squeezes out all of the potential value from delay.

We can investigate the impact of competition on the investment option value by using an explicit expression for the option premium inherent in equilibrium investment, using the base-case example of the previous section. Recall that the function $G(X, Q)$ represents the equilibrium value of a unit of investment. At the moment of investment (exercise), the net present value of investment equals $G\left[X^*(Q), Q\right] - K$. By dividing the equilibrium net present value of investment by the cost of exercise, we obtain an expression for the equilibrium investment option premium. Denote this option premium, expressed as a function of the number of competitors, by $OP(n)$. $OP(n)$ can be written as:

$$OP(n) \equiv \left(G[X^*(Q), Q] - K\right)/K. \tag{31}$$

For the base-case model, a closed-form solution is obtained. By substituting the solution for $X^*(Q)$ in Equation (21) into the function $G(X, Q)$ in Equation (30), we find a very simple expression for the option premium:

$$OP(n) = \frac{1}{n\gamma - 1}. \tag{32}$$

The option premium displays the following properties:

$$\lim_{n \to \infty} OP(n) \quad = \quad 0, \tag{33}$$

$$\begin{aligned} OP(1) &= \frac{1}{\gamma - 1} > 0, \\ OP'(n) &= -\frac{\gamma}{(n\gamma - 1)^2} < 0, \\ OP''(n) &= \frac{2\gamma^2}{(n\gamma - 1)^3} > 0. \end{aligned}$$

The well-known option premium from the real options literature is clearly a function of the amount of competition. For larger n, the option premium can be very small. Consider the option premium for the parameter value $\gamma = 1.5$. For a monopolist, equilibrium investment occurs only when the net present value of the investment is equal to 200% of the cost of investment. This is the type of large option premium that typically appears in real options models. Importantly, however, note that with even a small amount of competition, the option premium is drastically reduced. With just two firms, the option premium falls to 50%. With five firms, the option premium falls to 15%. The option premium converges to 0%, leaving the traditional net present value rule intact.

While this solution for the equilibrium option premium was derived for particular specifications of X and $D(X, Q)$, the general properties continue to hold under alternative specifications. That is, the option premium is declining in the degree of competition, and converges to 0. Under alternative specification, however, the solutions are more complex and will generally depend on other parameters, most notably σ.

Another important and related implication of the model is the impact of competition on the potential for future investment losses. In the traditional real options models of investment, firms invest with a large option premium over the cost of investment, resulting in a substantial cushion against future market downturns. If a firm is able to invest when the net present value is equal to a large multiple over the cost of investment, then it will be very unlikely that the investment value will ever fall below its cost. In this sense, the standard real option models predict that it should be a very rare event when real asset values fall below investment cost.

Clearly, this prediction is vastly at odds with the empirical evidence. Real asset values can fall substantially below their investment costs. In commercial real estate markets during the late 1980's and early 1990's, many properties were valued at only pennies on the dollar of initial investment. Similarly, there was a significant amount of foreclosure activity during this period, where the value of the properties fell below the loan amounts used for construction of the property. Obviously, the activities of the RTC in auctioning off vast amounts of foreclosed commercial properties evidences this phenomenon.

As was demonstrated, when competitive pressure is introduced into the real options framework, the cushion provided by the investment option premium falls dramatically. We shall find that competitive pressure greatly increases the likelihood of investment values falling below their initial cost.

To characterize the impact of competition on ex-post investment values, consider the following measure of the likelihood of ex-post investment losses. Suppose a firm invests in an asset at time zero. Then, what is the probability that, at some time over the next T years, the asset value falls below the investment cost? Mathematically, denote this probability by Ω_T, where:

$$\Omega_T = \Pr\left\{\inf\left[G(X(s), Q(s)) : 0 \leq s \leq T\right] < K \left| X(0) = X^*[Q(0)]\right.\right\}. \qquad (34)$$

Figure 2 provides simulation results for the likelihood of ex-post investment losses occurring at some time over the periods of 1 and 5 years following exercise, for varying levels of competition, again, using the base-case model.[19] In the standard real options case with only one firm, there is essentially a zero probability of incurring investment losses over either 1 or 5 years following exercise. This is due to the large option premium at the time of exercise. With five firms in the industry, there is still virtually no chance of an investment loss over the next 1 year, but the likelihood of an investment loss occurring over the next 5 years is 37%. With ten firms in the industry, the chance of an investment loss occurring over the next 1 year is 10%, and the likelihood of an investment loss occurring over the next 5 years is 75%. When there are 50 firms in the industry, the likelihood of future losses is very high. With 50 firms, the likelihood of an investment loss occurring over the following year is over 75%, and is virtually certain at some time over the next 5 years.

5 The Strategic Equilibrium as an Artificial Competitive Equilibrium

In this section I demonstrate that the equilibrium investment strategies and output price process in an oligopolistic industry are identical to those obtained in a transformed perfectly competitive equilibrium. That is, the oligopolistic equilibrium output prices and policies under a demand curve $D(X, Q)$ can be obtained under a competitive equilibrium with transformed demand curve:

$$\hat{D}(X, Q) \equiv D(X, Q) + \frac{Q}{n} D_Q(X, Q). \qquad (35)$$

In addition, as in Slade (1994), the structure of the present model permits the oligopolistic equilibrium outcome to be derived by a single agent maximizing a "fictitious" objective function.

[19] The simulation results are obtained through Monte Carlo simulation. The process for $X(t)$ is discretized as:

$$X_j = X_{j-1} + \mu \cdot X_{j-1} \cdot \Delta t + \sigma \cdot X_{j-1} \cdot \sqrt{\Delta t}\, \varepsilon_j,$$

where $\Delta t = 0.0025$ and ε_j is a random sample from a standard normal distribution. $1,000$ simulation runs were conducted.

This equivalence can prove quite useful. Using this analogy between the game-theoretic equilibrium under an oligopoly and a competitive equilibrium, one can choose to solve the oligopolistic problem using the established results from models of competitive investment under uncertainty. The competitive framework, in such a stochastic environment, has been carefully analyzed by Leahy (1993) and Dixit (1989b, 1991).

To demonstrate this equivalence result, I derive the competitive equilibrium under the transformed demand curve $\hat{D}$. A rational expectations competitive equilibrium can be determined as the solution to a specific maximization problem. As in Lucas and Prescott (1971) or Dixit (1989, 1991), the equilibrium evolves *as if* it were set to maximize the present discounted value of social welfare in the form of consumer surplus. That is, the equilibrium path of prices and quantities can be derived from the social planner's perspective.

The social planner's problem is to choose the path of new supply so as to maximize the value of the flow of consumer surplus. The total flow rate of social surplus, $\hat{S}[X(t), Q(t)]$, is equal to the area under the artificial demand curve:

$$\hat{S}[X(t), Q(t)] = \int_0^{Q(t)} \hat{D}[X(t), q]\, dq. \tag{36}$$

The social planner's problem is to choose the path $Q(t)$ so as to solve the following optimal control problem:

$$J(X, Q) = \sup_{\{Q(s):s>0\}} E\left\{\int_0^{\infty} e^{-rt}\hat{S}[X(t), Q(t)]\, dt - \int_0^{\infty} e^{-rt} K dQ(t)\right\}, \tag{37}$$

where $dQ(t)$ denotes the infinitesimal increase in industry capacity at instants t when investment is made. The solution to the planner's problem is denoted by the Bellman value function, $J(X, Q)$.

The planner's dynamic programming problem can be solved using the methods of option pricing. The equilibrium value, $J(X, Q)$, and the planner's optimal investment trigger function, $X^c(Q)$, must simultaneously solve the following differential equation and boundary condition:

$$0 = \frac{1}{2}\sigma(X)^2 J_{XX} + \mu(X) J_X - r \cdot J + \hat{S}(X, Q), \tag{38}$$

subject to:

$$\begin{aligned} \frac{\partial J}{\partial Q}[X^c(Q), Q] &= K, \qquad (39) \\ \frac{\partial^2 J}{\partial Q \partial X}[X^c(Q), Q] &= 0, \end{aligned}$$

where the two boundary conditions represent the value-matching and smooth-pasting conditions, respectively.

Let $j(X,Q) = \frac{\partial J}{\partial Q}$. Equation (38) holds identically in Q. Differentiating (38) with respect to Q, we obtain the following differential equation and boundary conditions:

$$0 = \frac{1}{2}\sigma(X)^2 j_{XX} + \mu(X) j_X - r \cdot j + D(X,Q) + \frac{Q}{n} D_Q(X,Q), \tag{40}$$

subject to:

$$\begin{aligned} j\left[X^c(Q), Q\right] &= K, \\ \frac{\partial j}{\partial X}\left[X^c(Q), Q\right] &= 0. \end{aligned} \tag{41}$$

Note that differential equation (40) and its associated boundary conditions (41) are identical to that of the system in Proposition 3 used to derive the trigger $X^*(Q)$. Therefore, $X^c(Q) = X^*(Q)$, and $j(X,Q) = m(X,Q)$. We have thus demonstrated that the competitive equilibrium investment strategies under the artificial demand curve, $\hat{D}(X,Q)$, are identical to the strategic equilibrium exercise strategies under the original demand curve, $D(X,Q)$.

Another useful implication of this result is that the oligopolistic equilibrium outcome can be derived by a single agent maximizing a "fictitious" objective function. In this case, the "fictitious" objective function is to maximize the flow of $\hat{S}(X,Q)$, by exerting control on the process $Q(t)$. This is an interesting result. While we know that a monopolist maximizes an objective function equal to the present value of future cash flows, and a competitive equilibrium can be derived through a social planner that optimizes the present value of future social welfare, it is also true that such an objective function exists for an oligopoly in the present framework. A number of articles have addressed this problem, including Spence (1976a, 1976b) in the context of monopolistic competition, Bergstrom and Varian (1985) in a Cournot-Nash market, and Slade (1994) who provides necessary and sufficient conditions for a Nash equilibrium to be observationally equivalent to a single optimization problem.

6 Equilibrium Exercise Strategies with Time-to-Build

Time-to-build is an important characteristic of many real investment markets. Examples of investments with significant construction lags abound. In commercial real estate development, the construction of large office towers or super-regional shopping malls can often take more than two years to complete (Wheaton, 1987). In mining and natural resource extraction, the construction of an underground mine can require at least five or six years (Majd and Pindyck, 1987). For these markets with time-to-build, investors must optimally take into account in their investment decisions the potential market fluctuations that may ensue over the intervening construction period. Real options models with time-to-build include Majd and Pindyck (1987) and

Bar-Ilan and Strange (1996). The inclusion of a time-to-build feature in real options models has thus far been confined to models of individual firm (rather than industry equilibrium) investment strategies, with the exception of Grenadier (2000a) that considers a perfectly competitive industry equilibrium with time-to-build.

The difficulty of deriving an equilibrium with time-to-build is that the state space of the industry can be of infinite dimension. Unless the model can be converted to one of finite dimensionality, the standard solution techniques that require a Markov state space cannot be employed. Consider the relevant variables that are needed to describe the current state of the industry. Suppose assets take δ years to build. At any time t, the current state of demand and the number of completed units on the market determine the current level of output prices. However, agents must also condition their information on the number of units currently in the pipeline as well as the times at which these units will be completed. The amount and arrival times of units currently in the pipeline are relevant because they will impact the prices over the ensuing δ years of construction. However, to know the arrival times of units under construction, agents need knowledge of the historical path of entry over the previous δ years.

To solve for the equilibrium in a strategic equilibrium setting, I rely on the results obtained by Grenadier (2000a) in a perfectly competitive equilibrium setting. Given the competitive equilibrium solution, I can then use the transformation result in Section 5 that forms an equivalence between the oligopolistic equilibrium and a competitive equilibrium under a modified demand curve. Grenadier (2000a) demonstrates that the competitive equilibrium exercise strategies (with time-to-build) are identical to those for an industry in which all units currently in the pipeline are assumed to be completed. This reduction in the state-space allows the equilibrium exercise strategies to be simple functions of "committed capacity", which equals all units currently under construction or completed. Thus, as established in Section 5, the resulting competitive equilibrium strategies are equivalent to the myopic strategies derived in Proposition 3, where in this case the equilibrium trigger strategies are functions of committed capacity.

Let $C(t)$ denote the amount of committed capacity, and $Q(t)$ the amount of completed capacity (as in the model). Since investment takes δ years to complete, the completed supply in δ years, $Q(t+\delta)$, is equal to $C(t)$. Consider a perfectly competitive industry with an "artificial" inverse demand curve of the form:

$$\hat{D}[X(t), Q(t)] = X(t) \cdot H[Q(t)], \tag{42}$$

with $H^{'} < 0$, and with $X(t)$ following a geometric Brownian motion as in (20). Then, Grenadier (2000a) proves that the competitive equilibrium exercise strategy is a trigger function of committed capacity, $C(t)$, such that investment occurs whenever $X(t)$ rises to the trigger $X^c(C)$, with:

$$X^c(C) = \frac{\beta}{\beta - 1}(r - \mu)K\frac{e^{(r-\mu)\delta}}{H(C)}. \tag{43}$$

To solve for an oligopolistic equilibrium with $D(X, Q) = X \cdot Q^{-\frac{1}{\gamma}}$, we simply need to let $H(Q) = \frac{n\gamma-1}{n\gamma} Q^{-\frac{1}{\gamma}}$. In that case, $\hat{D}(X, Q) = D(X, Q) + \frac{Q}{n} D_Q(X, Q)$, and the results of Section 5 are satisfied. Therefore, the equilibrium exercise strategy for the oligopolistic industry, with time-to-build, is:

$$X^*(C) = v_n e^{(r-\mu)\delta} \cdot C^{\frac{1}{\gamma}}. \tag{44}$$

There is a very simple interpretation of this result. With no time-to-build, the strategic equilibrium solution is to exercise the first moment that $P(t)$ rises to the trigger v_n. Now, with a time-to-build of δ years, the equilibrium exercise strategy is to exercise when the expected value of the price in δ years equals the "future value" of the trigger, $e^{r\delta} v_n$. To see this, the price in δ years is $P(t+\delta) = X(t+\delta) \cdot Q(t+\delta)^{-\frac{1}{\gamma}}$. Taking the expectation at time t, and noting that $C(t) = Q(t+\delta)$, the expected price, $E_t[P(t+\delta)]$, can be written as $X(t) \cdot e^{\mu\delta} \cdot C(t)^{-\frac{1}{\gamma}}$. Thus, the equilibrium trigger in (44) is equivalent to exercising the first moment that $E_t[P(t+\delta)]$ equals $e^{r\delta} v_n$.

It is important to emphasize the "path dependence" of the equilibrium with time-to-build. Two industries, with the exact same current shock term, X, and supply, Q, will have different exercise paths over the next δ years depending on their path of exercise over the previous δ years. Thus, in order to forecast the path of exercise over the future, it is not enough to know just the current supply and demand, but also the precise path of exercise over the previous δ years.

7 Equilibrium Results for Alternative Specifications

In this section I further illustrate the equilibrium solution approach by solving several examples. In each case, I solve for the equilibrium exercise trigger, $X^*(Q)$, by solving the differential equation (13) subject to boundary conditions (14). Solutions for the equilibrium value of the firm, $F(X, Q)$, and asset value, $G(X, Q)$, are also obtainable, but are not presented in this section.

In the examples, I consider three types of stochastic processes for the state variable $X(t)$, and two demand curves. The processes and demand curves that are selected are among the most common in the real options literature.

Stochastic processes for $X(t)$

1) *Geometric Brownian Motion (log-normal):* $dX = \mu X dt + \sigma X dz$

We assume that $r > \mu$ in order to ensure convergence (see footnote 4).

2) *Arithmetic Brownian Motion (normal):* $dX = \mu dt + \sigma dz$

3) *Square-Root:* $dX = \kappa(\theta - X)dt + \sigma\sqrt{X} dz$, $\kappa, \theta > 0$

This process permits mean reversion in $X(t)$, with θ representing the long-run mean and k representing the speed of adjustment.

Inverse Demand Functions $D(X, Q)$

1) *Constant Elasticity:* $D(X,Q) = X \cdot Q^{-\frac{1}{\gamma}}$

We assume that $\gamma > 1/n$ to ensure that marginal profits are increasing in X.

2) *Linear:* $D(X,Q) = aX - bQ$

We assume that $b > 0$ to ensure that $D_Q < 0$, and that $a > 0$ to ensure that marginal profits are increasing in X.

I now combine these processes and demand functions into six examples.[20]

Example 1: $X(t)$ follows a Geometric Brownian Motion and $D(X,Q)$ is constant elastic.

This example was already solved in Section 3. The equilibrium investment trigger, $X^*(Q)$, is presented in Equation (21).

Example 2: $X(t)$ follows a Geometric Brownian Motion and $D(X,Q)$ is linear.

This is the specific setting under which Baldursson (1997) derives his oligopolistic solution. The results are identical. The equilibrium investment trigger, $X^*(Q)$, can be expressed as:

$$X^*(Q) = \left(\frac{\beta}{\beta - 1}\right)\left(\frac{r-\mu}{a}\right)\left[K + \left(\frac{n+1}{n}\right) \cdot \frac{bQ}{r}\right]. \tag{45}$$

The equilibrium trigger is an increasing, affine function of Q, and is decreasing in n.

Example 3: $X(t)$ follows an Arithmetic Brownian Motion and $D(X,Q)$ is constant elastic.

The equilibrium investment trigger, $X^*(Q)$, can be expressed as:

$$X^*(Q) = -\left(\frac{r+\lambda}{r\lambda}\right) + r \cdot \left(\frac{n\gamma}{n\gamma - 1}\right) Q^{\frac{1}{\gamma}}. \tag{46}$$

The equilibrium trigger is an increasing, affine function of $Q^{\frac{1}{\gamma}}$, and is decreasing in n.

Example 4: $X(t)$ follows an Arithmetic Brownian Motion and $D(X,Q)$ is linear.

The equilibrium investment trigger, $X^*(Q)$, can be expressed as:

$$X^*(Q) = \frac{r}{a}\left[K - \frac{a\mu}{r^2} - \frac{a}{r\lambda} + \frac{n+1}{n} bQ\right], \tag{47}$$

where $\lambda = \left(-\mu + \sqrt{\mu^2 + 2r\sigma^2}\right)/\sigma^2 > 0$. The equilibrium trigger is an increasing, affine function of Q, and decreasing in n.

[20] The solutions to the partial differential equations require the imposition of a boundary condition at the lower limit of the space for the process of $X(t)$. For the Geometric Brownian Motion, zero is an absorbing barrier, and thus as $X \to 0$, the value of the option to expand becomes worthless. For the Arithmetic Brownian Motion, the lower limit for $X(t)$ is unbounded. As $X \to -\infty$, the value of the option to expand becomes sufficiently out of the money so that it becomes worthless. For square-root process, the appropriate boundary condition is that the derivative of the option value is finite at the lower boundary of zero.

Example 5: $X(t)$ follows a square-root process and $D(X,Q)$ is constant elastic.

While a closed-form solution does not exist, the equilibrium investment trigger, $X^*(Q)$, can be expressed as the solution to the following equation:

$$\frac{H\left[\frac{r}{\kappa}, \frac{2\kappa\theta}{\sigma^2}, \frac{2\kappa X^*(Q)}{\sigma^2}\right]}{H\left[1+\frac{r}{\kappa}, 1+\frac{2\kappa\theta}{\sigma^2}, \frac{2\kappa X^*(Q)}{\sigma^2}\right]} = \frac{r}{\kappa\theta}\left[X^*(Q) + \frac{\kappa\theta}{r} - K(r+\kappa)\frac{n\gamma}{n\gamma-1}Q^{\frac{1}{\gamma}}\right], \tag{48}$$

where the function $H(a,b,z)$ is the confluent hypergeometric function, which has the following series representation:

$$H(a,b,z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{z^3}{3!} + ... \tag{49}$$

Example 6: $X(t)$ follows a square-root process and $D(X,Q)$ is linear.

While a closed-form solution does not exist, the equilibrium investment trigger, $X^*(Q)$, can be expressed as the solution to the following equation:

$$\frac{H\left[\frac{r}{\kappa}, \frac{2\kappa\theta}{\sigma^2}, \frac{2\kappa X^*(Q)}{\sigma^2}\right]}{H\left[1+\frac{r}{\kappa}, 1+\frac{2\kappa\theta}{\sigma^2}, \frac{2\kappa X^*(Q)}{\sigma^2}\right]} = \frac{r}{\kappa\theta}\left[X^*(Q) + \frac{\kappa\theta}{r} - \frac{(r+\kappa)}{a}\left(K + \frac{n+1}{n}\frac{bQ}{r}\right)\right]. \tag{50}$$

8 Conclusion

The standard real options framework has proven to be quite useful in the analysis of investment under uncertainty. By using the analogy between real and financial options, the tools of option pricing theory can be applied to real investment analysis. However, the standard framework of formulating exercise strategies in isolation is not particularly relevant to many real-world applications. While one investor is (maybe only heuristically) solving a partial differential equation and its associated smooth-pasting condition, so too are its competitors. The true optimal exercise strategy cannot be solved individually, but must be part of a game-theoretic equilibrium.

This article provides a very general and tractable solution approach for deriving the equilibrium investment strategies of firms in a Cournot-Nash framework. The resulting equilibrium is found to be analytically simple, and has potentially wide applications. For example, a real estate developer contemplating the construction of an office building will choose his optimal time at which to begin construction, contingent upon his beliefs about the decisions of other developers in the market. Similarly, a firm considering entry or exit from an industry will choose its optimal strategy based on the analogous strategies it assumes its competitors will follow. While the derivation of equilibrium exercise strategies in a continuous-time stochastic setting can be quite difficult, the present model permits a rather simple solution approach.

The generalization of the real option approach to include strategic equilibrium exercise strategies provides very different implications from the standard individualistic setting. For example, perhaps the most well-known result of the real options literature is the invalidation of the standard net present value rule of investing in any project with a non-negative net present value. The optimal investment rule, as described in the real options literature, is to invest when the asset value exceeds the investment cost by a potentially large option premium. However, the inclusion of competitive access to the investment opportunity leads to a rapid erosion in the option to wait, and makes the standard net present value rule a much more accurate description of the actual investment threshold. Similarly, markets in which investment is initiated with a small option premium are more likely to experience subsequent periods in which the value of the investment turns out to be less than the original cost. While such an outcome is highly unlikely in the standard real options setting, the increased likelihood of ex-post investment losses in a strategic equilibrium more accurate reflects observed market behavior.

A useful result of the model is that the Nash equilibrium exercise strategies are identical to those obtained in an "artificial" perfectly competitive equilibrium, with a slightly modified demand function. Thus, one can borrow from the results of models of perfectly competitive investment under uncertainty, in order to derive results for the seemingly more complex case of oligopolistic settings. For example, the model of Dixit (1989b) analyzes the entry and exit strategies of domestic and foreign firms in a competitive industry with stochastic exchange rates. Similarly, Dixit (1991) analyzes investment behavior in a perfectly competitive market with exogenous price ceilings. A particular example of importing the results from perfectly competitive settings appears in this article where a time-to-build feature is added to the model, using the results of Grenadier (1999b). The addition of other features from models of perfectly competitive settings could follow along similar lines.

Appendix

A.1 Proof of Proposition 2

Proposition 2 *The symmetric Nash equilibrium exercise strategy described in Proposition 1 is characterized by each firm increasing output whenever $X(t)$ rises to the myopic trigger function $X^m(q_{i,}Q_{-i})$. That is, $\bar{X}(q_{i,}Q_{-i}) = X^m(q_{i,}Q_{-i})$.*

Proof. I first demonstrate that the myopic trigger, $X^m(q_{i,}Q_{-i})$, is optimal for a firm considering the exercise of its option to increase output from q_i to $q_i + dq$, when it assumes that all of its competitors will exercise their options at any time after the trigger $X^m(q_{i,}Q_{-i})$ is reached. The myopic strategy, $X^m(q_{i,}Q_{-i})$, represents the optimal exercise trigger for a firm considering increasing investment from q_i to q_i+dq, assuming that Q_{-i} will never change. The incremental payoff from this option exercise (per unit) is the perpetual cash flow $\frac{\partial \pi_i(X(t),q_i,Q_{-i})}{\partial q_i}$. However, it is easy to see that this is also the optimal exercise trigger in an environment in which Q_{-i} will increase by dQ_{-i} at any time after $X^m(q_{i,}Q_{-i})$ is reached. In essence, since there is no chance of preemption prior to the myopic trigger, the myopic strategy is "unconstrained" by the future exercise of its competitors. In this setting, at the time of competitor exercise, firm i will receive the incremental cash flow stream (which will be negative) of $\frac{\partial \pi_i(X(t),q_i,Q_{-i})}{\partial Q_{-i}}$. This incremental cash flow, however, is beyond the control of firm i, and does not change the incremental payoff of its option exercise, and the future change in Q_{-i} can be ignored. Thus, the myopic exercise policy is still optimal.

In section 2 we initially considered the optimal value of firm i, when it takes its competitors' exercise strategies as given. Recall that the function $F^i(X, q_i, Q_{-i}; X^{-i})$ represents the (maximal) value of firm i, conditional on all of its competitors exercising at the trigger function $X^{-i}(q_i, Q_{-i})$. Thus, consider the value of firm i when its competitors all exercise at the trigger $X^m(q_{i,}Q_{-i}) + \varepsilon$, for an infinitesimally small value of $\varepsilon > 0$. $F^i(X, q_i, Q_{-i}; X^m + \varepsilon)$, and its associated optimal trigger $X^i(q_i, Q_{-i})$, satisfy differential equation (7), subject to boundary conditions (8), (9), and (10). Therefore, $F^i(X, q_i, Q_{-i}; X^m + \varepsilon)$ and $X^i(q_i, Q_{-i})$ must solve:

$$0 = \frac{1}{2}\sigma(X)^2 F^i_{XX} + \mu(X)F^i_X - rF^i + \pi_i(X, q_i, Q_{-i}), \tag{51}$$

subject to:

$$\begin{aligned} \frac{\partial F^i}{\partial q_i}\left[X^i(q_{i,}Q_{-i}), q_i, Q_{-i}; X^m + \varepsilon\right] &= K, \\ \frac{\partial^2 F^i}{\partial q_i \partial X}\left[X^i(q_{i,}Q_{-i}), q_i, Q_{-i}; X^m + \varepsilon\right] &= 0, \\ \frac{\partial F^i}{\partial Q_{-i}}\left[X^m(q_{i,}Q_{-i}) + \varepsilon, q_i + dq_i, Q_{-i}; X^m + \varepsilon\right] &= 0. \end{aligned} \tag{52}$$

As was demonstrated above, the optimal exercise strategy for firm i, when there is no fear of preemption before the myopic trigger is reached, is to exercise at $X^m(q_{i,}Q_{-i})$. Thus, we can therefore replace $X^i(q_{i,}Q_{-i})$ with $X^m(q_{i,}Q_{-i})$ in boundary conditions (52). In addition, since the solution to differential equation (51), $F^i(X, q_i, Q_{-i}; X^m + \varepsilon)$, and the function $X^m(q_{i,}Q_{-i})$ are continuous in all arguments, as we let $\varepsilon \to 0$, the final boundary condition approaches $\frac{\partial F^i}{\partial Q_{-i}}\left[X^m(q_{i,}Q_{-i}), q_i, Q_{-i}; X^m\right] = 0$.[21]

Therefore, for $\varepsilon \to 0$, the constrained problem can be written as:

$$0 = \frac{1}{2}\sigma(X)^2 F^i_{XX} + \mu(X) F^i_X - rF^i + \pi_i(X, q_i, Q_{-i}), \tag{53}$$

subject to:

$$\begin{aligned} \frac{\partial F^i}{\partial q_i}\left[X^m(q_{i,}Q_{-i}), q_i, Q_{-i}; X^m\right] &= K, \\ \frac{\partial^2 F^i}{\partial q_i \partial X}\left[X^m(q_{i,}Q_{-i}), q_i, Q_{-i}; X^m\right] &= 0, \\ \frac{\partial F^i}{\partial Q_{-i}}\left[X^m(q_{i,}Q_{-i}), q_i, Q_{-i}; X^m\right] &= 0. \end{aligned} \tag{54}$$

Since $X^m(q_{i,}Q_{-i})$ is the optimal exercise response to competitors pursuing the same strategy, $X^m(q_{i,}Q_{-i}) = \bar{X}(q_{i,}Q_{-i})$, and $F^i(X, q_i, Q_{-i}; X^m) = V^i(X, q_i, Q_{-i})$, as outlined in Proposition 1. ■

A.2 Proof of Proposition 3

Proposition 3 *In a symmetric Nash equilibrium, each firm will exercise its investment option whenever $X(t)$ rises to the trigger $X^*(Q)$. Let $m(X, Q)$ denote the value of a myopic firm's marginal investment, with $m(X, Q) \equiv m^i(X, \frac{1}{n}Q, \frac{n-1}{n}Q)$. $X^*(Q)$ and $m(X, Q)$ are jointly determined by the following differential equation:*

$$0 = \frac{1}{2}\sigma(X)^2 m_{XX} + \mu(X) m_X - r \cdot m + D(X, Q) + \frac{Q}{n} D_Q(X, Q), \tag{55}$$

subject to:

$$\begin{aligned} m\left[X^*(Q), Q\right] &= K, \\ \frac{\partial m}{\partial X}\left[X^*(Q), Q\right] &= 0. \end{aligned} \tag{56}$$

Proof. Equation (11) holds identically in q_i. Differentiating equation (11) with respect to q_i, we obtain the following differential equation and boundary conditions:

[21] Continuity of F^i and X^m can be demonstrated in a manner similar to the Lemma in the appendix of Leahy (1993).

$$0 = \frac{1}{2}\sigma(X)^2 m^i_{XX} + \mu(X) m^i_X - r \cdot m^i + \frac{\partial \pi^i_i(X, q_i, Q_{-i})}{\partial q_i}, \tag{57}$$

subject to:

$$\begin{aligned} m^i \left[X^m(q_{i,}Q_{-i}), q_i, Q_{-i}\right] &= K, \\ \frac{\partial m^i}{\partial X}\left[X^m(q_{i,}Q_{-i}), q_i, Q_{-i}\right] &= 0. \end{aligned} \tag{58}$$

Note that there is no boundary condition for m^i that involves any partial derivatives with respect to q_i or Q_{-i}. Therefore, in a symmetric equilibrium we can write the common marginal option value as a function of just X and Q, by setting $m(X,Q) = m^i(X, Q/n, \frac{n-1}{n}Q)$, and the equilibrium trigger as $X^*(Q) = X^m\left(Q/n, \frac{n-1}{n}Q\right)$. Thus, $m(X,Q)$ and $X^*(Q)$ must satisfy the differential equation and boundary conditions outlined in the proposition 3. ■

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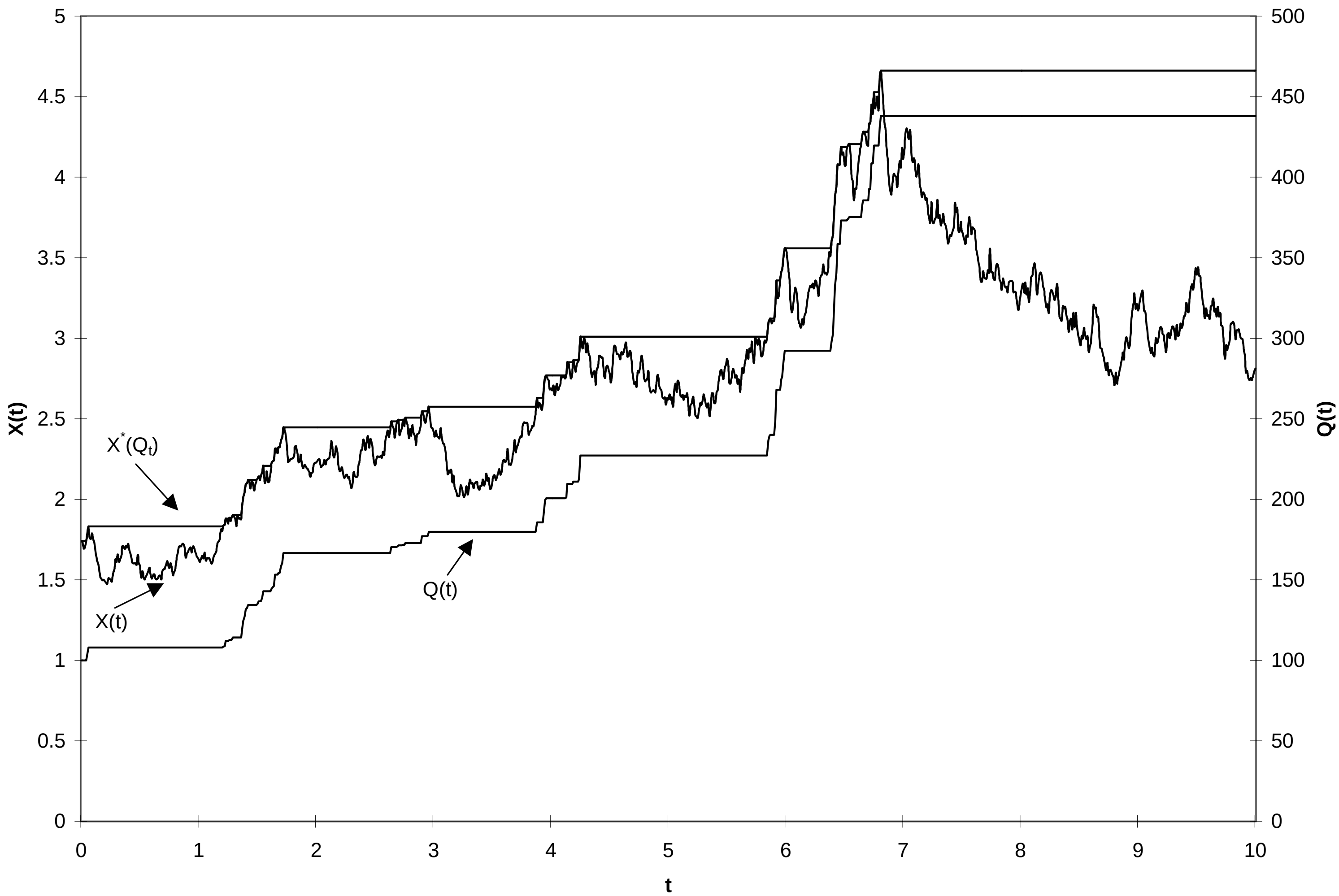

Fig. 1: Simulation of equilibrium investment strategies.

This figure shows the simulated path of investment, *Q(t)*, for a particular realization of the shock process, *X(t)*, for a hypothetical *10*-firm industry. The Nash equilibrium exercise strategy is for each firm to invest whenever *X(t)* reaches the upper trigger function $X^*(Q_t)$. At any such point, industry capacity increases incrementally through new investment. The top curve, appearing like a step-function, is the equilibrium exercise trigger function, $X^*(Q_t)$. The jagged curve below it is the path of the shock term, *X(t)*. The units for both *X(t)* and $X^*(Q_t)$ are displayed on the left axis. The third curve, initially appearing as the bottom curve with a step-function appearance, is the equilibrium path of investment supply, *Q(t)*. The units for *Q(t)* are displayed on the right axis. *Q(t)* increases only at those moments at which *X(t)* hits the trigger function, $X^*(Q_t)$. The default parameter values are $\mu=0.02$, $r=0.05$, $\sigma=0.175$, $\gamma=1.5$, $n=10$, $K=1$, $Q(0)=100$, and $X(0)=1.74$.

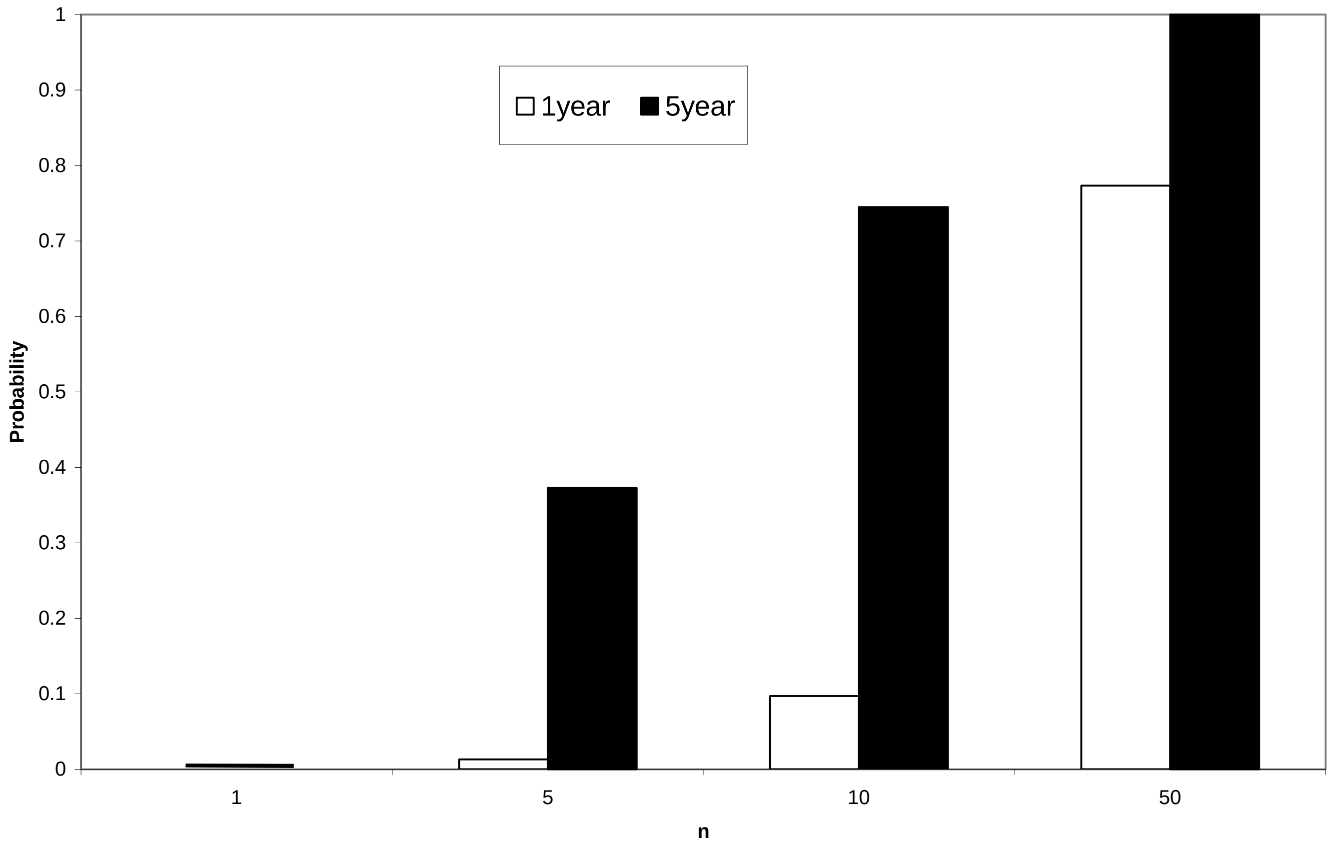

Fig. 2: Competition and the likelihood of ex-post investment losses.

An implication of standard real options models is that since investment is made only at a substantially positive net present value, there is little likelihood that the value of the investment over a future period will ever fall below its initial cost. However, this result is not generally true for markets under competition. Define the function Ω_T as the probability that the value of the investment ever falls below its initial cost over the *T*-year period following exercise. This figure displays simulation results for the likelihood of investment losses over the ensuing *1* and *5* year period, for various levels of *n*, where *n* denotes the number of firms in the industry. When there is only one firm in the industry, there is virtually no chance of an investment loss occurring over the *5*-year period following exercise. However, the likelihood of ex-post investment losses increases dramatically as the number of firms in the industry increases. When there are *50* firms in the industry, there is a 75% chance of an investment loss occurring at some time during the next year, and with virtual certainty over the future *5* years. The default parameter values are $\mu=0.02$, $r=0.05$, $\sigma=0.175$, $\gamma=1.5$, $K=1$, and $Q(0)=100$.