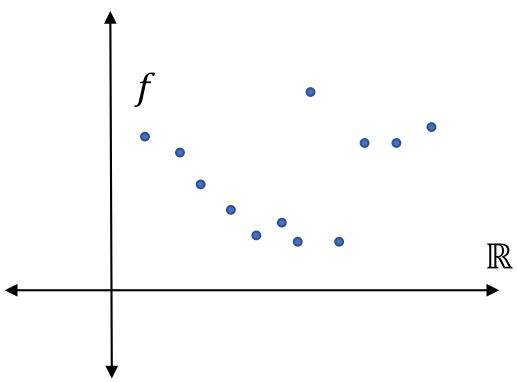
Introduction to Optimization Theory

Lecture #3 - 9/22/20

MS&E 213 / CS 2690



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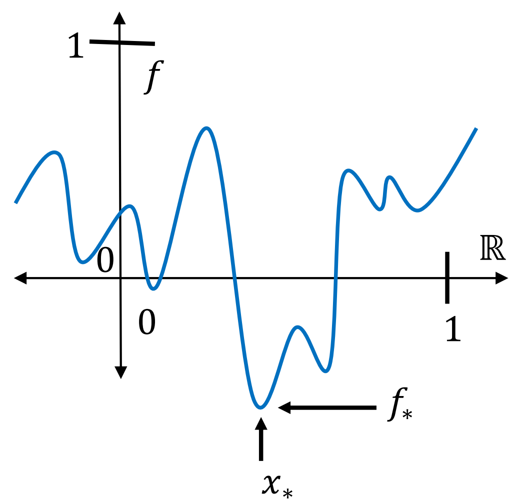


Lecture Plan

Recap

- Oracles, minimization, efficiency, and iterative methods
- Continuity, smoothness, and critical points

Material

- Continuity, ϵ-nets, and lower bounds

Thursday

- Smoothness revisited
- Convexity

Recap

Goal

- Objective function $f: \mathbb{R}^n \to \mathbb{R}$
- Constraint set $S \subseteq \mathbb{R}^n$

  (Next few lectures, unconstrained $S = \mathbb{R}^n$)
- Optimize

$$\min_{x \in S \subseteq \mathbb{R}^n} f(x)$$

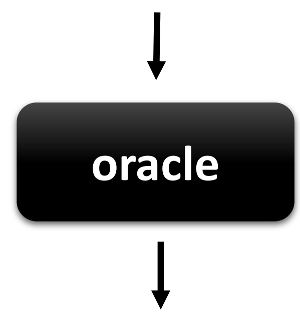
provably efficiently with few assumptions

Access to f?

- Through an "oracle"

query

e.g. $x \in \mathbb{R}^n$



output

e.g. $f(x) \in \mathbb{R}$ [value]
e.g. $\nabla f(x)$ [gradient]

Recap

Goal
$\min_{x \in \mathbb{R}^n} f(x)$ given by an oracle provably efficiently with few assumptions

Minimize? Progress Measure?

ϵ-(sub)optimal point or a point with **ϵ-additive function error**:

- $x \in S$ s.t. $f(x) \leq f_* + \epsilon$ where $f_* = \min_{x \in S} f(x)$

ϵ-critical point:

- $x \in S$ s.t. $\|\nabla f(x)\|_2 \leq \epsilon$ where $\|y\|_2 \stackrel{\text{def}}{=} \sqrt{\sum_{i \in [n]} y_i^2}$

Efficency?

- Oracle complexity = #calls to oracle
- Runtime = # oracle calls $\times$ (average computational cost per call)

Recap

Iterative Method Approach

- Start at initial point x_0
- For $t = 0, \dots, T-1$
  - Query oracle
  - Take “local step” to obtain x_{t+1}
  - Repeat
- Output aggregation of the x_t

e.g.

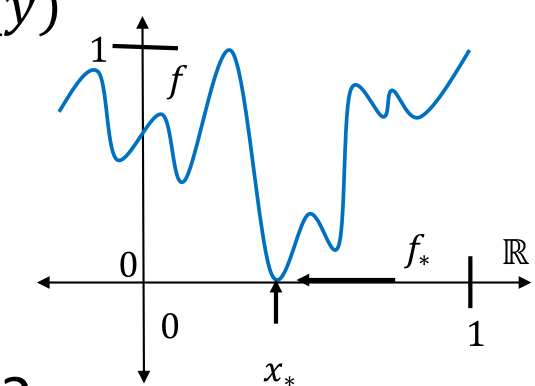
- **Last iterate**: x_{T-1}
- **Average iteration**: $\frac{1}{T}\sum_{k\in[T-1]} x_k$

Analysis

- Oracle complexity = # iterations
- Runtime = # iterations * cost per iteration (iteration complexity)

Recap: setting #0: impossible

- $f : \mathbb{R} \to \mathbb{R}$ *(one dimensional)*
- Have evaluation oracle *(can compute $f(x)$ with 1 query)*
- Promised $\exists x_* \in [0,1]$ such that $f(x) = f_* = \inf_{y \in \mathbb{R}} f(y)$
- Promised $f(x) \in [0,1]$ for all $x \in \mathbb{R}$
- Goal: compute 1/2-optimal point
  - i.e. compute x with $f(x) \leq f(x_*) + 1/2$

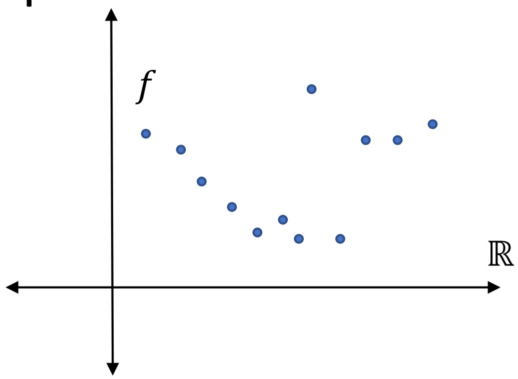


- **Question**: what oracle complexity achievable?
- **Answer**: ∞ is optimal

We will discuss this lower bound more formally today.

Recap

Problem: oracle gives only pointwise information, no local information.



Solution:

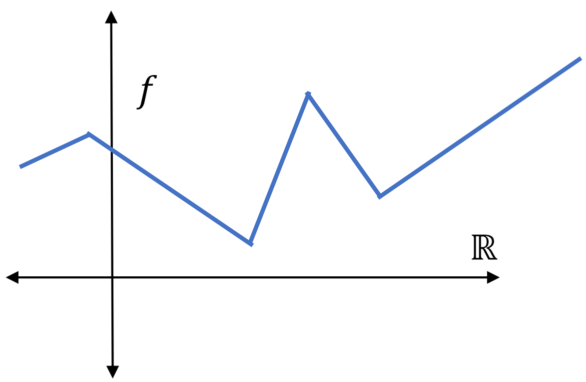
- This is a class on *continuous* optimization
- **Today**: assume more structure and analyze a working method

Last class discussed how continuity is not enough and will prove today.

Recap: assuming more structure

f is L_1-Lipschitz w.r.t. $\|\cdot\|$

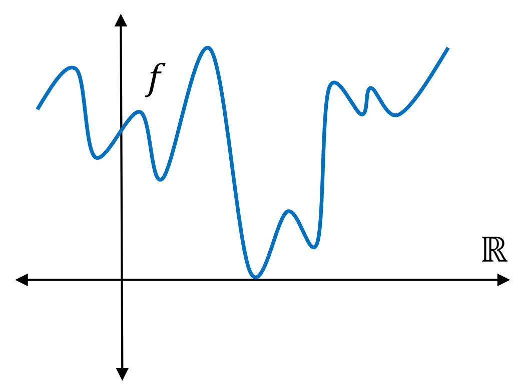
$$|f(x) - f(y)| \leq \|x - y\|$$
for all $x, y \in \mathbb{R}^n$



(bounded slope)
(bounded 1st derivatives)

f is L_2-Lipschitz

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L_2\|x - y\|_2$$
for all $x, y \in \mathbb{R}^n$

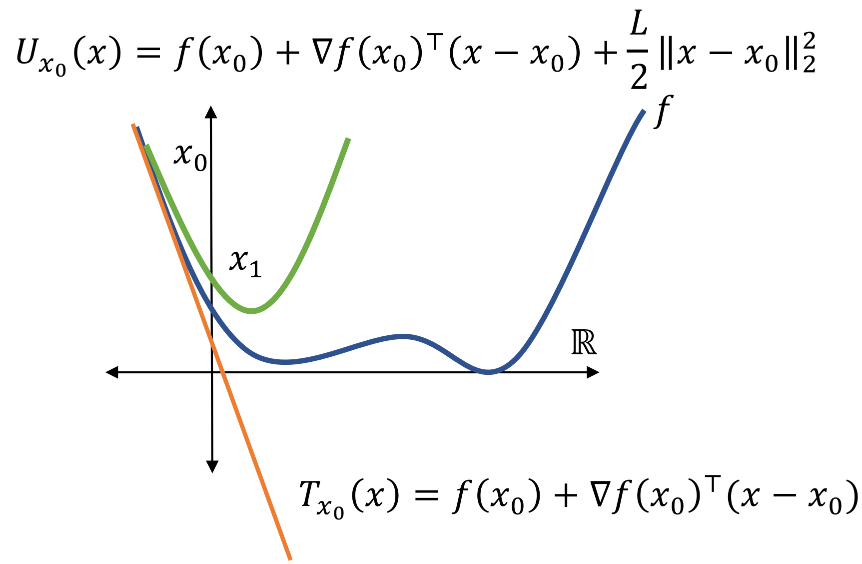


(bounded curvature)
(bounded 2nd derivative)

Recap: Gradient Descent Method for Critical Points

Algorithm / Method (for L-smooth f)

- Initial point: $x_0 \in \mathbb{R}^n$
- For $k = 0,1,2, \dots$
  - $x_{k+1} = x_k - \frac{1}{L} \nabla f(x_k)$
  - If $\|\nabla f(x_k)\|_2 \leq \epsilon$ then output x_k



Theorem

ϵ-critical point in $\leq 2L[f(x_0) - f_*]/\epsilon^2$ steps / queries for $f_* = \inf_{x \in \mathbb{R}^n} f(x)$

Today: ϵ-(sub)optimal points

Lecture Plan

Recap ✓

- Oracles, minimization, efficiency, and iterative methods
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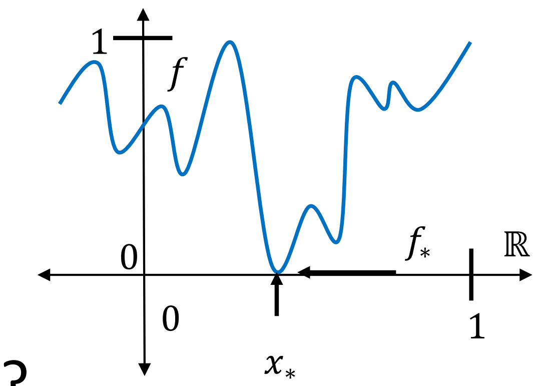
Thursday

- Smoothness revisited
- Convexity

Setting #1: 1d-Lipschitz Function Minimization

- $f: \mathbb{R} \to \mathbb{R}$ *(one dimensional)*
- Have evaluation oracle *(can compute $f(x)$ with 1 query)*
- $\exists x_* \in [0,1]$ such that $f(x) = f_* = \inf_{y \in \mathbb{R}} f(y)$
- $f(x) \in [0,1]$ for all $x \in \mathbb{R}$
- f is L-Lipschitz with respect to ℓ_∞
- **Goal**: compute ϵ-optimal point for $\epsilon \in (0,1)$



- **Question #1**: what oracle complexity achievable?
- **Question #0:** what does L-Lipschitz mean? Imply?

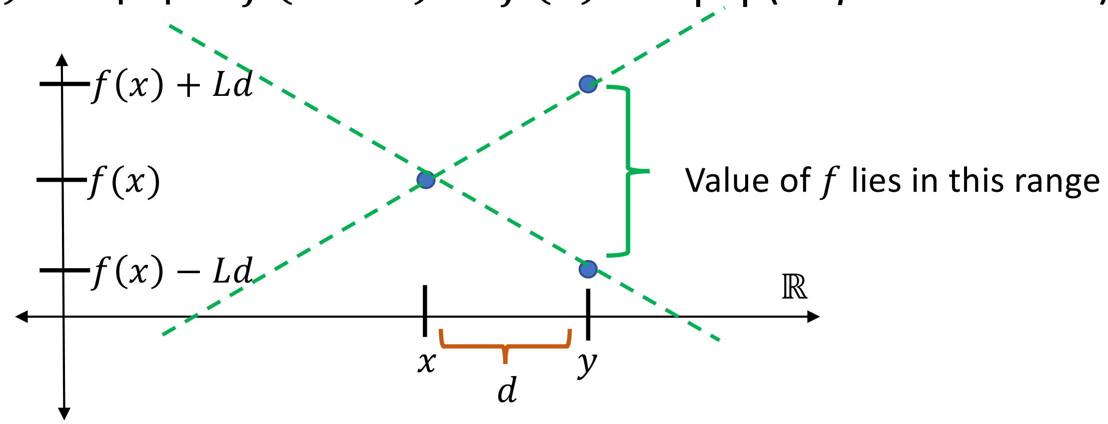
L-Lipschitz Function

f is L-Lipschitz w.r.t. $\|\cdot\|$ if $|f(x) - f(y)| \le L\|x - y\|$ for all $x, y \in \mathbb{R}^n$

- $\Leftrightarrow -L\|x-y\| \le f(y) - f(x) \le L\|x-y\|$ for all $x, y \in \mathbb{R}^n$
- $\Leftrightarrow f(x) - L\|x-y\| \le f(y) \le f(x) + L\|x-y\|$ for all $x, y \in \mathbb{R}^n$

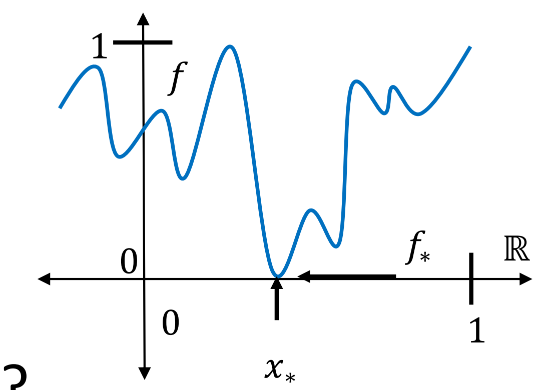
- If $n = 1$ and $\|\cdot\| = \|\cdot\|_p$ *(i.e.* $\|x\| = \|x\|_p = (|x|^p)^{1/p} = |x|$*)* then

  $\Leftrightarrow f(x) - L|d| \le f(x+d) \le f(x) + L|d|$ (*slope at most* L)



Setting #1: 1d-Lipschitz Function Minimization

- $f: \mathbb{R} \to \mathbb{R}$ *(one dimensional)*
- Have evaluation oracle *(can compute $f(x)$ with 1 query)*
- $\exists x_* \in [0,1]$ such that $f(x) = f_* = \inf_{y \in \mathbb{R}} f(y)$
- $f(x) \in [0,1]$ for all $x \in \mathbb{R}$
- f is L-Lipschitz with respect to ℓ_∞
- Goal: compute ϵ-optimal point for $\epsilon \in (0,1)$



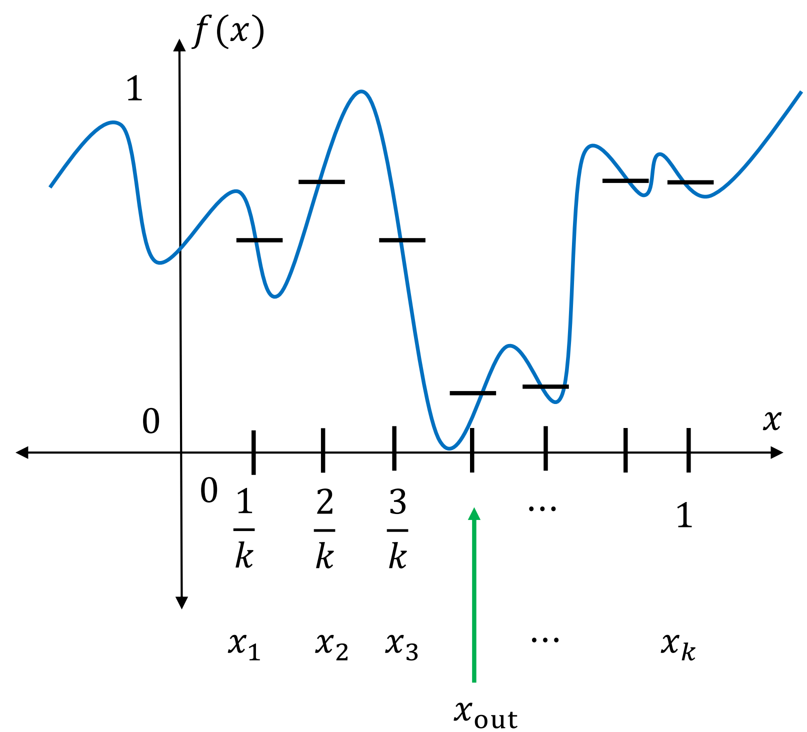
- **Question #1**: what oracle complexity achievable?
- ~~**Question #0:** what does L-Lipschitz mean? Imply?~~

Setting #1:

Theorem: there is method with query complexity $\lceil L/\epsilon \rceil$ for setting #1

Algorithm

- Pick $k \in \mathbb{Z}_{\geq 0}$
- For $i \in [k] = \{1, .., k\}$
  - Let $x_i = \frac{i}{k}$
  - Query $f(x_i)$ for all $i \in [k]$
- Return $x_{\text{out}} = \operatorname{argmin}_{x_i} f(x_i)$



Setting #1:

Theorem: there is method with query complexity $\lceil L/\epsilon \rceil$ for setting #1

Algorithm

- Pick $k \in \mathbb{Z}_{\geq 0}$
- For $i \in [k] = \{1, .., k\}$
  - Let $x_i = \frac{1}{k}$
  - Query $f(x_i)$ for all $i \in [k]$
- Return $x_{\text{out}} = \underset{x_i}{\operatorname{argmin}} f(x_i)$

Analysis

- $x_* \in \left[\frac{i-1}{k}, \frac{i}{k}\right]$ for some $i \in [k]$
- $\exists i_* \in [k]$ s.t. $\left|x_* - \frac{i_*}{k}\right| \leq \frac{1}{k}$
- $\left|f(x_{i_*}) - f(x_*)\right| \leq L \left\|x_* - \frac{i_*}{k}\right\|_\infty \leq \frac{L}{k}$
- $f(x_{i_*}) \leq f_* + \frac{L}{k}$
- $f(x_{\text{out}}) \leq f(x_{i_*})$
- $k \geq L/\epsilon \Rightarrow f(x_{out})$ is ϵ-optimal

Improvements?

Lower bound?

Setting #1:

Theorem: there is method with query complexity $1 + [L/2\epsilon]$ for setting #1

Algorithm

- Pick $k \in \mathbb{Z}_{\geq 0}$
- For $i \in \{0, 1, .., k\}$
  - Let $x_i = \frac{i}{k}$
  - Query $f(x_i)$ for all $i \in [k]$
- Return $x_{\text{out}} = \operatorname*{argmin}_{x_i} f(x_i)$

Improvements?

Lower bound?

Analysis

- $x_* \in \left[\frac{i-1}{k}, \frac{i}{k}\right]$ for some $i \in [k]$
- $\exists i_* \in \{0, \dots, k\}$ s.t. $\left|x_* - \frac{i_*}{k}\right| \leq \frac{1}{2k}$
- $\left|f(x_{i_*}) - f(x_*)\right| \leq L \left\|x_* - \frac{i_*}{k}\right\|_\infty \leq \frac{1}{2k}$
- $f(x_{i_*}) \leq f_* + \frac{L}{2k}$
- $f(x_{\text{out}}) \leq f(x_{i_*})$
- $k \geq L/(2\epsilon) \Rightarrow f(x_{out})$ is ϵ-optimal

Lower bound proof strategy

Called a **resisting oracle**

Arbitrary Algorithm

- For $k = 1, \dots, K$
  - Compute point x_k based on previous oracle output ~~(and randomness)~~
  - Query oracle at x_k
- Output a point x_{out} based on previous points, oracle

Lower Bound Strategy

- From oracle output at $x_1, \dots, x_{k-1}$ specify oracle output at x_k.
- Show that there are two valid functions f_1 and f_2 consistent with oracle output on $x_1, \dots, x_{k-1}$ with no common valid output point.

Any algorithm must take at least K steps.

Why?

Algorithm outputs incorrect answer on either f or g.

Setting #0

- $f: \mathbb{R} \to \mathbb{R}$ *via evaluation oracle*
- $\exists x_* \in [0,1]$ *such that* $f(x) = f_*$
- $f(x) \in [0,1]$ *for all* $x \in \mathbb{R}$
- *Goal: compute ½ -optimal point*

<u>*Candidate f_i*</u>

- *For all* $z \in [0,1]$ *let*

$$f_z(x) = \begin{cases} 1 & x \neq z \\ 0 & x = z \end{cases}$$

- *Note:* f_{z_1} *and* f_{z_2} *have disjoint ½-optimal points for* $z_1 \neq z_2$

Arbitrary Algorithm

- For $k = 1, \ldots, K$
  - Compute point x_k based on previous oracle output ~~(and randomness)~~
  - Query oracle at x_k
- Output a point x_{out} based on previous points, oracle

Lower Bound Strategy

- From oracle output at $x_1, \ldots, x_{k-1}$ specify oracle output at x_k.

Output = 1

- Show that there are two valid functions f_1 and f_2 consistent with oracle output on $x_1, \ldots, x_{k-1}$ with no common valid output point.

f_{z_1} and f_{z_2} for any $z_1 \neq z_2$ with $z_1, z_2 \notin \{x_1, \ldots, x_k\}$

Any algorithm must take at least K steps.

Since holds for all K, an infinite number of steps are needed.

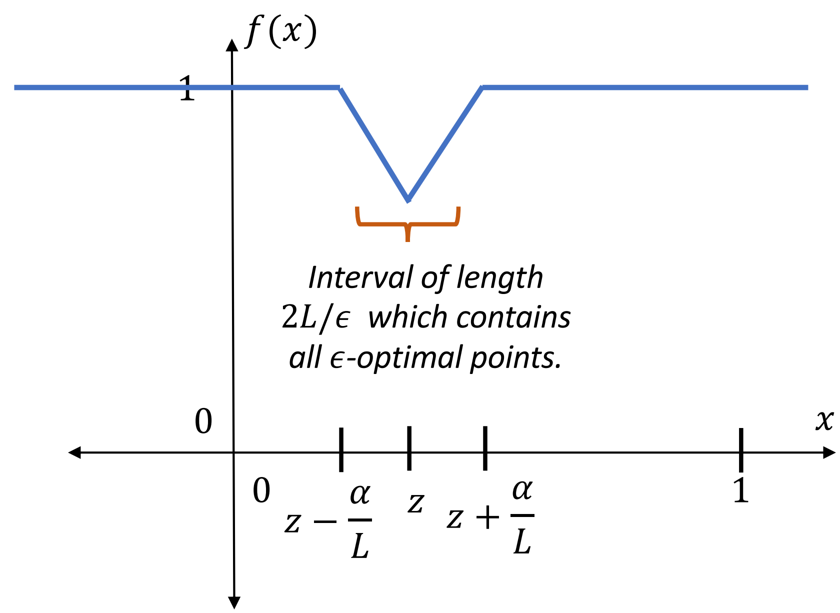
Setting #1

- $f: \mathbb{R} \to \mathbb{R}$ *via evaluation oracle*
- $\exists x_* \in [0,1]$ *such that* $f(x) = f_*$
- $f(x) \in [0,1]$ *for all* $x \in \mathbb{R}$
- f *is L-Lipschitz w.r.t* $\|\cdot\|_\infty$
- *Goal: compute* ϵ*-optimal point*

Valid functions with disjoint ϵ*-optimal points.*

What should the candidate f_i be?

- $f_{z,\alpha}(x) = \min\{1, -\alpha + L|x - z|\}$



Claims

- x' is ϵ-optimal for $f_{z,\alpha}$ for $\alpha > \epsilon$ if and only if $|x' - z| \leq L/\epsilon$
- $f_{z,\alpha}$ is L-Lipschitz w.r.t $\|\cdot\|_\infty$

Lower bound idea

- If oracle outputs 1 and not enough queries, consistent with two $f_{z,\alpha}$

Setting #1

- $f: \mathbb{R} \to \mathbb{R}$ *via evaluation oracle*
- $\exists x_* \in [0,1]$ *such that* $f(x) = f_*$
- $f(x) \in [0,1]$ *for all* $x \in \mathbb{R}$
- f *is L-Lipschitz w.r.t* $\|\cdot\|_\infty$
- *Goal: compute* ϵ*-optimal point*

Upper bound was $\frac{L}{2\epsilon} + 1$*. Can we improve?*

Lower Bound

At least $\frac{L}{4\epsilon} - 2$ queries are needed

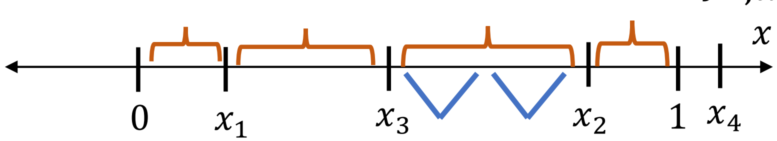
- $f_{z,\alpha}(x) = \min\{1, -\alpha + L|x - z|\}$

Claims

- x' is ϵ-optimal for $f_{z,\alpha}$ for $\alpha > \epsilon$ if and only if $|x' - z| \leq L/\epsilon$
- $f_{z,\alpha}$ is L-Lipschitz w.r.t $\|\cdot\|_\infty$

Lower bound idea

- If oracle outputs 1 and not enough queries, consistent with two $f_{z,\alpha}$



Lower bound proof

- Algorithm makes K-queries
- Can partition $[0,1]$ with $\leq K + 1$ intervals so points are on boundary
- At least one interval is length at least $1/(k+1)$
- If length is > $4\epsilon/L$ then there are two $f_{z,\alpha}$ consistent with disjoint ϵ-optimal points
- $\Rightarrow k + 1 > L/4\epsilon$

Improve

- $f: \mathbb{R} \to \mathbb{R}$ *via evaluation oracle*
- $\exists x_* \in [0,1]$ *such that* $f(x) = f_*$
- $f(x) \in [0,1]$ *for all* $x \in \mathbb{R}$
- f *is L-Lipschitz w.r.t* $\|\cdot\|_\infty$
- *Goal: compute* ϵ*-optimal point*

Upper bound was $\frac{L}{2\epsilon} + 1$. *Can we improve?*

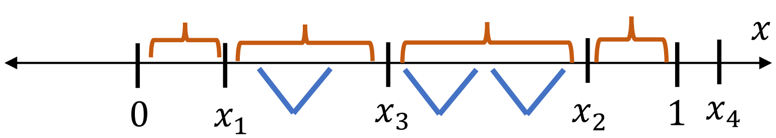
Lower Bound

At least $\frac{L}{4\epsilon} - 1$ queries are needed

- Algorithm also fails if there are two disjoint intervals of length $> 2\epsilon/L$
- To succeed the total length of the intervals (1) satisfies

$$< k\left(\frac{2\epsilon}{L}\right) + \frac{4\epsilon}{L}$$

- $k \geq \frac{L}{2\epsilon} - 2$
- Correct answer up to an additive 3!!!



Lower bound proof

- Algorithm makes K-queries
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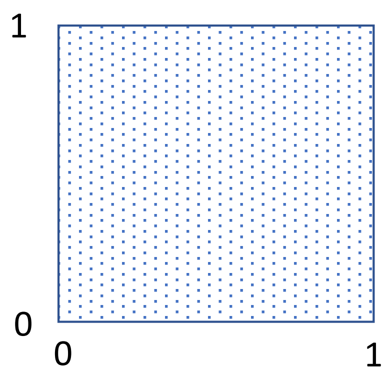
- At least one interval is length at least $1/(k+1)$
- If length is > $4\epsilon/L$ then there are two $f_{z,\alpha}$ consistent with disjoint ϵ-optimal points
- $\Rightarrow k + 1 \geq L/4\epsilon$

Setting #2: Higher Dimensions

- *$f: \mathbb{R}^n \to \mathbb{R}$ via evaluation oracle*
- *$\exists x_* \in [0,1]$ such that $f(x) = f_*$*
- *$f(x) \in [0,1]$ for all $x \in \mathbb{R}^n$*
- *f is L-Lipschitz w.r.t $\|\cdot\|_\infty$*
- *Goal: compute ϵ-optimal point*

Algorithm *(ϵ-net)*

- Pick $k \in \mathbb{Z}_{\geq 0}$
- Query $\left(\frac{i_1}{k}, \frac{i_2}{k}, \dots, \frac{i_k}{k}\right)^\top$ for all possible $i_j \in [k]$
- Return point of minimum value



Analysis

- $\forall i \in [n], \exists j \in [k]$ s.t. $\left|x_*(i) - \frac{j}{k}\right| \leq \frac{1}{k}$
- $\exists q$ queried s.t. $\|x_* - q\|_\infty \leq \frac{1}{k}$
- $f(q) \leq f(x_*) + \frac{L}{k}$
- Point output is $\frac{L}{k}$-optimal
- k^n queries are made
- $\left\lceil \frac{L}{\epsilon} \right\rceil^n$-queries suffice

How do we avoid this large dependence on dimension?

Optimal up to constants!
($(cL/\epsilon)^n$ queries are needed)

Lecture Plan

Recap ✓

- Oracles, minimization, efficiency, and iterative methods
- Continuity, smoothness, and critical points

Material ✓

- Continuity, ϵ-nets, and lower bounds

Thursday

- Lipschitzness and smoothness elaborated / revisited
- Convexity