

# MS&E 213 / CS 269O : Chapter 7

## Online Linear Optimization, Subgradient Descent, and Mirror Descent \*

By Aaron Sidford (sidford@stanford.edu)

November 18, 2020

### 1 Motivation

In the last section we proved that separation oracles and subgradient oracles exist for convex functions. In this chapter we explore how to obtain methods for minimizing non-smooth functions given a subgradient oracle. Rather than working with subgradient oracles directly, we will consider an intermediate problem of *online linear optimization*. This is a fundamental problem in learning and online algorithms. We show how to solve this problem efficiently and achieve efficient algorithms not just for subgradient descent, but also the learning from experts problem and more. This will in turn motivate fundamental algorithms of follow the regularized leader and mirror descent and we will discuss stochastic gradient descent and variants.

### 2 Subgradients

We start by recalling the definition of *subgradients*. As we have discussed these are close analogs of gradients of convex functions and when we minimize convex functions that are possibly non-differentiable, this is what we will assume we can compute in this chapter.

**Definition 1** (Subgradient). For a function  $f : S \rightarrow \mathbb{R}$  and  $x \in S$  we call  $g$  a *subgradient* of  $f$  at  $x$  if for all  $y \in S$  we have the property that

$$f(y) \geq f(x) + g^\top(y - x)$$

and we let  $\partial f(x)$  denote the set of *subgradients* of  $f$  at  $x$ .

In the previous chapter we proved that subgradients always exist for convex  $f$  at points in the interior of  $S$ . In the remainder of this section we prove various properties of subgradients and their connections to convex functions more broadly.

First we show that if a function has a subgradient at every point, then the function is convex.

**Lemma 2.** *Suppose for convex  $S \subseteq \mathbb{R}^n$  and  $f : S \rightarrow \mathbb{R}$  it is the case that  $\partial f(x) \neq \emptyset$  for all  $x \in S$  then  $f$  is convex.*

---

\*These notes are a work in progress. They are not necessarily a subset or superset of the in-class material, there may also be occasional *TODO* comments which demarcate material I am thinking of adding in the future, and citations are often omitted. These notes are intended converge to a superset of the class material that is TODO-free with a more complete set of citations and pointers to the literature. Your feedback is welcome and highly encouraged. If anything is unclear, you find a bug or typo, or if you would find it particularly helpful for anything to be expanded upon, please do not hesitate to post a question on the Piazza or contact me directly at sidford@stanford.edu.

*Proof.* Let  $x, y \in S$  and  $t \in [0, 1]$  be arbitrary. Let  $x_t \stackrel{\text{def}}{=} t \cdot x + (1 - t) \cdot y$  and  $g \in \partial f(x_t)$ , as  $x_t \in S$  by convexity. By definition of  $\partial f(x)$  we have that

$$\begin{aligned} f(x) &\geq f(x_t) + g^\top(y - x_t) = f(x_t) + (1 - t) \cdot g^\top(y - x) \text{ and} \\ f(y) &\geq f(y_t) + g^\top(y - x_t) = f(x_t) + t \cdot g^\top(y - x) \end{aligned}$$

adding a  $t$  multiple of the first inequality to a  $(1 - t)$  multiple of the second inequality yields that

$$t \cdot f(x) + (1 - t) \cdot f(y) \geq f(x_t).$$

□

Second, we show that the subgradients of a differentiable function at a point must be the gradient of the function at this point.

**Lemma 3.** *Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable at  $x \in \mathbb{R}^n$  and  $g \in \partial f(x)$ , then  $g = \nabla f(x)$ .*

*Proof.* We know that  $\nabla f(x) \in \partial f(x)$  since  $f(y) \geq f(x) + \nabla f(x)^\top(y - x)$  for all  $x$  and  $y$  convexity. What remains to show is that if  $g \in \partial f(x)$  then  $g = \nabla f(x)$ .

Let  $g \in \partial f(x)$  be arbitrary. By definition, we have that

$$f(y) \geq f(x) + g^\top(y - x)$$

for all  $y, x \in \mathbb{R}^n$ . However, since  $f$  is differentiable we know that

$$\lim_{y \rightarrow x} \frac{|f(y) - [f(x) + \nabla f(x)^\top(y - x)]|}{\|x - y\|_2} = 0.$$

Consequently, for all  $d \neq 0$ , letting  $x_{t,d} \stackrel{\text{def}}{=} x + t \cdot d$  we clearly have  $\lim_{t \rightarrow 0} x_{t,d} = x$ , and substituting these  $x_t$  as  $y$  in the above expressions yields that

$$0 \leq \lim_{t \rightarrow 0} \frac{f(x_{t,d}) - f(x) - t \cdot g^\top d}{t \cdot \|d\|_2} = \lim_{t \rightarrow 0} \frac{t \cdot \nabla f(x)^\top d - t \cdot g^\top d}{t \cdot \|d\|_2} = \frac{(\nabla f(x) - g)^\top d}{\|d\|_2}.$$

Consequently, we have that  $(\nabla f(x) - g)^\top d \geq 0$  for all  $d$  and thus  $\nabla f(x) = g$  (as otherwise we would get a different sign when  $g = -(\nabla f(x) - g)$ ). □

### 3 Online Linear Optimization

Now that we know what subgradients are, we provide our abstraction for how we will use having a subgradient oracle to perform minimization of smooth functions. Our abstraction is a problem known as the *online linear optimization problem*. It is more general than the problem of subgradient descent and we define it formally as follows.

**Definition 4** (Online Linear Optimization Problem). The *online linear optimization problem* is as follows: we are given a closed convex set  $S \subseteq \mathbb{R}^n$  and in each round  $t \geq 1$  we pick some  $x_t \in S$ . We are then revealed a penalty  $p_t \in \mathbb{R}^n$ . We wish to minimize *regret* defined as follows

$$\text{regret}(T) \stackrel{\text{def}}{=} \sum_{t \in [T]} p_t^\top x_t - \min_{x \in S} \sum_{t \in [T]} p_t^\top x.$$

The idea behind this game is that  $S$  should be thought of as some sort of space of actions we are allowed to take and  $p_t^\top x_t$  is to be thought of as the penalty we incur for picking that action. Now, we wish to minimize our total penalty,  $\sum_{t \in [T]} p_t^\top x_t$ , as compared to the minimal penalty we could have incurred if we had just fixed an action  $x \in S$  and stuck with it, throughout the game. We call this *regret*, as it captures how much we might regret simply picking a fixed strategy from  $S$ .

Note, that there are other notions of regret we might pick or different things we wish to optimize in such an online linear optimization problem. This notion of regret was chosen as it directly maps to subgradient descent and a variety of problems. It is one of the simplest problems in online optimization and learning, yet still highly nontrivial to solve.

Now, as we will see our goal will ultimately be to provide algorithms so that  $\lim_{T \rightarrow \infty} \frac{\text{regret}(T)}{T} = 0$ , i.e. the average regret over the rounds of the game converges to 0. This seems highly difficult to achieve, as in general the penalty functions  $p_t$  are allowed to depend on anything (including the  $x_t$  themselves).

Also, note that regret need not always be positive. Since  $x_t$  may change between rounds in principle it is possible that  $\sum_{t \in [T]} p_t^\top x_t$  is less than  $\min_{x \in S} \sum_{t \in [T]} p_t^\top x$ . While this is suggestive of picking a regret measure like  $\sum_{t \in [T]} p_t^\top x_t - \min_{x_1^*, \dots, x_T^* \in S} \sum_{t \in [T]} p_t^\top (x_t^*)$ , as we will see in the homework this in general impossible to have low average regret for.

## 4 Reduction to Convex Optimization

Before we discuss algorithm for solving the linear optimization problem, let's quickly see why this problem encompasses convex optimization with a subgradient oracle. The idea is quite simple, if we pick the penalty functions to be subgradients, then we can show that a low regret algorithm for the online linear optimization problem produces an  $\epsilon$ -optimal point. First, we formally define a subgradient oracle and then we show this claim.

**Definition 5** (Subgradient Oracle). For  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  as *subgradient oracle* is an oracle that given query point  $x \in \mathbb{R}^n$  returns  $g_x \in \partial f(x)$ .

In the next lemma we show that a solution to the online linear optimization problem plus a subgradient oracle can be used to compute  $\epsilon$ -optimal points for constrained convex function minimization.

**Lemma 6.** For closed convex set  $S$  and convex  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with minimizer  $x_* \in S$  if we have a solution to the online linear optimization problem that achieves some regret,  $\text{regret}(T)$ , when for the sequence of points  $x_1, \dots, x_T$  we have that  $p_t \in \partial f(x_t)$  then the point  $\frac{1}{T} \sum_{t \in [T]} x_t$  is  $\text{regret}(T)/T$ -optimal.

*Proof.* Note that by definition of  $\partial f(x_t)$  it is the case that  $f(x_*) \geq f(x_t) + p_t^\top (x_* - x_t)$ . Consequently, by convexity

$$\begin{aligned} f\left(\frac{1}{T} \sum_{t \in [T]} x_t\right) - f_* &\leq \frac{1}{T} \sum_{t \in [T]} f(x_t) - f_* \leq \frac{1}{T} \sum_{t \in [T]} p_t^\top (x_t - x_*) \\ &\leq \frac{1}{T} \left[ \sum_{t \in [T]} p_t^\top x_t - \min_{x \in S} \sum_{t \in [T]} p_t^\top x \right] = \frac{\text{regret}(T)}{T} \end{aligned}$$

□

Consequently, this shows that if we have a solution to the online linear optimization problem where average regret converges to 0 then this yields an algorithm for computing  $\epsilon$ optimal points with a subgradient oracle. Moreover, this shows that the exact query complexity of the problem depends on the rate at which average regret goes to 0.

## 5 Learning From Experts

Before we give our algorithm for solving the online linear optimization problem, as a warm-up to gain some intuition about the problem, let us consider a particular special case of the online linear optimization problem.

**Definition 7** (Learning From Experts Problem). The *learning from experts problem* is the online learning problem where  $S = \Delta^n = \{x \in \mathbb{R}_{\geq 0}^n \mid \sum_{i \in [n]} x_i = 1\}$  and for all  $t \geq 0$  and  $i \in [n]$  we have  $p_t(i) \in [0, 1]$ .

The set  $S = \Delta^n$  is known as the simplex. It is the set of non-negative vectors whose coordinates sum to 1. The set can be thought of as the set of probability vectors on  $n$  items. There is a natural bijection between  $\Delta^n$  and the marginals of a distribution on  $n$  items. For every distribution on  $n$  items if we let  $p_i$  be the probability of picking item  $i$  then  $p \in \Delta^n$ . Conversely, for  $x \in \Delta^n$  we let  $\mathcal{D}_x$  be the simple distribution on  $[n]$  where if  $\Pr_{i \sim \mathcal{D}_x}[i = j] = x_j$ . With this interpretation in mind, if we think of the penalty in a round of the learning from experts problem we have that

$$p_t^\top x_t = \sum_{i \in [n]} p_t(i) \cdot x_t(i) = \mathbb{E}_{i \sim \mathcal{D}_{x_t}} p_t(i)$$

in other words,  $p_t^\top x_t$  is the expected value of the  $i$ -th coordinate of  $p_t$  as sampled by the distribution induced by  $x_t$ .

Consequently, the learning from experts problem has the following nice interpretation. We can think there are  $n$  experts corresponding to each of the coordinates. We think that each day the experts are making some sort of prediction or giving some sort of advice and we would like to decide who to listen to. Maybe they are all predicting whether or not it will rain or whether the stock market will go up or down and  $p_t(i) = 1$  means they were incorrect and  $p_t(i) = 0$  means they were correct. Now, in each day we pick a distribution over the experts corresponding to the distribution over which expert we will listen to, i.e.  $x_t \in \Delta^n$ . Then at the end of the day, it is revealed how well each of the experts do  $p_t \in \mathbb{R}^n$  with  $p_t(i)$  large means did badly (large penalty) and  $p_t(i)$  small means they did well (small penalty).

Now,  $\sum_{t \in [T]} x_t^\top p_t$  is how well we do by following our strategy. It is the expected penalty we incur from picking experts as we did. However,  $\min_{x \in \Delta^n} \sum_{t \in [T]} p_t^\top x = \min_{i \in [n]} \sum_{t \in [T]} p_t(i)$ , i.e. the penalty incurred by the expert with minimum penalty. Consequently, regret is the difference between our total penalty and the penalty incurred by the best expert. Thus, achieving low average regret, means that we can do as well as the best expert does on average.

However, this is a highly non-trivial claim. Note that we are allowing the performance of the experts to depend on the distribution we pick. If we overcommit to listening to an expert, he might cease to perform well. However, if we pick a uniform distribution, then we might miss out on an expert performing particularly well. This learning from experts problem is about how to trade off these two issues while dealing with the fact that future performance of the experts might have nothing to do with their past performance. In the next section we take a closer look at some of these simple algorithms to help derive the actual algorithm that we will use.

## 6 Simple Learning from Expert Algorithms

Here we consider some simple algorithms to solve the learning from experts problem. We consider these algorithms to better understand the problem and ultimately motivate the efficient algorithms we consider for this problem.

## 6.1 Arbitrary

To start, to get an upper bound on performance. Let us consider the the penalty and possible regret of an arbitrary algorithm. Note that since  $p_t(i) \in [0, 1]$  and  $x_t(i) \geq 0$  we have that

$$p_t^\top x_t = \sum_{i \in [n]} p_t(i) \cdot x_t(i) \leq \sum_{i \in [n]} x_t(i) = 1$$

Consequently,  $\sum_{t \in [T]} p_t^\top x_t \leq T$ . Furthermore, for any  $x \in \Delta^n$  since  $x \geq \vec{0}$  coordinate-wise and  $p \geq \vec{0}$  coordinate wise we have

$$\min_{x \in \Delta} \sum_{t \in [T]} p_t^\top x \geq \sum_{t \in [T]} 0 = 0$$

and therefore, for any algorithm for solving the learning from experts problem we have  $\text{regret}_T(T) \leq T$ . Thus we can ensure average regret 1 easily, however getting an algorithm so average regret converges to 0 seems to require more work.

Note that it can actually be the case that  $\text{regret}_T(T) = T$ . If we just have two experts and  $p_t(1) = 1$  and  $p_t(2) = 0$  for all  $t \in [T]$  and we keep picking  $x_t = \vec{1}_1$ , i.e. we keep listening to the first expert, then we will get regret that is exactly  $T$ .

## 6.2 Random

Another simple algorithm to consider is a random strategy. We could simply pick  $x_t = \frac{1}{n} \vec{1}$  for all  $t \in [T]$ . Note that for this strategy we have that if  $j = \text{argmin}_{i \in [n]} \sum_{t \in [R]} p_t(i)$  then

$$\begin{aligned} \text{regret}(T) &= \sum_{t \in [T]} p_t^\top x_t - \min_{x \in \Delta^n} \sum_{t \in [T]} p_t^\top x = \sum_{t \in [T]} \left[ \sum_{i \in [n]} \frac{1}{n} p_t(i) - p_t(j) \right] \\ &\leq \sum_{t \in [T]} \sum_{j \neq i} \frac{1}{n} = T \cdot \frac{n-1}{n} = T \cdot \left( 1 - \frac{1}{n} \right). \end{aligned}$$

Furthermore, we can show that this is tight in the worst case. If  $x_t(1) = 0$  for all  $t$  and  $x_t(i) = 1$  for all  $i \neq 1$  then this is precisely the regret that will be achieved.

Thus, we can show that random does a little better than arbitrary, but how much better degrades as  $n$  increases and the average regret still stays at a constant when  $n \geq 2$ .

## 6.3 Follow the Leader

Each of the strategies we have seen so far have been oblivious to the actual penalties,  $p_t$  encountered. If we wish to do better the above analysis suggest we need to some how be focus in on experts doing better than the average. Another natural strategy follow to achieve this is known as *follow the leader (FTL)*. In each round  $t$  we can pick the strategy for which would have done the best on what we have observed in the past. In other words, we could let  $x_{t+1} = \text{argmin}_{x \in \Delta^n} \sum_{k \in [t]} p_k^\top x$ .

However, we can show that this strategy gets regret that converges to  $T/2$  in the worst case. To see this, consider a simple scenario where we have two experts. In the first round, expert 1 gets penalty  $\epsilon$  and expert 2 gets penalty 0. In the next round we pick expert 2 to listen to as they have done better in the past. However, expert 2 gets penalty 1 and expert 1 gets penalty 0. Now expert 1 has had total penalty  $\epsilon$  and expert 2 has had total penalty 1 so we pick expert 1. However, we then give expert 1 penalty 1 and expert 2 penalty 0. We continue in this way, penalizing only the expert we pick. Note that our total penalty after  $T$  rounds is

$\epsilon + (T - 1)$ . However, each of the experts individually have had total penalty at most  $\lceil \frac{T}{2} \rceil$ . Consequently, our regret converges to  $T/2$ .

In short, this strategy perform poorly as even though both experts have done about the same over time, by us switching experts every round, we realize a penalty that is much worse. Note that we can make the factor of 2 even worse by considering a similar counterexample with more experts.

## 7 Follow The Regularized Leader

In the previous section we saw several algorithms for the learning from experts problem. A random strategy failed as it didn't identify an expert performing much better than the others. However, the greedy follow the leader strategy also failed as it kept changing strategy missing out on the true leader. At first glance, this seems like an insurmountable problem, we want to learn what experts to listen to, however past performance says nothing about future performance and thus if we pick an expert, it might do worse down the road. How should we hope to surmount this?

One idea is that if we start to focus on an expert based on past performance, then if the expert ceases to do well, then there was likely not any strategy that does well. In our example where follow the leader does poorly, uniform does great, and in our example where uniform did poorly, greedy did great. Perhaps, the right algorithm is one that carefully interpolates between these two algorithms.

This turns out to be the case and is precisely the algorithm we will show in the case. A random strategy can be thought of picking  $x_t = \operatorname{argmin}_{x \in S} r(x)$  for some function  $r$  that depends only on  $S$  (not on the  $p_t$ ) and forced the algorithm to pick  $x_t = \frac{1}{n} \mathbf{1}$  every time. Here we consider an algorithm that interpolates between this algorithm and greedy. It is called *follow the regularized leader* and the term  $r(x)$  is called a *regularizer*.

**Definition 8** (Follow The Regularized Leader (FTRL) ). To solve the *online linear optimization problem* the *follow the regularized leader* (FTRL) algorithm is as follows, pick some function  $r : \mathbb{R}^n \rightarrow \mathbb{R}$ , called a *regularizer*, that is differentiable and  $\mu$ -strongly convex on  $S$  with respect to some norm  $\| \cdot \|$ . For all  $T \geq 0$  let

$$\Phi_T(x) \stackrel{\text{def}}{=} \eta \sum_{t \in [T]} p_t^\top x + r(x)$$

and (with  $[0] \stackrel{\text{def}}{=} \emptyset$  so  $\Phi_0(x) \stackrel{\text{def}}{=} r(x)$ ) and for all  $T \geq 0$  pick <sup>1</sup>

$$x_{T+1} \stackrel{\text{def}}{=} \operatorname{argmin}_{x \in S} \Phi_T(x).$$

This algorithm is quite simple, simply introduce a parameter  $\eta$  (sometimes called a learning rate) that trades off how much we care about optimizing past performance, i.e. minimizing  $\sum_{t \in [T]} p_t^\top x$  and being stable, i.e. minimizing  $r(x)$ . The algorithm simply takes the next action to be the minimizer over  $S$  of the  $\eta$  weighted combination of these two contributions. The idea is that by picking a well behaved strongly convex regularizer, we might ensure that we do not over commit to well performing experts while still be able to take advantage of their improved performance.

In the remainder of this section we show that this algorithm achieves average regret that converges to 0 in many scenarios. We show this by analyzing how the minimizer of  $\Phi_T$  changes in each iteration.

First, we recall the following basic lemma we proved earlier for this analysis.

**Lemma 9** ((Recap) Constrained Convex Minimizers). *If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a differentiable convex function and  $S \subseteq \mathbb{R}^n$  is a closed convex set, then  $x_*$  is a minimizer of  $f$  over  $S$ , i.e.  $x_* \in S$ , i.e.  $f(x_*) \leq f(x)$  for all  $x \in S$ , if and only if  $\nabla f(x_*)^\top (x - x_*) \geq 0$  for all  $x \in S$ .*

<sup>1</sup>Where exactly do we use the convexity of  $S$ ? (It is in the minimality.)

Using this we show that we bound how much  $\Phi_T$  changes as we move away from its minimizer.

**Lemma 10.** *In an instance of FTRL for all  $z \in S$  and  $t \geq 0$  we have*

$$\Phi_t(z) \geq \Phi_t(x_{t+1}) + \frac{\mu}{2} \|z - x_{t+1}\|^2.$$

*Proof.* Since  $r(z)$  is  $\mu$ -strongly convex and linear functions, e.g.  $\eta \sum_{k \in [t]} p_k^\top z$  is 0-strongly convex, and the sum of a  $\mu$ -strongly convex function and a 0-strongly convex function is  $\mu$ -strongly convex we have that  $\Phi_t$  is  $\mu$ -strongly convex. Consequently, as  $r$  is differentiable (and therefore  $\Phi$  is differentiable), we have

$$\Phi_t(z) \geq \Phi_t(x_{t+1}) + \nabla \Phi_t(x_{t+1})^\top (z - x_{t+1}) + \frac{\mu}{2} \|z - x_{t+1}\|^2.$$

However, since  $x_{t+1} = \operatorname{argmin}_{x \in S} \Phi_t(x)$  we have that  $\nabla \Phi_t(x_{t+1})^\top (z - x_{t+1}) \geq 0$  by Lemma 9 and the result follows.  $\square$

Using this we show bound how much  $\Phi_T$  increases in each iteration. Often the difficult part of analysis in online algorithms is showing that the optimal, e.g.  $\min_{x \in S} \sum_{t \in [T]} p_t^\top x$  is lower bounded by something involving our performance. Since, the minimizer of  $\Phi_T$  is approximately this quantity (up to scaling by  $\eta$  and addition of the regularizer), lower bounding  $\Phi_T$  serves this purpose.

**Lemma 11.** *In FTRL for all  $t \geq 1$  we have that*

$$\begin{aligned} \Phi_t(x_{t+1}) - \Phi_{t-1}(x_t) &\geq \eta \cdot p_t^\top x_t + \eta \cdot p_t^\top (x_{t+1} - x_t) + \frac{\mu}{2} \|x_t - x_{t+1}\|^2 \\ &\geq \eta \cdot p_t^\top x_t - \frac{\eta^2}{2\mu} \|p_t\|_*^2 \end{aligned}$$

*Proof.* First, note that by the definition of  $\Phi_t$  and Lemma 10 we have

$$\Phi_t(x_{t+1}) = \eta p_t^\top x_{t+1} + \Phi_{t-1}(x_{t+1}) \geq \eta p_t^\top x_{t+1} + \frac{\mu}{2} \|x_t - x_{t+1}\|^2 + \Phi_{t-1}(x_t).$$

Rearranging terms yields the first inequality and using that

$$\eta \cdot p_t^\top z + \frac{\mu}{2} \|z\|^2 \geq -\frac{\eta^2}{2\mu} \|p_t\|_*^2$$

yields the second inequality.  $\square$

This lemma shows that in every iteration the minimizer of  $\Phi_t$  increases by an  $\eta$  multiple of our realized penalty,  $p_t^\top x_t$ , minus  $\eta^2$  times  $\frac{1}{2\mu} \|p_t\|_*^2$ . Summing this up over each round lets us immediately bound the total regret.

**Theorem 12** (FTRL Guarantee). *If we implement FTRL for the online linear optimization problem and and  $G \geq \|p_t\|_*$  for all  $t \in [T]$  then for all  $z \in S$*

$$\sum_{t \in [T]} p_t^\top (x_t - z) \leq \frac{\eta}{2\mu} \cdot T \cdot G^2 + \frac{1}{\eta} \cdot \left[ r(z) - \min_{x \in S} r(x) \right] \quad (7.1)$$

and consequently for  $D \geq \max_{x \in S} r(x) - \min_{x \in S} r(x)$  and  $\eta = \sqrt{(2D\mu)/(TG^2)}$  we can achieve

$$\operatorname{regret}(T) \leq \sqrt{2DTG^2/\mu}. \quad (7.2)$$

*Proof.* Summing Lemma 11 yields

$$\Phi_T(x_{T+1}) - \Phi_0(x_1) = \sum_{t \in [T]} [\Phi_t(x_{t+1}) - \Phi_{t-1}(x_t)] \geq \sum_{t \in [T]} \left[ \eta \cdot p_t^\top x_t - \frac{\eta^2}{2\mu} \|p_t\|_*^2 \right].$$

Now clearly  $\Phi_T(z) \geq \Phi_T(x_{T+1})$  and since  $\|p_t\|_* \leq G$  and  $\Phi_0(x_1) = \min_{x \in S} r(x)$  we have

$$\eta \sum_{t \in [T]} p_t^\top z + \left[ r(z) - \min_{x \in S} r(x) \right] \geq \eta \sum_{t \in [T]} p_t^\top x_t - \frac{\eta^2 T \cdot G^2}{2\mu}.$$

Rearranging terms yields the first claim and the choice of  $\eta$  yields the second.  $\square$

Note that the analysis essentially says that each iteration we have that our regret grows by at most  $\frac{\eta}{2\mu} \|p_t\|_*^2$  and there is a  $\frac{1}{\eta}$  contribution from the regularizer. Trading off the two gives the final bound. Note that the rate gets worse as either  $D$  (how much the regularizer varies) or  $G$  (how large the subgradients are) increases. Also note, that the steps size or learning rate,  $\eta$  gets smaller as  $T$  increases,  $D$  decreases, or  $G$  increases. This says that as we run for longer, i.e.  $T$  increases, each step moves away from the regularizer less, i.e. we require great confidence in an experts performance to move towards them. It also says that the large the penalties the more cautious we should be (i.e. we should move away from the regularizer more slowly) and the large the regularizer the more we should move away from it (i.e. as it is less helpful and there is more space to explore).

Note also that this bound says we can achieve

$$\frac{\text{regret}(T)}{T} = \frac{\sqrt{2DTG^2/\mu}}{T} = \sqrt{\frac{2DG^2}{T\mu}}.$$

Consequently as  $T \rightarrow \infty$  we have  $\text{regret}(T)/T \rightarrow 0$ . Consequently, we can use this algorithm for our applications. We discuss using FTRL for the problems we considered more extensively in the next few sections.

## 8 Solving The Experts Problem

Here we show how to use FTRL to solve the learning from experts problem. Note that this is not immediate as FTRL does not specify what regularizer to use or how to compute  $x_{t+1} = \text{argmin}_{x \in S} \Phi_t(x)$ . Here we discuss these issues for the learning from experts problem.

Our main question is what regularizer  $r$  do we want to use. However, even more fundamental perhaps, we could ask what norm do we want to do our analysis in? Note that FTRL depends on the bounds on  $\|p_t\|_*$  and we have that  $\|p_t\|_\infty \leq 1$ . Since  $\|\cdot\|_\infty$  is the dual of  $\|\cdot\|_1$  (see homework) this would suggest that we may want to do our analysis in  $\|\cdot\|_1$ . Furthermore, note that since  $r$  is strongly convex in  $\|\cdot\|$  Lemma 10 shows that if there is some  $x \in S$  such that  $\|x - x_1\|$  is large then  $D$  will be large. However, since we know that  $\|x - y\|_1 \leq 2$  for all  $x, y \in \Delta$  (by triangle inequality) this would again suggest that we may want to do our analysis in  $\ell_1$  to keep  $D$  down as well.

Consequently, we will perform our analysis for the experts problem in the  $\|\cdot\|_1$  norm. The next question is what regularizer to use or equivalently what is a good strongly convex function with respect to  $\|\cdot\|_1$  over  $\Delta_n$ . Another, essentially equivalent question, is what is a good way to measure distances in the simplex? A natural choice here is entropy.

**Definition 13.** For  $x \in \mathbb{R}_{\geq 0}^n$  the *entropy regularizer* is define as  $e(x) = \sum_{i \in [n]} x_i \cdot \log x_i$  where we let  $0 \log 0 \stackrel{\text{def}}{=} 0$ .

We note that this choice of treatment of  $0 \log 0$  makes sense due to the following.

**Lemma 14.** *Note that  $\lim_{x \rightarrow 0} x \cdot \log x = 0$ .*

*Proof.* By L'Hopital we have

$$\lim_{x \rightarrow 0} x \cdot \log x = \lim_{x \rightarrow 0} \frac{\log x}{(1/x)} = \lim_{x \rightarrow 0} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0} -x = 0.$$

□

Next, we analyze the strong convexity of  $e(x)$  with respect to  $\|\cdot\|_1$ .

**Lemma 15.** *The function  $e(x) \stackrel{\text{def}}{=} \sum_{i \in [n]} x_i \cdot \log x_i$  is 1-strongly convex with respect to  $\|\cdot\|_1$  on  $\Delta^n$ .<sup>2</sup>*

*Proof.* Note that  $\nabla e(x)_i = 1 + \log x_i$  and  $\nabla^2 e(x) = \mathbf{X}^{-1}$  where  $\mathbf{X} = \mathbf{diag}(x)$ . Now for  $x \in \Delta^n$  we have by Cauchy Schwarz that

$$\|y\|_1^2 = \left( \sum_{i \in [n]} \sqrt{x_i} \cdot \frac{|y_i|}{\sqrt{x_i}} \right)^2 \leq \sum_{i \in [n]} x_i \cdot \sum_{i \in [n]} \frac{y_i^2}{x_i} = y^\top \mathbf{X}^{-1} y.$$

Consequently we have that  $r(x)$  is 1-strongly convex on  $\Delta^n$ .

□

Next, we characterize the minimizers of  $r(x)$  plus a linear function over the simplex.

**Lemma 16.** *For  $e(x) = \sum_{i \in [n]} x_i \cdot \log x_i$ ,  $p \in \mathbb{R}^n$  we have*

$$z = \operatorname{argmin}_{x \in \Delta^n} p^\top x + r(x)$$

*is given by*

$$z_i = \frac{\exp(-p_i)}{\sum_{j \in [n]} \exp(-p_j)}.$$

*Proof.* Note that clearly  $z \in \Delta^n$  and therefore from the characterization of the minimizer of a differentiable convex function over a closed convex set, it suffices to show that  $\nabla f(z)^\top (x - z) \geq 0$  for  $f(x) = p^\top x + r(x)$  and all  $x \in \Delta^n$ . However, by design, for all  $i \in [n]$  we have

$$\nabla f(z)_i = p_i + 1 + \log(z_i) = 1 - \log \left[ \sum_{j \in [n]} \exp(-p_j) \right],$$

i.e. each coordinate of  $\nabla f(z)$  has the same value. However since  $\sum_{i \in [n]} x_i = \sum_{i \in [n]} z_i = 1$  as  $x, z \in \Delta^n$  we have that  $\nabla f(z)^\top (x - z) = 0$ . □

Using this characterization of the minimizer of  $r(x)$  plus a linear over the simplex we bound  $D$ .

**Lemma 17.** *We have*

$$\max_{x \in \Delta^n} e(x) - \min_{x \in \Delta^n} e(x) \leq \log n.$$

---

<sup>2</sup>Technically, for  $f : S \rightarrow \mathbb{R}$  we have only defined strong convexity for functions where  $S = \mathbb{R}^n$  and proven the equivalence of  $\mu$ -strong convexity with respect to  $\|\cdot\|$  and  $z^\top \nabla^2 f(x) z \geq \mu \|z\|^2$  for all  $x \in S$  in this case; however the definitions and equivalence proof extend naturally.

*Proof.* Note that  $x \log x$  is convex on  $[0, 1]$  with  $0 \log 0 = 0$  and  $1 \log 1 = 0$  and consequently

$$\max_{x \in \Delta^n} e(x) \leq 0$$

Furthermore, we have that  $\arg \min_{x \in \Delta^n} e(x) = \frac{1}{n} \vec{1}$  and this

$$\min_{x \in \Delta^n} e(x) = e\left(\frac{1}{n} \vec{1}\right) = \sum_{i \in [n]} \left(\frac{1}{n}\right) \log\left(\frac{1}{n}\right) = -\log n.$$

□

Putting these pieces together gives the following result on solving the learning from experts problem.

**Theorem 18.** *Suppose in the learning from experts problem we start with  $w_1 = \vec{1}$  and in each round  $t \geq 1$  let  $x_t = w_t / \|w_t\|_1$ , i.e. pick expert  $i$  with probability proportional to  $w_t(i)$  and let  $w_{t+1}(i) := w_t(i) \cdot \exp(-\eta p_t(i))$  for all  $i \in [n]$  and  $\eta \stackrel{\text{def}}{=} \sqrt{2 \log n / T}$  then  $\text{regret}(T) \leq \sqrt{2T \log n}$ .*

*Proof.* We apply FTRL with  $r(x) = e(x)$ . We have shown that  $r$  is 1-strongly convex with respect to  $\|\cdot\|_1$  over  $\Delta$ , that  $\|p_t\|_* \leq 1$  for all  $t$  and  $\max_{x \in \Delta^n} r(x) - \min_{x \in \Delta^n} r(x) \leq \log n$ . Consequently, FTRL achieves the desired regret bound as we can pick  $G = 1$  and  $D = \log n$ . To see that the algorithm we have proposed is the same as FTRL note that for all  $t \geq 0$  and  $i \in [n]$  we have

$$x_{t+1}(i) = \frac{\exp\left(\sum_{k \in [t]} -\eta \cdot p_k(i)\right)}{\sum_{j \in [n]} \exp\left(\sum_{k \in [t]} -\eta \cdot p_k(j)\right)}$$

and therefore  $x_{t+1} = \arg \min_{x \in \Delta^n} \Phi_t(x)$  as required. □

Note that it can be shown that this bound cannot be improved up to constants in the worst case up to constants.

## 9 Solving Convex Optimization

Here we show how to use FTRL to solve  $L$ -Lipschitz continuous optimization problems. First we provide the following lemma showing that a function is  $L$ -Lipschitz with respect to some norm if and only if its subgradients are bounded with respect to the dual norm.

**Lemma 19.** *A convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $L$ -Lipschitz continuous with respect to a norm  $\|\cdot\|$ , i.e.  $|f(x) - f(y)| \leq L \cdot \|x - y\|$  for  $x, y \in \mathbb{R}^n$ , if and only if for all  $x$  and  $g \in \partial f(x)$  it is the case that  $\|g\|_* \leq L$ .*

*Proof.* First suppose that  $\|g\|_* \leq L$  for all  $g \in \partial f(x)$ . Then for  $x, y \in \mathbb{R}^n$  and  $g_x \in \partial f(x)$  and  $g_y \in \partial f(y)$  we have that

$$f(y) \geq f(x) + g_x^\top (y - x) \geq f(x) - \|g_x\|_* \|y - x\| \geq f(x) - L \|y - x\|$$

and similarly

$$f(x) \geq f(y) + g_y^\top (x - y) \geq f(y) - \|g_y\|_* \|y - x\| \geq f(y) - L \|y - x\|$$

and therefore we have that

$$-L \|y - x\| \leq f(y) - f(x) \leq L \|y - x\|.$$

On the other hand suppose that  $\|g\|_* > L$  for some  $g \in \partial f(x)$ . Then, since for some  $z \in \mathbb{R}^n$  with  $\|z\| = 1$  we have  $g_x^\top z = \|g_x\|_*$  it is the case that

$$f(x + z) \geq f(x) + g_x^\top (x + z - x) = f(x) + \|g_x\|_*$$

Consequently

$$|f(x+z) - f(x)| \geq \|g_x\|_* > L \cdot \|(x+z) - x\|.$$

□

Using this we show provide bounds for unconstrained Lipschitz continuous minimization in  $\|\cdot\|_2$ . The idea is quite simple, we simply use  $r(x) = \frac{1}{2}\|x - x_0\|_2^2$  as the regularizer. Note that  $D = \max_{x \in S} r(x) = \infty$  though in this case and consequently, we cannot simply use the regret bounds of FTR directly. Instead

**Lemma 20.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $L$ -Lipschitz function and  $S$  is a closed convex set known explicitly. Further suppose we have a subgradient oracle (and  $\partial f(x) \neq \emptyset$  for all  $x \in S$ ) and some point  $x_0 \in S$  and  $R \geq \|x_0 - x_*\|_2^2$  then if  $x_*$  is a minimizer of  $f$  we can design a method that with  $k$  call to the subgradient oracle produces  $x_k \in S$  such that*

$$f(x_k) - f_* \leq \sqrt{\frac{RL^2}{k}}$$

and therefore we can produce an  $\epsilon$ -optimal point with  $O(RL^2/\epsilon^2)$  queries to the subgradient oracle.

*Proof.* For all  $T$  let  $g_T \in \partial f(x_T)$  and let consider the algorithm

$$x_{T+1} = \operatorname{argmin}_{x \in S} \eta \sum_{t \in [T]} g_t^\top x + \frac{1}{2} \|x - x_0\|_2^2$$

we know that  $\|g_T\|_2 \leq L$  for all  $T$  and thus for  $r(x) = \frac{1}{2}\|x - x_0\|_2^2$  since  $r$  is 1 strongly convex Theorem 12 yields that

$$\sum_{t \in [T]} g_t^\top (x_t - x_*) \leq \frac{\eta}{2} \cdot T \cdot L^2 + \frac{1}{\eta} \cdot \left[ r(x_*) - \min_{x \in S} r(x) \right].$$

However, since clearly  $\min_{x \in S} r(x) = 0$  we have that

$$\frac{1}{T} \sum_{t \in [T]} g_t^\top (x_t - x_*) \leq \frac{\eta}{2} \cdot L^2 + \frac{R}{2\eta T}$$

Since  $f(x_*) \geq f(x_t) + g_t^\top (x_* - x_t)$  we have that for  $\eta = \sqrt{R/(L^2 T)}$

$$f \left( \frac{1}{T} \sum_{t \in [T]} x_t \right) - f_* \leq \frac{1}{T} \sum_{t \in [T]} g_t^\top (x_t - x_*) \leq \sqrt{\frac{RL^2}{T}}.$$

□

What exactly is this algorithm in the unconstrained case? Note that  $\vec{0} = \nabla \Phi_t(x_{t+1}) = \eta \sum_{k \in [t]} g_k + x_{t+1} - x_0$  and thus  $x_{t+1} = x_0 - \eta \sum_{k \in [t]} g_k$  and thus this algorithm is  $x_{t+1} = x_t - \eta g_t$ , i.e. it is essentially gradient descent, aka *subgradient descent*. In this case we have shown that the average of the iterates converges for this algorithm for decaying choice of  $\eta$ .

## 10 Mirror Descent

Here we consider another algorithm for solving the online linear optimization. This algorithm is known as *mirror descent* and the analysis will give us another interesting perspective on the problem.

Now rather than using our strongly convex function directly in a single minimization problem, we will use it through Bregman divergences defined as follows.

**Definition 21** (Bregman Divergence). For differentiable strongly convex function  $r : \mathbb{R}^n \rightarrow \mathbb{R}$  the Bregman Divergence,  $D_r(x||y)$  for  $x, y \in \mathbb{R}^n$  is given by  $D_r(x||y) \stackrel{\text{def}}{=} r(x) - [r(y) + \nabla r(y)^\top (x - y)]$ . Furthermore, for closed, convex, non-empty  $S \subseteq \mathbb{R}^n$  we let  $\pi_S^r(y) \stackrel{\text{def}}{=} \operatorname{argmin}_{x \in S} D_r(x||y)$ .

The Bregman divergence is a way of turning an arbitrary differentiable strongly convex function into a distance-like function (though triangle inequality need not hold of it). Indeed, so long as  $r$  is  $\mu$ -strongly convex with respect to  $\|\cdot\|$  for  $\mu > 0$  we can show that  $D_r(x||y) \geq \frac{\mu}{2} \|x - y\|^2$  and thus  $D_r(x||y) = 0$  if and only if  $x = y$ . (See homework.) Furthermore, this implies that it is reasonable to think of  $\pi_S^r(y)$  as the projection of  $y$  onto  $S$ .

The other notion we need is that of a *mirror map*.

**Definition 22** (Mirror Map). We call differentiable strongly convex  $r : \mathbb{R}^n \rightarrow \mathbb{R}$  a *mirror map* if for all  $x \in \mathbb{R}^n$  it is the case that  $x = \nabla r(y)$  for some  $y$  in the domain of  $r$ .

The idea of a mirror map is that it is a way to translate from the dual space, i.e. the space of gradients or subgradients, back to the domain. Note that we often measure differences between points in one norm but norms of gradients in the dual norm. While in finite dimensions all norms approximate each other up to multiplicative factors and therefore such a primal space can always be related to the dual space, this need not happen in infinite dimension. For example, if we have an infinite dimensional vector  $g \in \ell_\infty$  and  $x \in \ell_1$  the operation  $y = x + \eta g$  doesn't necessarily even make sense. A mirror map provides a way to avoid such issues by providing the mapping from dual space back to primal.

Note that clearly  $r(x) = \frac{1}{2} \|x - x_0\|_2^2$  is a mirror map as  $\nabla r(x) = x - x_0$ . Furthermore, clearly  $e(x) = x \log x$  for  $x \in \mathbb{R}_{\geq 0}^n$  is a mirror map as  $\nabla e(x) = \vec{1} + \log(x)$  and  $\nabla e(x) = y$  if and only if  $x = \exp(y - 1)$  which we can always ensure.

These notions give another interesting perspective on FTRL provided that  $r$  is a mirror map. Note that the step

$$x_{t+1} = \operatorname{argmin}_{x \in S} \eta \sum_{k \in [t]} p_k^\top x + r(x)$$

is trivially the same as

$$x_{t+1} = \operatorname{argmin}_{x \in S} D_r(x||y_t)$$

provided where  $t$  is chosen such that

$$\nabla r(y_t) = -\eta \sum_{k \in [t]} p_k.$$

Consequently, we see that FTRL is equivalent to the following. Start with  $y_0$  such that  $\nabla r(y_0) = 0$  and then for all  $t$

- Let  $x_{t+1} = \pi_S^r(y_t) = \operatorname{argmin}_{x \in S} D_r(x||y_t)$
- Let  $y_{t+1}$  be such that  $\nabla r(y_{t+1}) = \nabla r(y_t) - \eta p_{t+1}$

In other words, this algorithm is essentially performing gradient descent in the dual (on  $\nabla r(y_t)$ ) and then projecting back to  $x_t$ . This is known as *lazy mirror descent* or *dual averaging*, in the context of convex minimization. Here, the updates are occurring primarily in the dual,  $\nabla r$ , space and we simply do projection to get points in  $S$ .

Another natural modification of this algorithm would be to do exactly the same procedure, but project every iteration and therefore better maintain points in  $S$ . Here we could start with some  $y_0 = x_0 \in S$  and for all  $t$  repeat the following

- Let  $x_{t+1} = \operatorname{argmin}_{x \in S} D_r(x||y_t)$

- Let  $y_{t+1}$  be such that  $\nabla r(y_{t+1}) = \nabla r(x_{t+1}) - \eta p_{t+1}$ .

The algorithm is known as *mirror descent*. Here we perform the gradient descent update on the mapping of  $x_{t+1}$  back into the dual space. Note that in many cases, *mirror descent* and *lazy mirror descent*, can be shown to be equivalent. However, the analysis typically revolves around different potential functions. The analysis of mirror descent as the virtue that we can primarily measure progress by distances in the primal,  $S$ , space.

We analyze the performance of mirror descent in the remainder of this section.

...

**Lemma 23.** *In each iteration of mirror descent when  $r$  is a  $\mu$ -strongly convex mirror map with respect to  $\|\cdot\|$  and  $S$  is a closed set we have that for all  $z \in \mathbb{R}^n$*

$$\begin{aligned} D_r(z||y_t) - D_r(z||x_t) &\leq -\frac{\mu}{2}\|y_t - x_t\|^2 - \eta p_t^\top (y_t - x_t + x_t - z) \\ &\leq \frac{\eta^2}{2\mu}\|p_t\|_*^2 - \eta \cdot p_t^\top (x_t - z) \end{aligned}$$

*Proof.* Direct calculation reveals that for all  $t \geq 0$

$$\begin{aligned} D(z||y_t) - D(z||x_t) &= [r(z) - r(y_t) - \nabla r(y_t)^\top (z - y_t)] - [r(z) - r(x_t) - \nabla r(x_t)^\top (z - x_t)] \\ &= r(x_t) - r(y_t) - \nabla r(x_t)^\top (x_t - y_t) - \eta p_t^\top (y_t - z) \end{aligned}$$

Since  $r(y_t) \geq r(x_t) + \nabla r(x_t)^\top (y_t - x_t) + \frac{\mu}{2}\|y_t - x_t\|^2$  by strong convexity this implies

$$\begin{aligned} D(z||y_t) - D(z||x_t) &\leq -\frac{\mu}{2}\|y_t - x_t\|^2 - \eta p_t^\top (y_t - x_t + x_t - z) \\ &\leq \frac{\eta^2}{2\mu}\|p_t\|_*^2 - \eta \cdot p_t^\top (x_t - z). \end{aligned}$$

□

This lemma shows that we can bound the changed in Bregman divergence between any vector and  $y_t$  and  $x_t$ . However, we would really like to relate this to changes only in  $x_t$ . For this we can show a type of generalized Pythagorean theorem that holds for Bregman divergences.

**Lemma 24.** *For all differentiable  $r : S \rightarrow \mathbb{R}$  and any  $x, y, z \in S$*

$$D_r(x||z) - D_r(x||y) - D_r(y||z) = \nabla D_r(y||z)^\top (x - y)$$

*Proof.* Recall by definition that

$$\begin{aligned} D_r(x||z) &= r(x) - [r(z) + \nabla r(z)^\top (x - z)], \\ D_r(x||y) &= r(x) - [r(y) + \nabla r(y)^\top (x - y)], \text{ and} \\ D_r(y||z) &= r(y) - [r(z) + \nabla r(z)^\top (y - z)]. \end{aligned}$$

Consequently,

$$\begin{aligned} D_r(x||z) - D_r(x||y) - D_r(y||z) &= \nabla r(z)^\top (y - x) - \nabla r(y)^\top (x - y) \\ &= (\nabla r(y) - \nabla r(z))^\top (x - y). \end{aligned}$$

□

**Lemma 25.** *If  $r : T \rightarrow \mathbb{R}$  is strongly convex and  $S$  is a closed convex set then for  $y \in S$  and  $z \in \mathbb{R}^n$*

$$D_r(x||y) + D_r(y||z) \leq D_r(x||z) \text{ for all } x \in S$$

*if and only if  $y = \pi_S^r(z)$ .*

*Proof.* Let  $x, y \in S$  and  $z \in \mathbb{R}^n$  be arbitrary and recall that by definition

$$\begin{aligned} D_r(x||z) &= r(x) - [r(z) + \nabla r(z)^\top (x - z)] \\ D_r(x||y) &= r(x) - [r(y) + \nabla r(y)^\top (x - y)], \text{ and} \\ D_r(y||z) &= r(y) - [r(z) + \nabla r(z)^\top (y - z)]. \end{aligned}$$

Consequently,

$$\begin{aligned} D_r(x||z) - D_r(x||y) - D_r(y||z) &= \nabla r(z)^\top (y - x) - \nabla r(y)^\top (x - y) \\ &= (\nabla r(y) - \nabla r(z))^\top (x - y). \end{aligned}$$

However, note that  $\nabla D_r(y||z) = \nabla r(y) - \nabla r(z)$ . By our characterization of the minimization of convex functions over closed convex sets we know that for  $y \in S$  it is the case  $\nabla D_r(y||z)^\top (x - y) \geq 0$  for all  $x \in S$  if and only if  $y = \operatorname{argmin}_{x \in S} D_r(x||z) = \pi_S^r$ .  $\square$

**Lemma 26.** *If  $r : S \rightarrow \mathbb{R}$  is differentiable and convex for closed convex  $S$  and*

$$x_{t+1} = \operatorname{argmin}_{x \in S} \eta \cdot p_t^\top x + D_r(x||x_t)$$

*then for all  $z \in S$  we have*

$$D_r(z||x_{t+1}) - D_r(z||x_t) \leq \eta p_t^\top (z - x_t) + \eta p_t^\top (x_t - x_{t+1}) - D_r(x_{t+1}||x_t).$$

*Consequently, if  $r$  is  $\mu$ -strongly convex with respect to  $\|\cdot\|$  then*

$$D_r(z||x_{t+1}) - D_r(z||x_t) \leq \eta p_t^\top (z - x_t) + \frac{\eta^2}{2\mu} \|p_t\|_*^2.$$

*Proof.* By Lemma 24 we have

$$D_r(z||x_t) - D_r(z||x_{t+1}) - D_r(x_{t+1}||x_t) = \nabla D_r(x_{t+1}||x_t)^\top (z - x_{t+1}).$$

Further, by the optimality conditions for  $x_{t+1}$  we have

$$(\eta p_t + \nabla D_r(x_{t+1}||x_t))^\top (z - x_{t+1}) \geq 0.$$

Combining yields that

$$\begin{aligned} D_r(z||x_{t+1}) - D_r(z||x_t) &\leq -\nabla D_r(x_{t+1}||x_t)^\top (z - x_{t+1}) - D_r(x_{t+1}||x_t) \\ &\leq \eta p_t^\top (z - x_{t+1}) - D_r(x_{t+1}||x_t) \\ &= \eta p_t^\top (z - x_t) + \eta p_t^\top (x_t - x_{t+1}) - D_r(x_{t+1}||x_t). \end{aligned}$$

The result follows by the fact that when  $r$  is  $\mu$ -strongly convex  $D_r(x_{t+1}||x_t) \geq \frac{\mu}{2} \|x_{t+1} - x_t\|^2$  and

$$\max_{z \in \mathbb{R}} \eta p_t^\top (x_t - x_{t+1}) - \frac{\mu}{2} \|x_{t+1} - x_t\|^2 = -\frac{\eta^2}{2\mu} \|p_t\|_*^2.$$

$\square$

We call this a generalized Pythagorean theorem as if  $r = \frac{1}{2}\|x\|_2^2$  then the inequality in this lemma is  $\|x - y\|_2^2 + \|y - z\|_2^2 \leq \|x - z\|_2^2$ .

Using this we see that in mirror descent that for all  $z \in S$  clearly  $D(z||x_{t+1}) \leq D(z||y_t)$ .

**Theorem 27** (Mirror Descent). *Mirror Descent for  $\mu$ -strongly convex  $r$  with respect to  $\|\cdot\|$  where we let  $y_0 = x_0 \in S$  ensures that for all  $z \in S$*

$$\sum_{t \in [T]} p_t^\top (x_t - z) \leq \sum_{t \in [T]} \frac{\eta}{2\mu} \|p_t\|_*^2 + \frac{1}{\eta} D_r(z||x_0).$$

*Proof.* By the previous lemmas we have that for all  $t \in [T]$

$$D_r(z||x_{t+1}) - D_r(z||x_t) \leq \frac{\eta^2}{2\mu} \|p_t\|_*^2 - \eta \cdot p_t^\top (x_t - z).$$

Summing over  $t \in [T]$  and re-arranging terms yields that

$$\sum_{t \in [T]} p_t^\top (x_t - z) \leq \frac{\eta}{2\mu} \sum_{t \in [T]} \|p_t\|_*^2 + \frac{1}{\eta} [D_r(z||x_1) - D_r(z||x_{T+1})].$$

However,  $D_r(z||x_{t+1}) \geq 0$  and  $x_1 = x_0$  yielding the claim  $\square$

Thus we see that we can also use this theorem to get a  $\sqrt{T}$  regret growth rate with essentially the same bounds as those we got for FTRL.

Note that there is another equivalent way of writing mirror descent that makes it look more like gradient descent. Recall that in each iteration of mirror descent we let  $x_{t+1} = \operatorname{argmin}_{x \in S} D(x||y_t)$  where  $\nabla r(y_t) = \nabla r(x_t) - \eta p_t$ . Changing terms that do not affect the minimizer we have

$$\begin{aligned} x_{t+1} &= \operatorname{argmin}_{x \in S} r(x) - [r(y_t) + \nabla r(y_t)^\top (x - y_t)] \\ &= \operatorname{argmin}_{x \in S} r(x) - [r(x_t) + [\nabla r(x_t) - \eta p_t]^\top (x - x_t)] \\ &= \operatorname{argmin}_{x \in S} \eta p_t^\top (x - x_t) + D_r(x||x_t). \end{aligned}$$

Thus, each step is minimizing a weighted combination of the inner product with  $p_t$  and the distance to  $x_t$  as measured by the Bregman divergence. Note when  $p_t$  is a gradient and  $r$  is the  $\ell_2$  squared norm, then this is projected exactly gradient descent (i.e. composite gradient descent where the composite term is an indicator for  $x$  being an element of the set) and gradient descent when  $S = \mathbb{R}^n$ . This is in contrast to FTRL which if it used  $D_r(x||x_0)$  as the regularizer would have

$$x_{t+1} = \operatorname{argmin}_{x \in S} \eta \sum_{t \in [T]} p_t^\top (x - x_0) + D_r(x||x_0).$$

One of the nice things about FTRL is that to compute an  $x_t$  we only need the  $p_t$  and a single minimization over  $S$ . On the other hand, one of the nice things about mirror descent is we only need to know  $x_t$ , a point in  $S$ , and  $p_t$  to determine  $x_{t+1}$ . Each are useful for different reasons and in many cases they lead to the same algorithm.

## 11 Mirror Descent for Smooth Functions

Here we consider the performance of mirror descent on smooth functions. Suppose we ran the same algorithm as before for convex  $L$ -smooth  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and let  $p_t = \nabla f(x_t)$ . Then, letting  $z = x_*$ , the guarantees of

mirror descent yield that

$$\sum_{t \in [T]} [f(x_t) - f(x_*)] \leq \sum_{t \in [T]} p_t^\top (x_t - x_*) \leq \sum_{t \in [T]} \frac{\eta}{2\mu} \|p_t\|_*^2 + \frac{1}{\eta} \cdot D_r(x_* \| x_0)$$

However, we also have that  $\|p_t\|_*^2 = \|\nabla f(x_t)\|_2^2 \leq 2L \cdot [f(x_t) - f(x_*)]$  and thus

$$\sum_{t \in [T]} \left(1 - \frac{\eta L}{\mu}\right) [f(x_t) - f(x_*)] \leq \frac{1}{\eta} D_r(x_* \| x_0)$$

Consequently picking  $\eta = \frac{\mu}{2L}$  we have

$$f\left(\frac{1}{T} \sum_{t \in [T]} x_t\right) - f_* \leq \frac{1}{T} \sum_{t \in [T]} [f(x_t) - f_*] \leq \frac{4L}{\mu T} \cdot D_r(x_* \| x_0).$$

Thus we see that we now get a  $1/T$  rate for smooth minimization. Note that if instead of running mirror descent we ran FTRL, then we could obtain exactly the same guarantee. The  $1/T$  rate stems not from the analysis of mirror descent, but rather just an interpretation of its guarantees and picking  $\eta$  appropriately.

## 12 Stochastic Gradient Descent (SGD) and Finite Sums

Here we consider using the previous algorithms in the case where we pick  $p_t$  to be a random variable so that  $\mathbb{E}p_t \in \partial f(x_t)$ . Such algorithms are known as stochastic gradient descent (SGD). Note that if in each iteration we have that  $\mathbb{E}\|p_t\|_*^2 \leq G^2$  then we get exactly the same bounds as before with the only difference that our regret bounds become expected regret bounds and the function errors we obtain become expected function errors.

To demonstrate the power of this technique, here we show how to use SGD through a technique known as variance reduction to get faster algorithms for minimizing finite sums and more broadly, regression.

### 12.1 The Problem

The main problem we consider in this section is what is known as the *finite sum problem*.

**Definition 28** (Finite Sum Problem). We have a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  that is  $\mu$ -strongly convex such that  $f(x) = \frac{1}{n} \sum_{i \in [n]} f_i(x)$  where each  $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$  is  $L$ -smooth and convex. We wish to minimize  $f$  given some initial point  $x_0 \in \mathbb{R}^d$  and an oracle that when queried at  $x \in \mathbb{R}^d$  and  $i \in [n]$  outputs  $\nabla f_i(x)$ .

This is a commonly occurring problem and encompasses a common setting in machine learning, where we wish to optimize over some set of variables  $x$  and have  $n$  pieces of data each of which should be explained by  $x$ . The terms  $f_i(x)$  are then used to measure how well data point  $x_i$  is explained by  $x$  and  $f$  yields the average explanation quality. Due to the popularity of this problem there are many variants considered, however we will restrict our analysis to this relatively simple form.

A particularly popular (and perhaps simplest) instance of the finite sum problem is the *regression problem*.

**Definition 29** (Regression Problem). Given  $\mathbf{A} \in \mathbb{R}^{n \times d}$  with rows  $a_1, \dots, a_n \in \mathbb{R}^d$  we wish to solve  $\min_{x \in \mathbb{R}^d} f(x)$  where

$$f(x) = \frac{1}{2} \|\mathbf{A}x - b\|_2^2 = \frac{1}{n} \sum_{i \in [n]} \frac{n}{2} (a_i^\top x - b_i)^2 = \frac{1}{2} x^\top \mathbf{A}^\top \mathbf{A} x - b^\top \mathbf{A} x + \frac{1}{2} b^\top b.$$

such that  $\lambda_{\min}(\mathbf{A}^\top \mathbf{A}) \geq \mu$  and  $\|a_i\|_2^2 \leq L$  for all  $i \in [n]$ .

Note that  $\nabla^2 f(x) = \mathbf{A}^\top \mathbf{A}$  and thus  $\lambda_{\min}(\mathbf{A}^\top \mathbf{A}) \geq \mu$  if and only if  $f$  is  $\mu$ -strongly convex. Also note that if we let  $f_i(x) = \frac{n}{2}(a_i^\top x - b_i)^2$  then  $\nabla^2 f_i(x) = n \cdot a_i a_i^\top$  which has eigenvector  $a_i$  of eigenvalue  $n\|a_i\|_2^2$  and all other eigenvectors 0. Consequently  $f_i$  is convex and  $L$ -smooth if and only if  $\|a_i\|_2^2 \leq L/n$  and thus this is an instance of the finite sum problem where the oracle can be implemented in  $O(d)$  time.

Now note that in the finite sum problem clearly  $f$  is  $L$  smooth and since a gradient can be computed with  $n$  oracle evaluations, an  $\epsilon$ -optimal point can be computed by gradient descent with  $O(n \cdot \frac{L}{\mu} \cdot \log((f(x_0) - f_*)/\epsilon))$  calls to the oracle. Moreover, this can be improved by AGD to  $O(n \cdot \sqrt{\frac{L}{\mu}} \cdot \log((f(x_0) - f_*)/\epsilon))$  calls to the oracle. Note that in the case of regression these correspond to running times of  $O(nd \cdot \frac{L}{\mu} \cdot \log((f(x_0) - f_*)/\epsilon))$  and  $O(nd \cdot \sqrt{\frac{L}{\mu}} \cdot \log((f(x_0) - f_*)/\epsilon))$ .

The main question of this section is can these be improved. We will show that the factor of  $n$  can be improved using SGD and sampling techniques at least over regression. Accelerated gains can be made as well, however we will simply provide references for those improvements. Again, this is an active area of research and there are more results than we can present here, the goal in this section is simply to show how sampling can provide general improvements over gradient descent.

## 12.2 Warm-up: Direct SGD

So how should we try to improve over gradient descent for the finite sum problem? Perhaps the most natural idea would simply be to perform SGD, i.e in iteration  $k$  pick  $i_k \in [n]$  at random and let  $p_k = \nabla f_{i_k}(x_k)$  and use this for mirror descent. Note that

$$\mathbb{E} p_k = \frac{1}{n} \sum_{i \in [n]} \nabla f_i(x_k) = \nabla f(x_k)$$

While we could do this in the full generality of the situation we do mirror descent or FTRL, here for completeness we analyze this directly in the simple case where our regularizer is  $r(x) = \frac{1}{2}\|x\|_2^2$  and we work with  $\|\cdot\|_2^2$ , i.e. our algorithm is  $x_{k+1} = x_k - \eta \cdot p_k$ . We provide this lemma with these settings but a slightly more general formulation for how  $p_k$  is chosen.

**Lemma 30** (Specialized SGD). *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a differentiable convex function and for all  $k \geq 1$  let*

$$x_{k+1} = x_k - \eta g_k$$

*where conditioned on all  $g_j$  for  $j \leq k$  we have  $\mathbb{E} g_k = \nabla f(x_k)$  and  $\mathbb{E} \|g_k\|_2^2 \leq \sigma_k^2$  for some  $\sigma_k$ . Then, for each iteration  $k$  conditioned on the value of  $x_k$  we have*

$$2\eta \cdot [\mathbb{E} f(x_k) - f_*] \leq \|x_k - x_*\|_2^2 - \mathbb{E}_{g_k \sim \mathcal{D}} \|x_{k+1} - x_*\|_2^2 + \eta^2 \cdot \mathbb{E}_{g_k \sim \mathcal{D}} \|\nabla g_k(x_k)\|_2^2$$

*Let  $\mathcal{D}$  be a distribution over convex  $L$ -smooth functions  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  and let  $f(x) \stackrel{\text{def}}{=} \mathbb{E}_{g \sim \mathcal{D}} g(x)$ . Suppose for some  $x_1 \in \mathbb{R}^d$  and all  $k \geq 1$  we let  $x_{k+1} = x_k - \eta \nabla g_k(x_k)$  where  $g_k \sim \mathcal{D}$  independently at random. Then, for each iteration  $k$  conditioned on the value of  $x_k$  we have*

$$2\eta \cdot [\mathbb{E} f(x_k) - f_*] \leq \|x_k - x_*\|_2^2 - \mathbb{E}_{g_k \sim \mathcal{D}} \|x_{k+1} - x_*\|_2^2 + \eta^2 \cdot \mathbb{E}_{g_k \sim \mathcal{D}} \|\nabla g_k(x_k)\|_2^2$$

*and therefore for all  $T \geq 0$  we have*

$$\sum_{k \in [T]} \frac{1}{T} \mathbb{E} f(x_k) - f_* \leq \frac{\eta}{2T} \sum_{k \in [T]} \mathbb{E} \|\nabla g_k(x_k)\|_2^2 + \frac{1}{2\eta T} \|x_1 - x_*\|_2^2.$$

*If we further assume that  $f$  is  $\mu$ -strongly convex this simplifies to*

$$\sum_{k \in [T]} \frac{1}{T} \mathbb{E} f(x_k) - f_* \leq \frac{\eta}{2T} \sum_{k \in [T]} \mathbb{E} \|\nabla g_k(x_k)\|_2^2 + \frac{1}{\eta \mu T} [f(x_1) - f_*].$$

*Proof.* Direct algebraic manipulation yields that

$$\begin{aligned}\mathbb{E}_{g_k \sim \mathcal{D}} \|x_{k+1} - x_*\|_2^2 &= \mathbb{E}_{g_k \sim \mathcal{D}} \|x_k - x_* - \eta \nabla g_k(x_k)\|_2^2 \\ &= \mathbb{E}_{g_k \sim \mathcal{D}} [\|x_k - x_*\|_2^2 - 2\eta \nabla g_k(x_k)^\top (x_k - x_*) + \eta^2 \|\nabla g_k(x_k)\|_2^2] \\ &= \|x_k - x_*\|_2^2 - 2\eta \nabla f(x_k)^\top (x_k - x_*) + \eta^2 \mathbb{E}_{g_k \sim \mathcal{D}} \|\nabla g_k(x_k)\|_2^2\end{aligned}$$

Since  $f(x_*) \geq f(x_k) + \nabla f(x_k)^\top (x_* - x_k)$  by convexity we have that

$$2\eta \cdot [f(x_k) - f_*] \leq \|x_k - x_*\|_2^2 - \mathbb{E}_{g_k \sim \mathcal{D}} \|x_{k+1} - x_*\|_2^2 + \eta^2 \cdot \mathbb{E}_{g_k \sim \mathcal{D}} \|\nabla g_k(x_k)\|_2^2$$

yielding the first inequality. Summing, taking expectation over all the  $g_k$ , and noting that the  $\mathbb{E}\|x_k - x_*\|_2^2$  cancel yields

$$2\eta \sum_{k \in [T]} [f(x_k) - f_*] \leq \sum_{k \in [T]} \eta^2 \mathbb{E} \|\nabla g_k(x_k)\|_2^2 + \|x_1 - x_*\|_2^2 - \mathbb{E} \|x_{T+1} - x_*\|_2^2.$$

Minor algebraic manipulations and the fact that  $f(x_1) \geq f(x_*) + \frac{\mu}{2} \|x_1 - x_*\|_2^2$  when  $f$  is strongly convex yields the result.  $\square$

This lemma reduces much of the analysis of SGD to bounding  $\mathbb{E} \|\nabla g_k(x_k)\|_2^2$ . As we have seen throughout our analysis of mirror descent the smaller we can show this quantity is, the faster we can converge. Moreover, as we have seen in the analysis of mirror descent for smooth functions, if we can somehow relate this quantity to  $f(x_k) - f_*$  then we can simply have our error decay as the difference from  $x_0$  to  $x_*$  over a linear term. Since, when our function is strongly convex, we can in turn relate this to our initial function error and hope to halve the error after some fixed number of iterations and achieve an algorithm a  $O(\log \epsilon^{-1})$  dependence on error.

In short, to have any hope of competing with GD we need to ensure  $\mathbb{E} \|\nabla g_k(x_k)\|_2^2$  decays as  $f(x_k) - f_*$  goes to 0. However, note that it might be the case that  $\mathbb{E} \|\nabla g_k(x_*)\|_2^2 \neq 0$  and consequently as we get closer to the minimizer we still might have a fixed lower bound on how big  $\mathbb{E} \|\nabla g_k(x_k)\|_2^2$  thereby impinging on our ability to get a  $O(\log \epsilon^{-1})$  rate. In the next section we show how to deal with this issue. For now we explore what rates we can get depending on what  $\sigma^2 \stackrel{\text{def}}{=} \mathbb{E} \|\nabla g_k(x_*)\|_2^2$ .

The first step in this analysis is to bound  $\mathbb{E} \|\nabla g_k(x_k) - \nabla g_k(x_*)\|_2^2$ . This will allow us to break the contribution of  $\mathbb{E} \|\nabla g_k(x_k)\|_2^2$  into the contribution to this term and  $\sigma^2$ . Note that if there was no distribution, we would have that this is at most  $2L \cdot [f(x_k) - f_*]$  by smoothness. Below we show the same holds.

**Lemma 31** (Variance Bound). *Let  $\mathcal{D}$  be a distribution over convex  $L$ -smooth functions  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  and let  $f(x) \stackrel{\text{def}}{=} \mathbb{E}_{g \sim \mathcal{D}} g(x)$  and  $x_*$  be a minimizer of  $f$ . Then for all  $x \in \mathbb{R}^n$  we have  $\mathbb{E}_{g \sim \mathcal{D}} \|\nabla g(x) - \nabla g(x_*)\|_2^2 \leq 2L \cdot [f(x) - f_*]$ .*

*Proof.* For all  $g \in \text{supp}(\mathcal{D})$  since  $g$  is convex and  $L$ -smooth then by Lemma 8 in Chapter 5 we have

$$\mathbb{E}_{g \sim \mathcal{D}} \|\nabla g(x) - \nabla g(x_*)\|_2^2 \leq \mathbb{E}_{g \sim \mathcal{D}} 2L \cdot [g(x) - [g(x_*) - \nabla g(x_*)^\top (x - x_*)]]$$

and the result follows from the definition of  $f$  and the fact that  $\mathbb{E}_{g \sim \mathcal{D}} \nabla g(x_*) = \nabla f(x_*) = 0$  by the optimality of  $x_*$ .

Using this, we show that if  $\sigma^2 = 0$  then we can achieve our desired rate. Note that  $\sigma^2 = 0$  implies that for all  $g \in \text{supp}(\mathcal{D})$  it is the case that  $\nabla g(x_*) = 0$ . In the case of regression, where  $g_k(x_k) = \frac{n}{2} \cdot (a_{i_k}^\top x_k - b_{i_k})^2$  for some  $i_k \in [n]$  we have that  $\nabla g_k(x_k) = n \cdot a_{i_k} \cdot (a_{i_k}^\top x_k - b_{i_k})$  and thus assuming that the  $a_i \neq 0$  this implies that  $a_i^\top x_* = b_i$  for all  $i \in [n]$  or in other words,  $\mathbf{A}x_* = b$ . In other words, the following gives us bounds for solving linear systems  $\mathbf{A}x = b$  when there is a  $x_*$  such that  $\mathbf{A}x_* = b$ .  $\square$

**Lemma 32.** *In the setup of Lemma 30 if  $\sigma^2 = \mathbb{E}_{g \sim \mathcal{D}} \|\nabla g(x_*)\|_2^2 = 0$  then for all  $k \geq 1$  if  $\eta = \frac{1}{2L}$*

$$\mathbb{E} \|x_k - x_*\|_2^2 \leq \left(1 - \frac{\mu}{4L}\right) \mathbb{E} \|x_{k-1} - x_*\|_2^2.$$

Consequently,

$$\mathbb{E} \|x_{k+1} - x_*\|_2^2 \leq \left(1 - \frac{\mu}{4L}\right)^k \|x_1 - x_*\|_2^2$$

and therefore can compute an expected  $\epsilon$ -optimal point with  $O\left(\frac{L}{\mu} \cdot \log\left(\frac{L}{\mu} \cdot \frac{f(x_1) - f_*}{\epsilon}\right)\right)$  oracle evaluations.

*Proof.* By Lemma 30 we have

$$\mathbb{E}_{g_k \sim \mathcal{D}} \|x_{k+1} - x_*\|_2^2 \leq \|x_k - x_*\|_2^2 - 2\eta \cdot [\mathbb{E}_{g_k \sim \mathcal{D}} f(x_k) - f_*] + \eta^2 \cdot \mathbb{E}_{g_k \sim \mathcal{D}} \|\nabla g_k(x_k)\|_2^2$$

However, since  $\sigma^2 = 0$  by Lemma 31 we have

$$\mathbb{E}_{g_k \sim \mathcal{D}} \|\nabla g_k(x_k)\|_2^2 = \mathbb{E}_{g_k \sim \mathcal{D}} \|\nabla g_k(x_k) - \nabla g_k(x_*)\|_2^2 \leq 2 \cdot L \cdot [f(x_k) - f_*].$$

Consequently we have

$$\mathbb{E}_{g_k \sim \mathcal{D}} \|x_{k+1} - x_*\|_2^2 \leq \|x_k - x_*\|_2^2 - 2\eta(1 - L\eta) \cdot \mathbb{E}_{g_k \sim \mathcal{D}} [f(x_k) - f_*]$$

Since for  $\eta = 1/(2L)$  we have  $2\eta(1 - L\eta) = 1/(2L)$  and since  $f(x_k) - f_* \geq \frac{\mu}{2} \|x_k - x_*\|_2^2$  by strong convexity the first result holds. Since  $f$  is  $L$ -smooth we have that  $f(x_k) - f_* \leq \frac{L}{2} \|x_k - x_*\|_2^2$  and therefore  $\mathbb{E} f(x_{k+1}) - f_* \leq \frac{L}{\mu} \left(1 - \frac{\mu}{4L}\right)^k [f(x_1) - f_*]$  yielding the second claim.  $\square$

Note that this says we can achieve our desired running time for regression and minimizing finite sums when  $\sigma^2 = 0$ . What about when  $\sigma^2 \neq 0$ ? We analyze this case in the remainder of this section.

In this case we have the following.

**Lemma 33.** *In the setup of Lemma 30 if  $\sigma^2 = \mathbb{E}_{g \sim \mathcal{D}} \|\nabla g(x_*)\|_2^2 > 0$  and  $f$  is  $\mu$ -strongly convex then for all  $k \geq 1$  we have*

$$(1 - 2\eta \cdot L) \left[ \frac{1}{T} \sum_{k \in [T]} \mathbb{E} f(x_k) - f_* \right] \leq \eta \cdot \sigma^2 + \frac{1}{\eta \mu T} [f(x_1) - f_*]$$

*Proof.* By Lemma 30 we have

$$\sum_{k \in [T]} \frac{1}{T} \mathbb{E} f(x_k) - f_* \leq \frac{\eta}{2T} \sum_{k \in [T]} \mathbb{E} \|\nabla g_k(x_k)\|_2^2 + \frac{1}{\eta \mu T} [f(x_1) - f_*].$$

However, since  $\|a + b\|_2^2 \leq 2\|a\|_2^2 + 2\|b\|_2^2$  (as  $\|a + b\|_2^2 + \|a - b\|_2^2 = 2 \cdot \|a\|_2^2 + 2\|b\|_2^2$ ) we have that

$$\sum_{k \in [T]} \mathbb{E} \|\nabla g_k(x_k)\|_2^2 = \sum_{k \in [T]} \mathbb{E} [2\|\nabla g_k(x_k) - \nabla g_k(x_*)\|_2^2 + 2\|\nabla g_k(x_*)\|_2^2] \leq 4L \cdot \sum_{k \in [T]} [\mathbb{E} f(x_k) - f_*] + 2T \cdot \sigma^2$$

yielding the desired result.  $\square$

Thus we see that we can get fast rates proved we can related  $\sigma^2$  to  $f(x_1) - f_*$ . This is precisely what we show how to do in the next section.

### 12.3 Variance Reduction

From the analysis in the previous section it seems that if we could find a way to relate  $\sigma^2$  to  $f(x_1) - f_*$  in the finite sum problem the we can achieve fast rates. Another way to say this, is that the problem with making SGD faster than gradient descent for regression is finding away to ensure that  $\mathbb{E}_{g \sim \mathcal{D}} \|\nabla g(x_*)\|_2^2$  decays over time.

To do this we simply want to write  $f(x) = \mathbb{E}_{g \sim \mathcal{D}} g(x)$  for better  $g$ . We have already seen the effects of picking  $g = f_i$  with probability  $\frac{1}{n}$ . The question is how to do better? One natural idea is to pick  $g(x) = f_i(x) - \nabla f_i(x_*)^\top x$  with probability  $\frac{1}{n}$ . Note that  $\mathbb{E}_{g \sim \mathcal{D}} g(x) = f(x) - \nabla f(x_*)^\top x = f(x)$  and thus this is a valid transformation. Furthermore, we have  $\nabla g(x_*) = 0$  for all  $g$  in the distribution. This works quite well, but unfortunately has the problem of requiring  $x_*$  to be computed. Another, idea would be to perform the same transformation, but using  $x_0$  as a proxy for  $x_*$ , this would yield  $g(x) = f_i(x) - \nabla f_i(x_1)^\top x$  with probability  $\frac{1}{n}$ . However, unfortunately,  $\mathbb{E}g(x) = f(x) - \nabla f(x_1)^\top x$ , which is not necessarily  $f(x)$ , and therefore is biased.

To fix all these issues what we actually do is let  $g(x) = f_i(x) - \nabla f_i(x_1)^\top x + \nabla f(x_1)^\top x$  with probability  $\frac{1}{n}$ . This has that  $\mathbb{E}g(x) = f(x)$  as desired. The corresponding algorithm is known as stochastic variance reduced gradient. To analyze it, all we have to do is bound  $\sigma^2$  for this particular  $g$ , i.e.  $\mathbb{E}_{g \sim \mathcal{D}} \|\nabla g(x_*)\|_2^2 = \sum_{i \in [n]} \frac{1}{n} \|\nabla f_i(x_*) - \nabla f_i(x_1) + \nabla f(x_1)\|_2^2$ .

**Lemma 34.** *Let  $\mathcal{D}$  be a distribution over convex  $L$ -smooth functions  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  and let  $f(x) \stackrel{\text{def}}{=} \mathbb{E}_{g \sim \mathcal{D}} g(x)$ ,  $x_1 \in \mathbb{R}^d$ , and  $x_*$  be a minimizer of  $f$  then*

$$\mathbb{E}_{g \sim \mathcal{D}} \|\nabla g(x_*) - \nabla g(x_1) + \nabla f(x_1)\|_2^2 \leq 2L \cdot [f(x_1) - f_*].$$

*Proof.* Note that for any random  $X \in \mathbb{R}^d$  we have

$$\mathbb{E}\|X - \mathbb{E}X\|_2^2 = \mathbb{E}\|X\|_2^2 - 2\mathbb{E}X^\top \mathbb{E}X + \|\mathbb{E}X\|_2^2 = \mathbb{E}\|X\|_2^2 - \|\mathbb{E}X\|_2^2 \leq \mathbb{E}\|X\|_2^2.$$

and consequently

$$\begin{aligned} \mathbb{E}_{g \sim \mathcal{D}} \|\nabla g(x_*) - \nabla g(x_0) + \nabla f(x_0)\|_2^2 &= \mathbb{E}_{g \sim \mathcal{D}} \|\nabla g(x_*) - \nabla g(x_0) - [\nabla f(x_*) - \nabla f(x_0)]\|_2^2 \\ &\leq \mathbb{E}_{g \sim \mathcal{D}} \|\nabla g(x_*) - \nabla g(x_0)\|_2^2. \end{aligned}$$

The result then follows from Lemma 31. □

This lemma then says we can bound  $\sigma^2$  by  $2L \cdot [f(x_1) - f_*]$ .

**Theorem 35** (SVRG). *Suppose in the finite sum problem for some  $x_1 \in \mathbb{R}^d$  and all  $k \geq 0$  we let*

$$x_{k+1} = x_k - \eta [\nabla f_i(x_k) - \nabla f_i(x_1) + \nabla f(x_1)]$$

*then for  $\eta \in (0, \frac{1}{2L})$  we have*

$$\frac{1}{T} \sum_{k \in [T]} \mathbb{E}f(x_k) - f_* \leq \frac{1}{1 - 2\eta \cdot L} \left[ 2L\eta + \frac{1}{\eta\mu T} \right] \cdot [f(x_1) - f_*].$$

*and consequently we can compute an expected  $\epsilon$ -optimal point with  $O((n + \frac{L}{\mu}) \log(\frac{f(x_1) - f_*}{\epsilon}))$  queries to the oracle*

*Proof.* As discussed, this algorithm is SGD where in each iteration  $k$  the stochastic gradient is  $\nabla g_k(x)$  for  $g_k \sim \mathcal{D}$  where  $g_k(x) \stackrel{\text{def}}{=} f_{i_k}(x) - [\nabla f_{i_k}(x_1) - \nabla f(x_1)]^\top x$  for  $i_k \in [n]$  chosen uniformly at random. Now, note that each  $g_k$  is convex and  $L$ -smooth. Further, by Lemma 34 this implies

$$\sigma^2 = \mathbb{E}_{g_k \sim \mathcal{D}} \|\nabla g_k(x_*)\|_2^2 = \mathbb{E}_{i_k} \|\nabla f_{i_k}(x_*) - \nabla f_{i_k}(x_1) + \nabla f(x_1)\|_2^2 \leq 2L \cdot [f(x_1) - f_*].$$

Consequently, Lemma 33 implies that

$$(1 - 2\eta \cdot L) \sum_{k \in [T]} \left[ \frac{1}{T} \mathbb{E} f(x_k) - f_* \right] \leq 2L\eta \cdot [f(x_1) - f_*] + \frac{1}{\eta\mu T} [f(x_1) - f_*]$$

yielding the first claim. Now, if we pick  $\eta = \frac{1}{8L}$  and  $T = 64\frac{L}{\mu}$  we have

$$\frac{1}{T} \sum_{k \in [T]} \mathbb{E} f(x_k) - f_* \leq 2 \cdot \left[ \frac{1}{4} + \frac{1}{8} \right] \cdot [f(x_1) - f_*] \leq \frac{3}{4} \cdot [f(x_1) - f_*]$$

Consequently we can decrease the function error by a constant factor with  $O(\frac{L}{\mu})$  iterations. Repeating this (i.e. taking either the average point or a random point as the next  $x_1$ ) and repeating then yields the claim. Each time it is repeated we need  $n$  oracle calls to get  $\nabla f(x_1)$  and an additional  $\frac{L}{\mu}$ , 1 for each of the iterations.  $\square$

Consequently, we can generically improve upon gradient descent. Furthermore, it can be shown that there are cases where  $L/\mu < 1$  and thus the number of oracle calls needed scales just linearly with  $n$  in these cases.

## 12.4 Extensions

This is an incredibly well studied problem and a very active area of research. There are numerous extensions that can and are considered. Here, we briefly state a few. First, if  $f_i$  is  $L_i$  smooth (rather than  $L$  smooth) non-uniform sampling can be applied to achieve an  $\tilde{O}(n + \frac{1}{n} \sum_{i \in [n]} \frac{L_i}{\mu})$  query algorithm. Furthermore, this can be accelerated to yield a  $\tilde{O}(n + \sum_{i \in [n]} \sqrt{\frac{L_i}{n\mu}})$  query algorithm, note that this is always an improvement as by Cauchy Schwarz and  $|ab| \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$  we have

$$\sum_{i \in [n]} \sqrt{\frac{L_i}{n\mu}} \leq \sqrt{\left( \sum_{i \in [n]} \frac{L_i}{n\mu} \right) \cdot \left( \sum_{i \in [n]} 1 \right)} \leq \frac{1}{2} \left( \frac{1}{n} \sum_{i \in [n]} \frac{L_i}{\mu} + n \right).$$

There are also composite analogs and it can be shown that there are interesting connections between these algorithms and coordinate descent (in some sense SVRG is an analog of running coordinate descent in the dual of the problem). If you would like further references to these, please post on Piazza.