Inviscid stability analysis of parallel bubbly flows

Suhas S Jain

ME451B Final Project Presentation,
Stanford University

Wednesday 12$^{\text{th}}$ December, 2018

Stanford University

Introduction and Motivation

motivation

- bubbly flows are ubiquitous in nature.
- even at low void fractions, their presence can significantly change
  - sound speed
  - attenuation characteristics
  - inertia of the medium
- crucial to understand the dynamical properties of the medium.

objectives

- derive governing equations and disturbance relations for bubble-liquid mixture.
- perform stability analysis of spacewise problem of inviscid bubbly shear flow, following d'Agostino et al., *JFM*, 1997.
- study the effect of presence of bubbles.

Stanford University

Basic equations

Individual phase continuity equation (IPCE):

$$\frac{\partial \rho_i \alpha_i}{\partial t} + \vec{\nabla} \cdot (\rho_i \vec{u}_i \alpha_i) = 0,$$

ρ_i and α_i are the density and volume fraction of phase i.

Individual phase momentum equation (IPME):

$$\frac{\partial \rho_i \vec{u}_i}{\partial t} + \vec{\nabla} \cdot (\rho_i \vec{u}_i \otimes \vec{u}_i + p_i \mathbb{1}) = 0,$$

Stanford University

Derivation of mixture continuity equation

Starting with IPCE for liquid phase:

$$\frac{\partial \rho_l \alpha_l}{\partial t} + \vec{\nabla} \cdot (\rho_l \vec{u}_l \alpha_l) = 0,$$

Rewrite,

$$\frac{1}{\rho_l} \frac{D\rho_l}{Dt} + \frac{1}{\alpha_l} \frac{D\alpha_l}{Dt} + \vec{\nabla} \cdot \vec{u}_l = 0,$$

If $p = f(\rho, s)$, for an isentropic process,

$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt}$$

where, c is the speed of sound. Using this above,

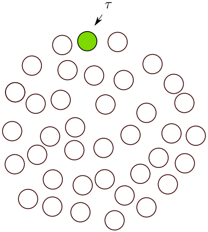
$$\frac{1}{\rho_l c_l^2} \frac{Dp_l}{Dt} + \frac{1}{\alpha_l} \frac{D\alpha_l}{Dt} + \vec{\nabla} \cdot \vec{u}_l = 0,$$

Stanford University

Introducing terminologies

Let,

- β - number of bubbles per unit liquid volume
- n - number of bubbles per unit total volume
- τ - individual bubble volume
- α_b - volume fraction of bubbles $= n\tau$



Now,

$$1 + \beta\tau = 1 + (\frac{\#}{liq.\ vol.})\tau = 1 + \frac{gas\ vol.}{liq.\ vol.} = \frac{tot.\ vol.}{liq.\ vol.}$$

$$n = \frac{\#}{tot.\ vol.} = \frac{\#}{liq.\ vol.} * \frac{liq.\ vol}{tot.\ vol} = \frac{\beta}{1+\beta\tau}$$

$$\alpha_b = \frac{\beta\tau}{1+\beta\tau}$$

Substituting in the equation,

Mixture continuity equation:

$$\left(\frac{1}{1+\beta\tau}\right)\frac{D\beta\tau}{Dt} - \frac{1}{\rho_l c_l^2}\frac{Dp_l}{Dt} = \vec{\nabla}\cdot\vec{u}_l,$$

where, $D/Dt = \partial/\partial t + \vec{u}_l \cdot \vec{\nabla}$ and $\tau = 4/3\pi R^3$

(assuming spherical bubbles)

Stanford University

Mixture momentum equation

Start with IPME for liquid phase:

$$\frac{\partial \rho_l \vec{u}_l}{\partial t} + \vec{\nabla} \cdot (\rho_l \vec{u}_l \otimes \vec{u}_l + p_l \mathbb{1}) = 0,$$

Rewriting,

$$\vec{u}_l \left\{ \frac{\partial \rho_l \alpha_l}{\partial t} + \vec{\nabla} \cdot (\rho_l \alpha_l \vec{u}_l) \right\} + \rho_l \alpha_l \left\{ \frac{\partial \vec{u}_l}{\partial t} + \vec{u}_l \cdot \vec{\nabla} \vec{u}_l \right\} = -\vec{\nabla} p_l,$$

Mixture momentum equation:

$$\rho_l (1 - \alpha_b) \frac{D\vec{u}_l}{Dt} = -\vec{\nabla} p_l$$

Stanford University

Closure

Assuming volumetric mode of oscillation of the bubbles,

modified Rayleigh-Plesset equation (also called as Keller-Miksis equation)

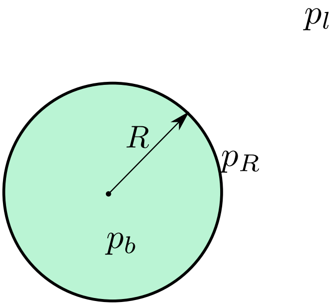
$$\left(1-\frac{1}{c_l}\dot{R}\right)R\ddot{R}+\frac{3}{2}\dot{R}^2\left(1-\frac{1}{3c_l}\dot{R}\right)=\left(1+\frac{1}{c_l}\dot{R}\right)\left\{\frac{p_R(t)+p_l(t+R/c_l)}{\rho_l}\right\}+\frac{R}{\rho_l c_l}\dot{p_R}(t)$$

where, dots are D/Dt, p_R is the liquid pressure at bubble surface and p_l is the driving pressure.

Boundary condition

$$p_b(t)=p_R(t)+2\frac{\sigma}{R}+\frac{4\mu\dot{R}}{R}$$

where, p_b is the uniform bubble internal pressure, σ is the surface tension and μ is the liquid viscosity.



[Keller & Miksis, *J. Acoust. Soc. Am.*, 1980.]

Stanford University

Final system

Mixture continuity equation:

$$\left(\frac{1}{1+\beta\tau}\right)\frac{D\beta\tau}{Dt} - \frac{1}{\rho_l c_l^2}\frac{Dp_l}{Dt} = \vec{\nabla}\cdot\vec{u}_l,$$

Mixture momentum equation:

$$\rho_l(1-\alpha_b)\frac{D\vec{u}_l}{Dt} = -\vec{\nabla}p_l$$

modified Rayleigh-Plesset equation (also called as Keller-Miksis equation):

$$\left(1-\frac{1}{c_l}\dot{R}\right)R\ddot{R} + \frac{3}{2}\dot{R}^2\left(1-\frac{1}{3c_l}\dot{R}\right) = \left(1+\frac{1}{c_l}\dot{R}\right)\left\{\frac{p_R(t)+p_l(t+R/c_l)}{\rho_l}\right\} + \frac{R}{\rho_l c_l}\dot{p_R}(t)$$

Boundary condition:

$$p_b(t) = p_R(t) + 2\frac{\sigma}{R} + \frac{4\mu\dot{R}}{R}$$

Stanford University

Stability analysis of 2D parallel flows

Let,

- $u_l = U(y)\hat{e}_x + \tilde{u}(x, y, t)$ and $v_l = \tilde{v}(x, y, t)$
- $p_l = p_0 + \tilde{p}(x, y, t)$
- $R_l = R_0 + \tilde{R}(x, y, t)$

Let, $\alpha_b \to \alpha$, $\rho_l \to \rho$, $c_l \to c$,

Substituting these in mass and momentum equations, linearizing and subtracting base flow and (assuming β to be uniform),

disturbance mass equation

$$\left(\frac{3\alpha}{R_0}\right)\frac{\hat{D}\tilde{R}}{\hat{D}t} - \frac{1}{\rho c^2}\frac{\hat{D}\tilde{p}}{\hat{D}t} = \vec{\nabla}\cdot\vec{\tilde{u}},$$

where, $\hat{D}/\hat{D}t = \partial/\partial t + U\partial/\partial x$

disturbance momentum equation

$$\rho(1-\alpha)\left\{\frac{\partial\tilde{u}}{\partial t} + U\frac{\partial\tilde{u}}{\partial x} + U'\tilde{v}\right\} = -\frac{\partial\tilde{p}}{\partial x}$$

$$\rho(1-\alpha)\left\{\frac{\partial\tilde{v}}{\partial t} + U\frac{\partial\tilde{v}}{\partial x}\right\} = -\frac{\partial\tilde{p}}{\partial y}$$

where, prime denotes $\partial/\partial y$

Stanford University

Continued

If gas is assumed to behave polytropically, then,

$$p_b = p_{b0}\left(\frac{R_0}{R}\right)^{3\gamma}$$

Linearizing,

$$p_b = p_{b0}\left(1 - 3\gamma\frac{R_0}{R}\right)$$

Substituting these in Keller-Miksis equation and the boundary condition, linearizing and subtracting base flow,

disturbance equation for bubble dynamic response

$$\rho\ddot{\tilde{R}} + p_{b0}\frac{3\gamma\tilde{R}}{R_0^2} - \frac{2\sigma\tilde{R}}{R_0^3} + \frac{4\mu\dot{\tilde{R}}}{R_0^2} + p_{b0}\frac{3\gamma\dot{\tilde{R}}}{cR_0} - \frac{2\sigma\dot{\tilde{R}}}{cR_0^2} + \frac{4\mu\ddot{\tilde{R}}}{R_0 c} = -\frac{\tilde{p}}{R_0}$$

where, dots represent $\hat{D}/\hat{D}t$

Stanford University

Normal mode assumption

Now, making an ansatz for the disturbance,

- $\tilde{u} = \hat{u}(y)e^{i(kx-\omega t)}$
- $\tilde{v} = \hat{v}(y)e^{i(kx-\omega t)}$
- $\tilde{p} = \hat{p}(y)e^{i(kx-\omega t)}$
- $\tilde{R} = \hat{R}(y)e^{i(kx-\omega t)}$

Substituting these in the disturbance equations,

disturbance mass equation

$$ik\hat{u} + \hat{v}' = -i\frac{3\gamma}{R_0}\omega_L\hat{R} + i\frac{\omega_L}{\rho c^2}\hat{p}$$

disturbance momentum equation

$$\rho(1-\alpha)(-i\omega_L\hat{u} + U'\hat{v}) = -ik\hat{p}$$

$$\rho(1-\alpha)(i\omega_L\hat{v}) = \hat{p}'$$

where, $\omega_L = \omega - Uk$ is the Lagrangian frequency.

Stanford University

Continued

disturbance equation for bubble dynamic response

$$\Big(-\underbrace{\omega_L^2}_{inertial}-\underbrace{i\omega_L\lambda}_{damping}+\underbrace{\omega_b^2}_{compressiblity}\Big)\hat{R}=-\Big(1+i\frac{\omega_L R_0}{c}\Big)\frac{\hat{p}}{\rho R_0}$$

where,

$$\lambda=\underbrace{\frac{\omega_L^2 R_0}{c}}_{acoustical}-\underbrace{\frac{4\mu}{\rho R_0^2}}_{viscous}+\big(thermal=0\big)$$

is the damping coefficient and

$$\omega_b^2=\frac{p_{b0}3\gamma}{\rho R_0^2}-\frac{2\sigma}{R_0^3}$$

is the natural frequency of the bubble

Stanford University

Dispersion relation for homogeneous medium

Let both x and y be homogeneous.

Repeating the whole process again,

Then eliminating $\hat{u}$, $\hat{v}$ and $\hat{R}$ from the 4 disturbance equations $\Rightarrow$ wave equation for $\hat{p}$,

$$\underbrace{\left[\left\{\frac{3\alpha}{R_0^2}\frac{(1-\alpha)(1+i\omega\frac{R_0}{c})}{(-\omega^2-i\omega\lambda+\omega_b^2)}+\frac{(1-\alpha)}{c^2}\right\}\omega^2-\left\{k_x^2+k_y^2\right\}\right]}_{=0\ \Rightarrow\ dispersion\ relation}\hat{p}=0$$

speed of propagation of harmonic disturbance ω in the bubbly mixture medium

$$\frac{1}{c_m^2(\omega)}=\frac{3\alpha}{R_0^2}\frac{(1-\alpha)(1+i\omega\frac{R_0}{c})}{(-\omega^2-i\omega\lambda+\omega_b^2)}+\frac{(1-\alpha)}{c^2}$$

Stanford University

Going back to the parallel flow setup

Eliminating $\hat{R}$ and $\hat{p}$ from the 4 disturbance equations,

equivalent Rayleigh system for bubbly flows

$$\hat{u}' = ik\hat{v} - i\frac{U''}{\omega_L}\hat{v} - i\frac{U'}{kc_m^2(\omega_L)}\left(i\omega_L\hat{u} - U'\hat{v}\right)$$

$$\hat{v}' = -ik\hat{u} + \frac{\omega_L}{kc_m^2(\omega_L)}\left(i\omega_L\hat{u} - U'\hat{v}\right)$$

In the limit of $c_m \to \infty$,

$$\hat{u}' = ik\hat{v} - i\frac{U''}{\omega_L}\hat{v}$$

$$\hat{v}' = -ik\hat{u}$$

Eliminating $\hat{u}$, it reduces to the classical Rayleigh equation,

$$(U - c)(D^2 - k^2)\hat{v} - U''\hat{v} = 0$$

where, D is $\partial/\partial y$

Stanford University

Base flow and boundary conditions

Base flow:

Inviscid shear layer

$$U(y) = \frac{U_1 + U_2}{2} + \frac{U_2 - U_1}{2} tanh(\frac{y}{\delta})$$

Boundary conditions:

When $y \to \pm\infty \Rightarrow U = const$, then the system reduces to,

$$\hat{u}' = ik\hat{v}$$

$$\hat{v}' = -ik\hat{u} + i\frac{\omega_L^2}{kc_m^2(\omega_L)}\hat{u}$$

This admits a close form solution as,

Asymptotic solutions

$$\hat{v} = Ae^{\pm y(k^2 - \omega_L^2/c_m^2)^{1/2}}$$

$$\hat{u} = \pm A\frac{ik}{(k^2 - \omega_L^2/c_m^2)^{1/2}} e^{\pm y(k^2 - \omega_L^2/c_m^2)^{1/2}}$$

where A is an arbitrary complex constant.

Stanford University

Solution procedure

Method: Shooting method.
Spacewise problem: complex k and real ω is assumed.

Procedure

- Guess a complex eigenvalue k.
- Choose A such that initial conditions at $y = -n\delta$ simplifies, where $n \gg 1$ ($n = 5$ in this project)

$$\hat{v} = 1$$

$$\hat{u} = \frac{ik}{(k^2 - \omega_L^2/c_m^2)^{1/2}}$$

- Integrate upto $y = n\delta$ (using RK4 in this project).
- Check if the solution is continuous with the asymptotic solution at $y = n\delta$

$$\hat{u} = -\frac{ik}{(k^2 - \omega_L^2/c_m^2)^{1/2}}\hat{v}$$

- Iteratively correct eigenvalue k until convergence (using 2D Newton-Raphson method in this project)

Stanford University

Verification of the solver

- d'Agostino et al. (1997) verified their solver against Michalke, *JFM*, (1965) results for $c_m \to \infty$.
- same reference used here.

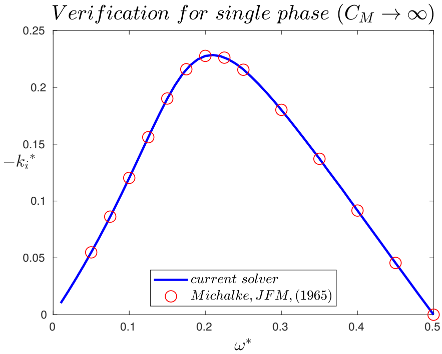


Figure 1: Verification against values from Table 1. of Michalke, *JFM*, (1965). In the present project, ω^* values from 0 to 0.5 has been used with a step size of 0.005. Solution is computed from lower ω^* to higher. $-k_i^*$ values from previous ω^* is used as an initial guess for next ω^*.

Stanford University

Effect of presence of bubbles

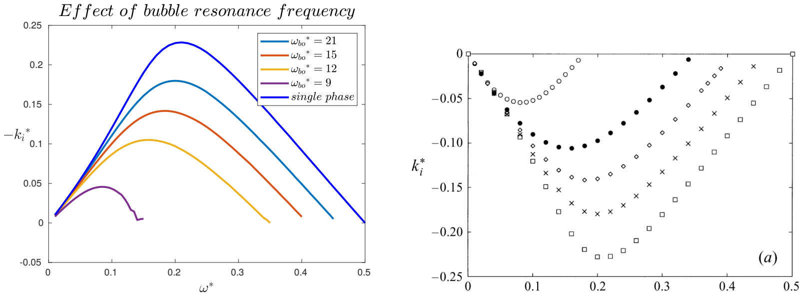


Figure 2: Presence of bubbles have a stabilizing effect on the flow. Left: current work, Right: d'Agostino et al., *JFM*, (1997). In all cases $\alpha = 0.01$ and $R_0 = 0.01$.

- as ω_{b0}^* decreases (approaches towards excitation frequency ω^*), flow stabilizes.
- $\omega_{b0}^* \gg \omega^* \Rightarrow$ fluid behaves barotropically $\rightarrow$ asymptotes to single-phase behavior.

Stanford University

Maximum amplification rate

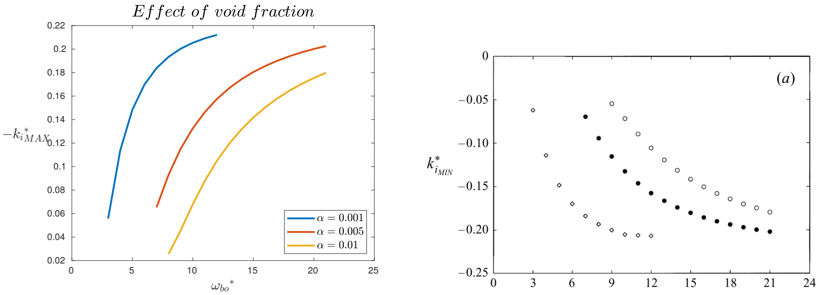


Figure 3: Maximum amplification rate $-k^*_{MAX}$ as a function of natural frequency of the bubbles ω^*_{b0} for different values of void fraction α. Left: current work, Right: d'Agostino et al., *JFM*, (1997). In all cases $R_0 = 0.01$.

- as ω^*_{b0} decreases, $-k^*_{MAX}$ reduces and flow stabilizes as observed before.
- ω^*_{b0} also decreases as α increases hence stabilizing the flow for higher void fractions.

Stanford University

Effect of bubble resonance

- Numbers for ω_{b0}^* in previous two cases were picked that are relevant to practical applications.

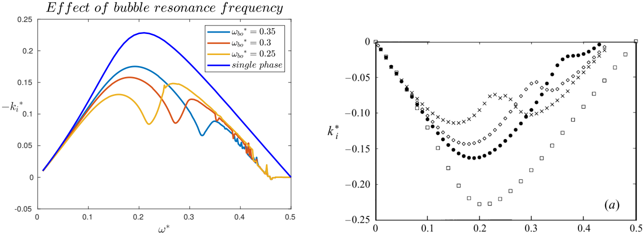


Figure 4: Left: current work, Right: d'Agostino et al., *JFM*, (1997). In all cases $\alpha = 0.003$.

- doesn't match the paper exactly, since R_0^* values are missing in the paper, but the results are close (I use $R_0^* = 0.32$).
- as ω_{b0}^* decreases, flow again stabilizes.
- at resonance $\omega_{b0}^* \approx \omega^*$, flow is more stable (a local minimum can be seen).

Stanford University

Reasons for stabilizing effect of bubbles

- Compressibility effect: "a certain amount basic flow energy must be used to do work against the force due to the elasticity of the medium, before it becomes available to initial instability" (Blumen et al. (1975)).
- Bubble dynamic damping: this provides another source of energy absorption, which at resonance is significant due to the large amplitude of bubble response.

Stanford University

Conclusion

summary

- Governing equations of bubble-liquid mixtures along with the closure were formally derived.
- Disturbance relations were derived making appropriate assumptions.
- Inviscid stability analysis of a bubbly inviscid shear flow was performed.
- Stabilizing effect of presence of bubbles was studied.
- Stability as a function of natural frequency of bubbles and void fraction was also studied.
- Stabilizing effect of bubbles at resonance was also investigated.
- Work of d'Agostino et al., *JFM*, (1997) was successfully reproduced.

Stanford University

THANK YOU

References:

d'Agostino, L; d'Auria, F & Brennen, C (1997), 'On the inviscid stability of parallel bubbly flows', *Journal of Fluid Mechanics*, Vol. 339, pp. 261-274.

d'Agostino, L. & Brennen, C. E. (1989), 'Linearized dynamics of spherical bubble clouds',*Journal of Fluid Mechanics*, 199, 155–176.

d'Auria, F., d'Agostino, L. & Brennen, C. E. (1995), 'Inviscid stability of bubbly jets',*AIAA*.

Blumen, W., Drazin, P. G. & Billings, D. F. (1975), 'Shear layer instability of an inviscid compressible fluid. Part 2.',*Journal of Fluid Mechanics*, 71, 305–316.

Keller, B, J & Miksis, M, (1980), 'Bubble oscillations of large amplitude', *The Journal of the Acoustical Society of America*, 68, 628.

Michalke, A. (1965), 'On spatially growing disturbances in an inviscid shear layer', *Journal of Fluid Mechanics*, 23, 521–544.