Inviscid stability analysis of parallel bubbly flows

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Abstract

Bubbly flows are ubiquitous in nature. Stability analysis of parallel bubbly flows is crucial to understand the dynamical properties of the medium, since the presence of bubbles can significantly change the properties of the medium, such as sound speed, attenuation and inertia, even at very low void fractions (d'Auria et al. 1995, d'Agostino et al. 1997). Hence, the aim of this project is to perform the stability analysis of parallel bubbly flows in simple configurations (such as an inviscid shear flow) to study the effect of presence of bubbles on the dynamics of the medium by comparing it against the stability analysis of the single-phase medium.

1 Introduction

Stability of parallel single-phase flows for various configurations, such as jets, wakes, shear layer, boundary layer, and internal flows, have been studied for over six decades (Betchov 2012, Drazin et al. 1982) for both incompressible and compressible flows (Mack 1987). However, the literature is still limited for two-phase flows. Hence, the focus of the current study is to perform stability analysis of bubbly flows, following the works of d'Auria et al. (1995), d'Agostino et al. (1997), d'Agostino & Brennen (1989) to study the effect of bubbles on the dynamical properties of the underlying medium by comparing against single-phase flows.

In the current study, the governing equations (continuity and momentum equations) for the bubble-liquid mixtures will be formally derived along with the closure relations (modified Rayleigh-Plesset equation). Then, assuming small linear disturbances and normal mode representation for disturbances, disturbance relations for all the governing equations will be derived. Assuming parallel, two-dimensional base flow, the disturbance relations will be simplified, which can be later solved using a shooting method to perform the spacewise stability analysis.

2 Governing equations

To derive the governing equations for the bubbly-liquid mixtures, we can start with the continuity equation for individual phase i, as

$$\frac{\partial \rho_i \alpha_i}{\partial t} + \vec{\nabla} \cdot (\rho_i \vec{u}_i \alpha_i) = 0, \tag{1}$$

where, ρ_i and α_i are the density and volume fraction of phase i. Now, rewriting this for liquid phase l, and expressing partial derivatives in terms of total derivatives, gives

$$\frac{1}{\rho_l}\frac{D\rho_l}{Dt} + \frac{1}{\alpha_l}\frac{D\alpha_l}{Dt} + \vec{\nabla} \cdot \vec{u}_l = 0. \tag{2}$$

Now, if we express pressure as, $p = f(\rho, s)$, where, s is the entropy, then for an isentropic process, taking total derivatives, we can write

$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt} \tag{3}$$

where, c is the speed of sound. Using this in the Eq. (2), we can write it as

$$\frac{1}{\rho_l c_l^2}\frac{Dp_l}{Dt} + \frac{1}{\alpha_l}\frac{D\alpha_l}{Dt} + \vec{\nabla} \cdot \vec{u}_l = 0, \tag{4}$$

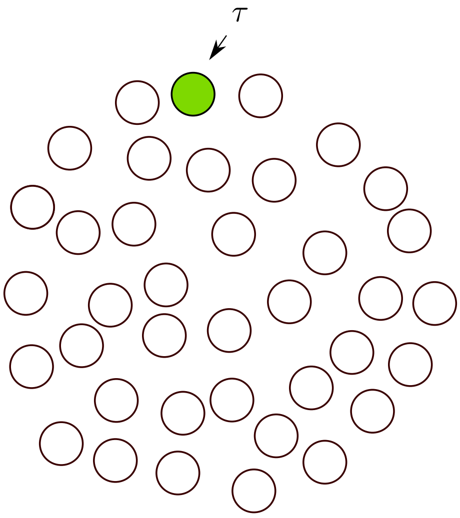


Figure 1: Schematic of a bubble cloud. τ represents individual bubble volume and is equal to $4/3\pi R^3$.

Now, before proceeding further, we can introduce some terminologies. Consider the bubble cloud shown in Figure 1. Let, β be the number of bubbles per unit liquid volume, n be the number of bubbles per unit total volume, τ be the individual bubble volume as shown in Figure 1 and is equal to $4/3\pi R^3$, where $R(\vec{x}, t)$ is the radius of the bubble (assuming spherical bubbles), and $\alpha_b = n\tau$ be the volume fraction of bubbles.

Now, we relate α_b to β and τ as follows. If we start with $1 + \beta\tau$, then it can be written as

$$1 + \beta\tau = 1 + (\frac{\#\ of\ bubbles}{liquid\ volume})\tau = 1 + \frac{gas\ volume}{liquid\ volume} = \frac{total\ volume}{liquid\ volume}. \tag{5}$$

Now, expressing number density n as

$$n = \frac{\#\ of\ bubbles}{total\ volume} = \frac{\#\ of\ bubbles}{liquid\ volume} * \frac{liquid\ volume}{total\ volume} \tag{6}$$

and using the above relation in Eq. (5), we get

$$n = \frac{\beta}{1 + \beta\tau}. \tag{7}$$

Hence void fractions α_b, α_l can be related to β and τ as

$$\alpha_b = \frac{\beta\tau}{1 + \beta\tau} = 1 - \alpha_l \tag{8}$$

Substituting this relation in Eq. (4) and simplyfying (see, Appendix A), we get the final form of the continuity equation for bubble-liquid mixtures as

$$\Big(\frac{1}{1 + \beta\tau}\Big)\frac{D\beta\tau}{Dt} - \frac{1}{\rho_l c_l^2}\frac{Dp_l}{Dt} = \vec{\nabla} \cdot \vec{u}_l, \tag{9}$$

where, $D/Dt = \partial/\partial t + \vec{u}_l \cdot \vec{\nabla}$.

Similarly, for the mixture momentum equation, starting with the momentum equation for liquid phase,

$$\frac{\partial \rho_l \vec{u}_l}{\partial t} + \vec{\nabla} \cdot (\rho_l \vec{u}_l \otimes \vec{u}_l + p_l \mathbb{1}) = 0, \tag{10}$$

and expanding the derivatives,

$$\vec{u}_l\Big\{\frac{\partial \rho_l \alpha_l}{\partial t} + \vec{\nabla} \cdot (\rho_l \alpha_l \vec{u}_l)\Big\} + \rho_l \alpha_l \Big\{\frac{\partial \vec{u}_l}{\partial t} + \vec{u}_l \cdot \vec{\nabla}\vec{u}_l\Big\} = -\vec{\nabla} p_l, \tag{11}$$

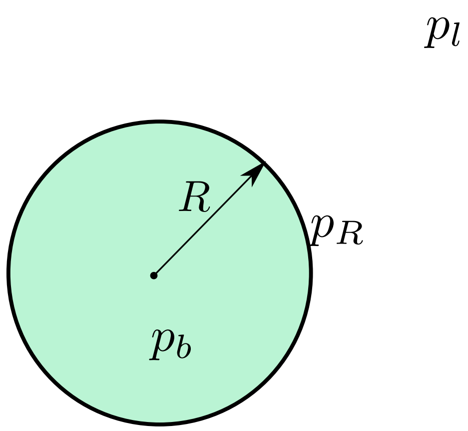


Figure 2: Oscillating single bubble, driven by external pressure.

and using Eq. (1), we get the final form of the momentum equation for bubble-liquid mixtures as

$$\rho_l(1-\alpha_b)\frac{D\vec{u}_l}{Dt} = -\vec{\nabla}p_l \tag{12}$$

Now, to close the system, we use a modified Rayleigh-Plesset equation, also called as Keller-Miksis equation (Keller & Miksis 1980) that describes the bubble response under the influence of external pressure. This equation is given by

$$\Big(1-\frac{1}{c_l}\dot{R}\Big)R\ddot{R} + \frac{3}{2}\dot{R}^2\Big(1-\frac{1}{3c_l}\dot{R}\Big) = \Big(1+\frac{1}{c_l}\dot{R}\Big)\Big\{\frac{p_R(t)+p_l(t+R/c_l)}{\rho_l}\Big\} + \frac{R}{\rho_l c_l}\dot{p_R}(t) \tag{13}$$

where dots represent D/Dt, p_R is the liquid pressure at bubble surface and p_l is the driving pressure as shown in Figure 2.

Now, to relate the liquid pressure at the bubble surface p_R with the internal pressure p_b, we can balance the forces at the surface of the bubble, which gives us the boundary condition

$$p_b(t) = p_R(t) + 2\frac{\sigma}{R} + \frac{4\mu\dot{R}}{R} \tag{14}$$

where, p_b is the uniform bubble internal pressure, σ is the surface tension and μ is the liquid viscosity. Finally, if we assume that the gas behaves polytropically, we can relate p_b to the radius of the bubble as

$$p_b = p_{b0}\Big(\frac{R_0}{R}\Big)^{3\gamma} \tag{15}$$

where, R_0 is the equilibrium radius of the bubble, p_{b0} is the bubble pressure in equilibrium, and γ is the polytropic coefficient. Summarizing, the final system can be written as

$$\Big(\frac{4/3\pi}{1+\beta\tau}\Big)\frac{D\beta R^3}{Dt} - \frac{1}{\rho_l c_l^2}\frac{Dp_l}{Dt} = \vec{\nabla}\cdot\vec{u}_l, \tag{16}$$

$$\rho_l(1-\alpha_b)\frac{D\vec{u}_l}{Dt} = -\vec{\nabla}p_l \tag{17}$$

$$\Big(1-\frac{1}{c_l}\dot{R}\Big)R\ddot{R} + \frac{3}{2}\dot{R}^2\Big(1-\frac{1}{3c_l}\dot{R}\Big) = \Big(1+\frac{1}{c_l}\dot{R}\Big)\Big\{\frac{p_R(t)+p_l(t+R/c_l)}{\rho_l}\Big\} + \frac{R}{\rho_l c_l}\dot{p_R}(t) \tag{18}$$

$$p_b(t) = p_R(t) + 2\frac{\sigma}{R} + \frac{4\mu\dot{R}}{R} \tag{19}$$

$$p_b = p_{b0}\Big(\frac{R_0}{R}\Big)^{3\gamma} \tag{20}$$

where, R, $\vec{u}_l$ and p_l are the unknown variables.

3 Disturbance equations

To study the linear stability of the system in Eqs. (16)-(20), typically the variables are decomposed into mean and disturbance quantities as

$$\begin{aligned} u_l &= U(y)\hat{e}_x + \tilde{u}(x,y,t) \\ v_l &= \tilde{v}(x,y,t) \\ p_l &= p_0 + \tilde{p}(x,y,t) \\ R_l &= R_0 + \tilde{R}(x,y,t) \end{aligned} \tag{21}$$

where $\tilde{}$ represent the disturbance quantities, $U(y)$ is the base flow velocity along x direction, p_0 and R_0 are the mean pressure and radius values. Substituting this in the continuity equation (Eq. 16) and making an assumption that β is uniform, and linearizing the equation assuming that the disturbances are small and subtracting the base flow, we arrive at the form

$$\Big(\frac{3\alpha}{R_0}\Big)\frac{\hat{D}\tilde{R}}{\hat{D}t} - \frac{1}{\rho c^2}\frac{\hat{D}\tilde{p}}{\hat{D}t} = \vec{\nabla}\cdot\vec{\tilde{u}}, \tag{22}$$

where, $\hat{D}/\hat{D}t = \partial/\partial t + U\partial/\partial x$. Similarly for the momentum equation, we arrive at the form

$$\rho(1-\alpha)\Big\{\frac{\partial\tilde{u}}{\partial t} + U\frac{\partial\tilde{u}}{\partial x} + U'\tilde{v}\Big\} = -\frac{\partial\tilde{p}}{\partial x} \tag{23}$$

and

$$\rho(1-\alpha)\Big\{\frac{\partial\tilde{v}}{\partial t} + U\frac{\partial\tilde{v}}{\partial x}\Big\} = -\frac{\partial\tilde{p}}{\partial y} \tag{24}$$

where, prime denotes $\partial/\partial y$. See Appendix B for the derivation of Eqs. (22)-(24). Similarly, combining the modified Rayleigh-Plesset equation Eq. (18), boundary condition Eq. (19) and gas law Eq. (20) and substituting the decomposition in Eq. (21), linearizing and subtracting the base flow, we obtain the equation of the form

$$\rho\ddot{\tilde{R}} + p_{b0}\frac{3\gamma\tilde{R}}{R_0^2} - \frac{2\sigma\tilde{R}}{R_0^3} + \frac{4\mu\dot{\tilde{R}}}{R_0^2} + p_{b0}\frac{3\gamma\dot{\tilde{R}}}{cR_0} - \frac{2\sigma\dot{\tilde{R}}}{cR_0^2} + \frac{4\mu\ddot{\tilde{R}}}{R_0 c} = -\frac{\tilde{p}}{R_0} \tag{25}$$

where, dots represent $\hat{D}/\hat{D}t$. See Appendix C for the derivation of Eq. (25).

4 Linear stability equations

Making an ansatz for the disturbance as

$$\begin{aligned} \tilde{u} &= \hat{u}(y)e^{i(kx-\omega t)} \\ \tilde{v} &= \hat{v}(y)e^{i(kx-\omega t)} \\ \tilde{p} &= \hat{p}(y)e^{i(kx-\omega t)} \\ \tilde{R} &= \hat{R}(y)e^{i(kx-\omega t)} \end{aligned} \tag{26}$$

where, k is the wavenumber and ω is the frequency. Substituting this ansatz in the disturbance equations, Eq. (22)-(25), we obtain the final system as

$$ik\hat{u} + \hat{v}' = -i\frac{3\gamma}{R_0}\omega_L\hat{R} + i\frac{\omega_L}{\rho c^2}\hat{p} \tag{27}$$

$$\rho(1-\alpha)(-i\omega_L\hat{u} + U'\hat{v}) = -ik\hat{p} \tag{28}$$

$$\rho(1-\alpha)(i\omega_L\hat{v}) = \hat{p}' \tag{29}$$

and

$$\Big(-\underbrace{\omega_L^2}_{inertial} - \underbrace{i\omega_L\lambda}_{damping} + \underbrace{\omega_b^2}_{compressiblity}\Big)\hat{R} = -\Big(1 + i\frac{\omega_L R_0}{c}\Big)\frac{\hat{p}}{\rho R_0} \tag{30}$$

where, $\omega_L = \omega - Uk$ is the Lagrangian frequency as seen by the bubbles moving with the mean flow. Inertial, damping and compressibility contributions to the bubble dynamic response has been highlighted, where

$$\lambda = \underbrace{\frac{\omega_L^2 R_0}{c}}_{acoustical} - \underbrace{\frac{4\mu}{\rho R_0^2}}_{viscous} + \big(thermal = 0\big)$$

is the damping coefficient. The thermal contribution to the damping coefficient is zero due to the assumption of adiabatic law for the gas in the bubble.

$$\omega_b^2 = \frac{p_{b0}3\gamma}{\rho R_0^2} - \frac{2\sigma}{R_0^3}$$

is the natural frequency of the bubble. See Appendix D for the derivation of the disturbance equation for the bubble dynamic response in Eq. (30).

Now, if both x and y directions were homogeneous, then we can repeat the whole process of deriving four disturbance relations for $\hat{u}$, $\hat{v}$, $\hat{R}$ and $\hat{p}$, and then eliminating $\hat{u}$, $\hat{v}$ and $\hat{R}$, a wave equation for $\hat{p}$ can be derived as

$$\underbrace{\left[\left\{\frac{3\alpha}{R_0^2}\frac{(1-\alpha)(1+i\omega\frac{R_0}{c})}{(-\omega^2 - i\omega\lambda + \omega_b^2)} + \frac{(1-\alpha)}{c^2}\right\}\omega^2 - \left\{k_x^2 + k_y^2\right\}\right]}_{=0\ \Rightarrow\ dispersion\ relation}\hat{p} = 0 \tag{31}$$

see Appendix E for the derivation of this wave equation for pressure $\hat{p}$. The coefficient of pressure is equal to zero and is called as the dispersion relation for the mixture medium. Using this, the speed of sound in bubbly mixture medium $c_m(\omega)$ for a harmonic disturbance ω can be written as

$$\frac{1}{c_m^2(\omega)} = \frac{3\alpha}{R_0^2}\frac{(1-\alpha)(1+i\omega\frac{R_0}{c})}{(-\omega^2 - i\omega\lambda + \omega_b^2)} + \frac{(1-\alpha)}{c^2} \tag{32}$$

Now, going back to the parallel flow setup and eliminating $\hat{R}$ and $\hat{p}$ from the four disturbance equations Eqs. (27)-(30) and using the mixture sound speed in Eq. (32), we arrive at the final equivalent Rayleigh system for bubbly flows as (see Appendix F for the derivation),

$$\hat{u}' = ik\hat{v} - i\frac{U''}{\omega_L}\hat{v} - i\frac{U'}{kc_m^2(\omega_L)}\big(i\omega_L\hat{u} - U'\hat{v}\big) \tag{33}$$

and

$$\hat{v}' = -ik\hat{u} + \frac{\omega_L}{kc_m^2(\omega_L)}\big(i\omega_L\hat{u} - U'\hat{v}\big) \tag{34}$$

Clearly, this equation is non-linear in k since the mixture sound speed c_m is a function of $\omega_L = \omega - Uk$. The equivalence of this system with the classical Rayleigh system can be easily seen in the limit of $c_m \to \infty$ (see Appendix G for this equivalence).

5 Inviscid shear layer

In the current work, a velocity profile of inviscid shear layer is chosen as the base flow,

$$U(y) = \frac{U_1 + U_2}{2} + \frac{U_2 - U_1}{2}tanh(\frac{y}{\delta}) \tag{35}$$

where U_1 and U_2 are the velocities as $y \to +\infty$ and $y \to -\infty$ respectively, and δ is the shear layer thickness. Substituting these asymptotic limits to the equivalent Rayleigh system in Eqs. (33)-(34), the system reduces to,

$$\hat{u}' = ik\hat{v} \tag{36}$$

and

$$\hat{v}' = -ik\hat{u} + i\frac{\omega_L^2}{kc_m^2(\omega_L)}\hat{u} \tag{37}$$

This can be integrated analytically and it admits a closed form solutions for $\hat{u}$ and $\hat{v}$ as

$$\hat{v} = Ae^{\pm y(k^2 - \omega_L^2/c_m^2)^{1/2}} \tag{38}$$

and

$$\hat{u} = \pm A \frac{ik}{(k^2 - \omega_L^2/c_m^2)^{1/2}} e^{\pm y(k^2-\omega_L^2/c_m^2)^{1/2}} \tag{39}$$

where A is an arbitrary complex constant, and $\pm$ signs in Eqs. (38)-(39) are chosen such that the solution decays as y approaches $\pm\infty$.

The focus of the current work is to study the spacewise problem. Hence k is complex and ω is real. The equivalent Rayleigh system in Eqs. (33)-(34) along with the boundary conditions in Eqs. (38)-(39) is solved using a shooting method and the solution procedure can be described as follows,

1. Guess a complex eigenvalue k.
2. Choose A such that initial conditions at $y = -n\delta$, where $n \gg 1$, (lower domain boundary) simplifies as,
$$\hat{v} = 1$$
$$\hat{u} = \frac{ik}{(k^2 - \omega_L^2/c_m^2)^{1/2}}$$
3. Integrate the system upto $y = n\delta$ (upper domain boundary).
4. Check if the solution is continuous with the asymptotic solution at $y = n\delta$
$$\hat{u} = -\frac{ik}{(k^2 - \omega_L^2/c_m^2)^{1/2}}\hat{v}$$
5. Iteratively correct eigenvalue k until convergence.

A larger value for n will result in bigger domain and hence, more accurate solution but is harder to converge and computationally more expensive. Hence, it is a trade-off. A value of $n = 5$ is used in the current project. A fourth-order Runge-Kutta integration scheme is used to integrate the solution from lower boundary to the upper boundary in this work and a two-dimensional Newton-Raphson method is used to iteratively correct the eigenvalue k after each integration.

To verify the numerical method, d'Agostino et al. (1997) used the work of Michalke (1965) to compare the growth rate in the limit of infinite mixture sound speed ($c_m \to \infty$). Hence, in the current project, we use the same reference to verify our numerical method by comparing the growth rates in the limit of $c_m \to \infty$ as shown in Figure 3. Clearly, the growth rates agree very well with the values from the reference. We use ω^* values from 0 to 0.5 with a step size of 0.005. Computation is performed from lower ω^* to higher and the converged k^* values from previous ω^* is used as an initial guess to compute k^* for the next ω^*. The code is written in MATLAB and has been uploaded on to the github repository, and can be accessed at this link: "https://github.com/suhasjains/Stability_for_bubbly_flow".

6 Results

In this section, the effect of presence of bubbles on the stability properties of the medium is studied by solving a spacewise problem. Fluid inside the bubbles is assumed to be ideal air with a polytropic coefficient $\gamma = 1.4$. Liquid is taken to be water with density $\rho = 1485\text{Kgm}^{-3}$, dynamic viscosity $\mu = 0.001\text{Nsm}^{-2}$, surface tension of the air-water interface to be $\sigma = 0.0728\text{N/m}$. All quantities are expressed in dimensionless form (using asterisk symbol) and the non-dimensionalization is performed using shear-layer thickness δ and shear-layer velocity difference ΔU as the reference velocity and length scale.

The effect of presence of bubbles can be clearly seen in Figure 4, where the amplification factor $-k_i^*$ (negative of imaginary part of the eigenvalue k) is plotted as a function of excitation frequency ω^* for both single-phase flow and bubbly flows. Clearly, the presence of bubbles stabilizes the flow since $-k_i^*$ decreases when compared to the single-phase flow. In the same figure, $-k_i^*$ is plotted for different values of natural frequency of the bubbles ω_{b0}^*. As ω_{b0}^* decreases (and approaches towards excitation frequency ω^*), the flow stabilizes. On the other hand, for far enough $\omega_{b0}^* >> \omega^*$, fluid behaves barotropically and hence asymptotes to the single-phase behavior.

Now, to study the effect of void fraction of the bubbles α on the stability, maximum amplification rate $-k_{i_{MAX}}^*$ is plotted as a function of natural frequency of the bubbles ω_{b0}^* in Figure 5

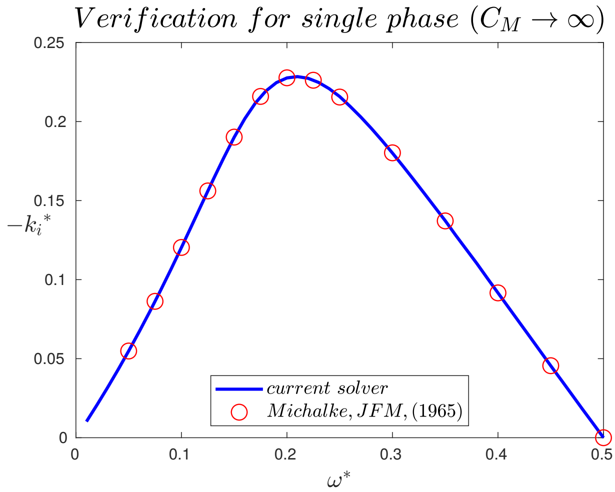
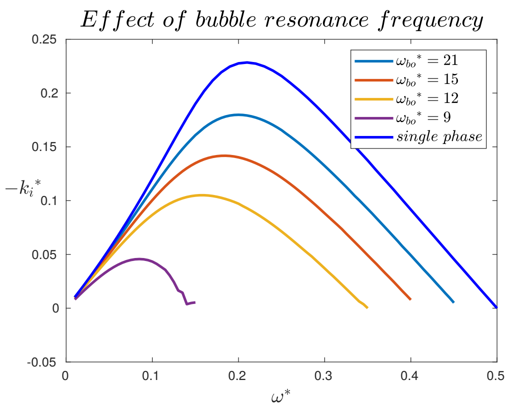


Figure 3: Verification against values from Table 1. of Michalke, *JFM*, (1965).



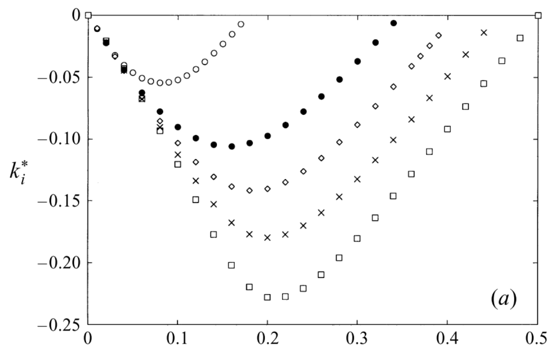


Figure 4: Plot of amplification factor $-k_i^*$ as a function of excitation frequency ω^*. Presence of bubbles have a stabilizing effect on the flow. Left: current work, Right: d'Agostino et al., *JFM*, (1997). In all cases $\alpha = 0.01$ and $R_0 = 0.01$.

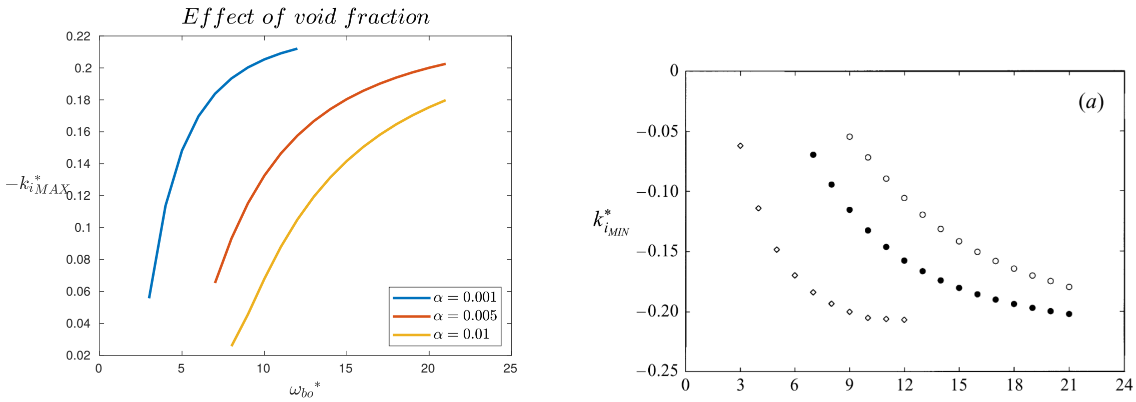


Figure 5: Maximum amplification rate $-k^*_{i_{MAX}}$ as a function of natural frequency of the bubbles ω^*_{b0} for different values of void fraction α. Left: current work, Right: d'Agostino et al., *JFM*, (1997). In all cases $R_0 = 0.01$.

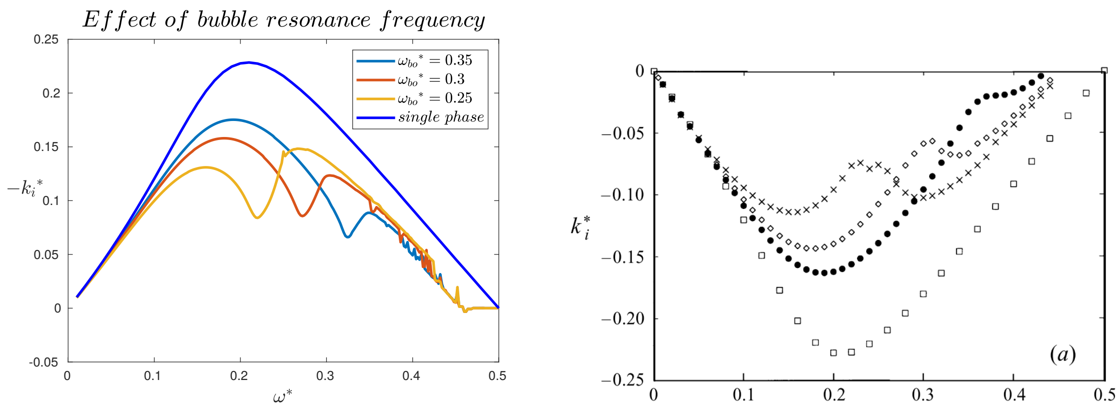


Figure 6: Plot of amplification factor $-k^*_i$ as a function of excitation frequency ω^* for values of $\omega^*_{b0} \approx \omega^*$. Left: current work, Right: d'Agostino et al. (1997). In all cases $\alpha = 0.003$. Value of R^*_0 used in this case is missing in the paper d'Agostino et al. (1997), and a value of $R^*_0 = 0.32$ is used in the current project. Hence the results for this test case are not matching the results from d'Agostino et al. (1997) exactly, but qualitatively the results are same.

for different values of void fraction α. As ω^*_{b0} decreases, $-k^*_{i_{MAX}}$ reduces and the flow stabilizes as already observed in Figure 4. maximum amplification rate $-k^*_{i_{MAX}}$ also decreases as α increases hence stabilizing the flow for higher void fractions.

The parameters used in the above test cases represent values that are relevant for practical applications. But, to study the effect of resonance of bubbles, the natural frequency of bubbles ω^*_{b0} can be further reduced such that it is approximately equal to the excitation frequency, $\omega^*_{b0} \approx \omega^*$. Results for this are shown in Figure 6. As ω^*_{b0} decreases, flow again stabilizes as seen before in above two test cases. Close to resonance, the flow is much more stable, as can be seen in the Figure 6 for all three values of ω^*_{b0}, where a local minimum can be seen in the amplification factor $-k^*_i$ for excitation frequencies ω^* close to natural frequency of the bubbles ω^*_{b0}.

Hence the effect of presence of bubbles is to stabilize the flow. The reason for this stabilization can be explained as follows. Bubbles add compressiblity to the liquid, and Blumen et al. (1975) explained that the effect of compressibility on the dynamics of the flow is to stabilize the flow. He stated that "a certain amount basic flow energy must be used to do work against the force due to the elasticity of the medium, before it becomes available to initiate instability". Further, the increased stability for frequencies close to natural frequencies of the bubbles was explained by d'Agostino et al. (1997). They hypothesized that the "bubble dynamic damping" provides another source of energy absorption, which at resonance is significant due to the large amplitude of bubble response.

7 Summary and conclusion

In this project, we studied the stability of parallel inviscid bubbly shear flow. Starting with the derivation of governing equations of bubble-liquid mixtures along with the closure, disturbance relations were derived making appropriate assumptions. The system was solved using a shooting method and was verified for the case of single-phase flow against the results from Michalke (1965).

The stabilizing effect of presence of bubbles was studied by investigating the amplification rate of the disturbance as a function of excitation frequency for various natural frequencies of the bubbles and void fractions. Futhermore, improved stabilization for resonant frequencies was investigated along with some possible reasons for the stabilization effect. This project was performed following the work of d'Agostino et al. (1997) and the results from the paper have been reproduced.

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Appendix A:- Mixture continuity equation,

Starting from,

$$\frac{1}{\rho_\ell C_\ell^2}\frac{DP_\ell}{Dt} + \frac{1}{\alpha_\ell}\frac{D\alpha_\ell}{Dt} + \left(\vec{\nabla}\cdot\vec{u}_\ell\right) = 0$$

Now rewriting $\alpha_\ell = 1-\alpha_b$,

$$\frac{1}{\alpha_\ell}\frac{D\alpha_\ell}{Dt} = \left(\frac{1}{1-\alpha_b}\right)\left(-\frac{D\alpha_b}{Dt}\right) = \frac{1}{\left(1-\frac{\beta\tau}{1+\beta\tau}\right)}\left\{-\frac{D}{Dt}\left(\frac{\beta\tau}{1+\beta\tau}\right)\right\}$$

$$= \left(\frac{1+\beta\tau}{1+\cancel{\beta\tau}-\cancel{\beta\tau}}\right)\left\{-\frac{D}{Dt}\left(\frac{\beta\tau}{1+\beta\tau}\right)\right\}$$

$$= -\left(\cancel{1+\beta\tau}\right)\times\frac{\left\{\frac{D\beta\tau}{Dt}(1+\beta\tau) - \beta\tau\frac{D(1+\beta\tau)}{Dt}\right\}}{(1+\beta\tau)^{\cancel{2}}}$$

$$= -\left(\frac{1}{1+\beta\tau}\right)\left\{\frac{D\beta\tau}{Dt}\left[1+\cancel{\beta\tau}-\cancel{\beta\tau}\right]\right\}$$

$$= -\left(\frac{1}{1+\beta\tau}\right)\frac{D\beta\tau}{Dt} \quad //$$

Substituting above,

$$\boxed{\begin{aligned} \therefore\ \left(\vec{\nabla}\cdot\vec{u}_\ell\right) &= \left(\frac{1}{1+\beta\tau}\right)\frac{D\beta\tau}{Dt} - \frac{1}{\rho_\ell C_\ell^2}\frac{DP_\ell}{Dt} \\ \text{where}\quad \frac{D}{Dt} &= \frac{\partial(\,)}{\partial t} + \vec{u}_\ell\cdot\vec{\nabla}(\,) \end{aligned}}$$

Appendix B : Derivation of disturbance continuity and momentum equation

Continuity :-

$$\therefore \left(\vec{\nabla}\cdot\vec{u}_\ell\right) = \left(\frac{1}{1+\beta\tau}\right)\frac{D\beta\tau}{Dt} - \frac{1}{\rho_\ell c_\ell^2}\frac{DP_\ell}{Dt}$$

Assuming β to be uniform,

$$\left(\vec{\nabla}\cdot\vec{u}_\ell\right) = \tau\underbrace{\left(\frac{\beta\tau}{1+\beta\tau}\right)}_{\alpha_b}\frac{D\tau}{Dt} - \frac{1}{\rho_\ell c_\ell^2}\frac{DP_\ell}{Dt}$$

$$\left(\vec{\nabla}\cdot\vec{u}_\ell\right) = \frac{\alpha_b}{\tau}\frac{D\tau}{Dt} - \frac{1}{\rho_\ell c_\ell^2}\frac{DP_\ell}{Dt}$$

let $\alpha_b \to \alpha$, $\rho_\ell \to \rho$, $c_\ell \to c$

Substituting these,

$$u_\ell = U(y)\,\hat{e}_x + \tilde{u}(x,y,t),$$
$$v_\ell = \tilde{v}(x,y,t),$$
$$P_\ell = P_0 + \tilde{p}(x,y,t),$$
$$R = R_0 + \tilde{R}(x,y,t)$$

$$\vec{\nabla}\cdot\vec{\tilde{u}} = \frac{\alpha}{(R_0+\tilde{R})^3}\frac{D(R_0+\tilde{R})^3}{Dt} - \frac{1}{\rho c^2}\frac{D\tilde{p}}{Dt}$$

Linearizing,

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{\tilde{u}} = \frac{\alpha}{R_0^3}\times\frac{D\left(R_0^3 + 3R_0^2\tilde{R}\right)}{Dt} - \frac{1}{\rho c^2}\frac{D\tilde{p}}{Dt}$$

disturbance mass equation. $\longrightarrow$

$$\boxed{\vec{\nabla}\cdot\vec{\tilde{u}} = \frac{3\alpha}{R_0}\frac{\tilde{D}\tilde{R}}{\tilde{D}t} - \frac{1}{\rho c^2}\frac{\tilde{D}\tilde{p}}{\tilde{D}t}}$$

where $\frac{\tilde{D}}{\tilde{D}t} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$

Now, for the momentum equation :-

$$\rho\,(1-\alpha)\left\{\frac{\partial \tilde{u}}{\partial t} + (U+\tilde{u})\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\,\frac{\partial (U+\tilde{u})}{\partial y}\right\} = -\vec{\nabla}\tilde{p}$$

and

$$\rho\,(1-\alpha)\left\{\frac{\partial \tilde{v}}{\partial t} + (U+\tilde{u})\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\,\frac{\partial \tilde{v}}{\partial y}\right\} = -\vec{\nabla}\tilde{p}$$

Linearizing,

disturbance momentum equation $\longrightarrow$

$$\boxed{\begin{aligned} \rho(1-\alpha)\left\{\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + U'\tilde{v}\right\} &= -\frac{\partial \tilde{p}}{\partial x} \\ \rho\,(1-\alpha)\left\{\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x}\right\} &= -\frac{\partial \tilde{p}}{\partial y} \end{aligned}}$$

Appendix C :- Derivation of disturbance Keller - Miksis equation

Consider the gas law :-

$$P_b = P_{bo} \left(R_o / R \right)^{3\gamma}$$

Substituting, $R = R_o + \tilde{R}(x,y,t)$

$$\Rightarrow P_b = P_{bo} \left(\frac{R_o + \tilde{R}}{R_o} \right)^{-3\gamma} = P_{bo} \left(1 + \frac{\tilde{R}}{R_o} \right)^{-3\gamma}$$

$$= P_{bo} \left\{ 1 - 3\gamma \frac{\tilde{R}}{R_o} + 6\gamma \left(\frac{\tilde{R}}{R_o} \right)^2 + \cdots \right\}$$

Linearizing, $\Rightarrow P_b = P_{bo} \left\{ 1 - 3\gamma \frac{\tilde{R}}{R_o} \right\} \longrightarrow$ (C1)

Now, consider Keller - Miksis equation :-

$$\left(1 - \frac{1}{C_l} \mathring{R} \right) R \ddot{\mathring{R}} + \frac{3}{2} \dot{R}^2 \left(1 - \frac{1}{3C_l} \mathring{R} \right) = \left(1 + \frac{1}{C_l} \mathring{R} \right) \left\{ \frac{P_R(t) + P_l(t + R/C_l)}{\rho} \right\} + \frac{R}{\rho C_l} \mathring{P}_R(t)$$

Substituting for P_R using the boundary condition,

$$P_b(t) = P_R(t) + \frac{2\sigma}{R} + \frac{4\mu \mathring{R}}{R}$$

and let, $C_l \to c$,

$$\left(1 - \frac{1}{c} \mathring{R} \right) R \ddot{\mathring{R}} + \frac{3}{2} \dot{R}^2 \left(1 - \frac{1}{3c} \mathring{R} \right) = \left(1 + \frac{1}{c} \mathring{R} \right) \left\{ \frac{P_b}{\rho} - \frac{2\sigma}{\rho R} - \frac{4\mu \mathring{R}}{\rho R} + \frac{P_l(t + R/c)}{\rho} \right\}$$

$$+ \frac{R}{\rho c} \left\{ \mathring{P}_b - 2\sigma \frac{D(R^{-1})}{Dt} - 4\mu \frac{D}{Dt} \left(\frac{\mathring{R}}{R} \right) \right\}$$

Now, substituting for P_b from Eq. (C1) and substituting

$$P_\ell = P_0 + \tilde{P}(x,y,t)$$

$\Rightarrow$

$$\left(1-\frac{\dot{\tilde{R}}}{c}\right)(R_0+\tilde{R})\ddot{\tilde{R}} + \frac{3}{2}\dot{\tilde{R}}^2\left(1-\frac{\dot{\tilde{R}}}{3c}\right) =$$

$$\left(1+\frac{\dot{\tilde{R}}}{c}\right)\left\{\frac{P_{b0}}{\rho} - \frac{P_{b0}3\gamma\tilde{R}}{\rho R_0} - \frac{2\sigma}{\rho}(R_0+\tilde{R})^{-1} - \frac{4\mu}{\rho}\dot{\tilde{R}}(R_0+\tilde{R})^{-1} - \left(\frac{P_0+\tilde{P}(t+R/c)}{\rho}\right)\right\}$$

$$+\left(\frac{R_0+\tilde{R}}{\rho c}\right)\left[-P_{b0}3\gamma\frac{\dot{\tilde{R}}}{R_0} - 2\sigma\frac{D\left(1/(R_0+\tilde{R})\right)}{Dt} - 4\mu\frac{D\left\{\dot{\tilde{R}}(R_0+\tilde{R})^{-1}\right\}}{Dt}\right]$$

Base flow can be written as,

$$P_{b0} = \frac{2\sigma}{R_0} + P_0(t+R_0/c)$$

Subtracting this base flow from the above equation,

$\Rightarrow$

$$\left(1-\frac{\dot{\tilde{R}}}{c}\right)(R_0+\tilde{R})\ddot{\tilde{R}} + \frac{3}{2}\dot{\tilde{R}}^2\left(1-\frac{\dot{\tilde{R}}}{3c}\right) =$$

$$\left(1+\frac{\dot{\tilde{R}}}{c}\right)\left[\cancel{\frac{P_{b0}}{\rho}} - \frac{P_{b0}3\gamma\tilde{R}}{\rho R_0} - \frac{2\sigma}{\rho R_0}\left\{\cancel{1} - \frac{\tilde{R}}{R_0} + \left(\frac{\tilde{R}}{R_0}\right)^2 + \dots HOT\right\} - \frac{4\mu\dot{\tilde{R}}}{\rho R_0}\left\{1 - \frac{\tilde{R}}{R_0} + \left(\frac{\tilde{R}}{R_0}\right)^2 + \dots HOT\right\} - \left(\frac{\cancel{P_0}+\tilde{P}(t+R/c)}{\rho}\right)\right]$$

$$+\left(\frac{R_0+\tilde{R}}{\rho c}\right)\left\langle -P_{b0}3\gamma\frac{\dot{\tilde{R}}}{R_0} - \frac{2\sigma}{R_0}\left\{-\frac{\dot{\tilde{R}}}{R_0} + \left(\frac{\dot{\tilde{R}}}{R_0}\right)^2 + \dots HOT\right\} - \frac{4\mu}{R_0}\frac{D}{Dt}\left[\dot{\tilde{R}}\left\{1 - \frac{\tilde{R}}{R_0} + \left(\frac{\tilde{R}}{R_0}\right)^2 + \dots HOT\right\}\right]\right\rangle$$

Linearizing,

$$R_0 \ddot{\tilde{R}} = -\frac{P_{b0} 3\gamma \tilde{R}}{\rho R_0} + \frac{2\sigma \tilde{R}}{\rho R_0^2} - \frac{4\mu \dot{\tilde{R}}}{\rho R_0} - \frac{\tilde{P}}{\rho}(t + R_0/c)$$

$$+ \frac{R_0}{\rho c}\left\{ -P_{b0} 3\gamma \frac{\dot{\tilde{R}}}{R_0} + \frac{2\sigma \dot{\tilde{R}}}{R_0^2} - \frac{4\mu \ddot{\tilde{R}}}{R_0} \right\}$$

$\Rightarrow$

disturbance $\rightarrow$

Keller-Miksis equation

$$\boxed{\rho \ddot{\tilde{R}} + \frac{P_{b0} 3\gamma \tilde{R}}{R_0^2} - \frac{2\sigma \tilde{R}}{R_0^3} + \frac{4\mu \dot{\tilde{R}}}{R_0^2} + \frac{P_{b0} 3\gamma \dot{\tilde{R}}}{c R_0} - \frac{2\sigma \dot{\tilde{R}}}{c R_0^2} + \frac{4\mu \ddot{\tilde{R}}}{R_0 c} = -\frac{\tilde{P}(t + R_0/c)}{R_0}}$$

Appendix D :- Derivation of final disturbance equation for dynamic bubble response

Normal mode :-

$$\tilde{u} = \hat{u}(y)\, e^{i(kx - \omega t)} , \quad \tilde{v} = \hat{v}(y)\, e^{i(kx-\omega t)}$$

$$\tilde{p} = \hat{p}(y)\, e^{i(kx - \omega t)} , \quad \tilde{R} = \hat{R}(y)\, e^{i(kx-\omega t)}$$

Substituting NMR in Keller-Miksis

$$-\omega_L^2 \hat{R} + \frac{P_{b0} 3\gamma \hat{R}}{\rho R_0^2} - \frac{2\sigma \hat{R}}{\rho R_0^3} - \hat{i}\frac{4\mu \omega_L \hat{R}}{\rho R_0^2}$$

$$-\hat{i}\omega_L \frac{P_{b0} 3\gamma \hat{R}}{\rho c R_0} + \hat{i}\omega_L \frac{2\sigma \hat{R}}{\rho c R_0^2} - \omega_L^2 \frac{4\mu \hat{R}}{R_0 \rho c} = -\frac{\hat{p}}{\rho R_0}$$

where $\omega_L = \omega - kU$ is the Lagrangian frequency.

$$\Rightarrow -\omega_L^2 \hat{R} + \left\{ \frac{P_{b0} 3\gamma \hat{R}}{\rho R_0^2} - \frac{2\sigma \hat{R}}{\rho R_0^3} - \hat{i}\frac{4\mu \omega_L \hat{R}}{\rho R_0^2} \right\} \left(1 - \hat{i}\frac{\omega_L R_0}{c} \right) = -\frac{\hat{p}}{\rho R_0}$$

$$\Rightarrow -\omega_L^2 \hat{R} \left(1 + \hat{i}\frac{\omega_L R_0}{c} \right) + \frac{P_{b0}\phi \hat{R}}{\rho R_0^2} - \frac{2\sigma \hat{R}}{\rho R_0^3} + \hat{i}\frac{4\mu\omega_L \hat{R}}{\rho R_0^2} = -\frac{\hat{p}}{\rho R_0}\left(1 + \hat{i}\frac{\omega_L R_0}{c} \right)$$

$\Rightarrow$

$$\left\{ -\omega_L^2 - \hat{i}\,\omega_L \left(\frac{\omega_L^2 R_0}{c} - \frac{4\mu}{\rho R_0^2} \right) + \frac{P_{b0}\, 3\gamma}{\rho R_0^2} - \frac{2\sigma}{\rho R_0^3} \right\} \hat{R}$$

$$= -\left(1 + \hat{i}\frac{\omega_L R_0}{c} \right) \frac{\hat{p}}{\rho R_0}$$

Rewriting,

$\Rightarrow$ disturbance equation for bubble dynamic response

$$\Big(\underbrace{-\omega_L^2}_{\text{inertial}} - \hat{i}\,\underbrace{\omega_L \lambda}_{\text{damping}} + \underbrace{\omega_B^2}_{\text{compressibility}}\Big)\hat{R} = -\left(1 + \frac{\hat{i}\,\omega_L R_0}{c}\right)\frac{\hat{P}}{\rho R_0}$$

where

phase factor b/w $\hat{R}$ and $\hat{P}$ = 0 (here)

$$\therefore\ \lambda = \underbrace{\frac{\omega_L^2 R_0}{c}}_{\text{acoustical}} - \underbrace{\frac{P_{b0}\, \mathrm{Im}\{\phi\}}{\omega_L \rho R_0^2}}_{\text{thermal dissipation}} - \underbrace{\frac{4\mu}{\rho R_0^2}}_{\text{viscous}}$$

damping coefficient

Eq 3.4 in (D'Agostino 1997)

$$\omega_B^2 = \frac{P_{b0}\, 3\gamma}{\rho R_0^2} - \frac{2\sigma}{\rho R_0^3}$$

natural frequency

Appendix E :- Derivation of dispersion relation for a homogeneous medium

disturbance mass $\rightarrow$

$$\hat{i} k_x \hat{u} + \hat{i} k_y \hat{v} = -\frac{\hat{i}\, 3\alpha}{R_0} \omega_L \hat{R} + \frac{\hat{i}\, \omega_L}{\rho c^2} \hat{p}$$

disturbance momentum $\rightarrow$

$$\rho(1-\alpha)\, \cancel{\hat{i}}\, \omega\, \hat{u} = \cancel{\hat{i}}\, k_x \hat{p}$$

$$\rho(1-\alpha)\, \cancel{\hat{i}}\, \omega\, \hat{v} = \cancel{\hat{i}}\, k_y \hat{p}.$$

disturbance bubble dynamic equation :-

$$\hat{R} = \frac{-\left(1 + \hat{i}\omega \frac{R_0}{c}\right) \frac{\hat{p}}{\rho R_0}}{\left(-\omega^2 - \hat{i}\omega\lambda + \omega_B^2\right)}$$

Substituting this for $\hat{R}$ in the disturbance mass equation,

$$\hat{i} k_x \hat{u} + \hat{i} k_y \hat{v} = \frac{\hat{i}\, 3\alpha}{R_0} \omega \left\{ \frac{\left(1 + \hat{i}\omega \frac{R_0}{c}\right) \frac{\hat{p}}{\rho R_0}}{\left(-\omega^2 - \hat{i}\omega\lambda + \omega_B^2\right)} \right\} + \frac{\hat{i}\, \omega}{\rho c^2} \hat{p}$$

$$\cancel{\hat{i}} k_x \hat{u} + \cancel{\hat{i}} k_y \hat{v} = \frac{\cancel{\hat{i}}\, \omega}{\rho} \left\{ \frac{3\alpha}{R_0^2} \frac{\left(1 + \hat{i}\omega \frac{R_0}{c}\right)}{\left(-\omega^2 - \hat{i}\omega\lambda + \omega_B^2\right)} + \frac{1}{c^2} \right\} \hat{p}$$

Substituting here for $\hat{u}$ and $\hat{v}$ from disturbance momentum equation,

$$\left\{ \frac{k_x^2}{\cancel{\rho}\, \omega (1-\alpha)} + \frac{k_y^2}{\cancel{\rho}(1-\alpha)\omega} \right\} \hat{p} = \frac{\omega}{\cancel{\rho}} \left\{ \frac{3\alpha}{R_0^2} \frac{\left(1 + \hat{i}\omega \frac{R_0}{c}\right)}{\left(-\omega^2 - \hat{i}\omega\lambda + \omega_B^2\right)} + \frac{1}{c^2} \right\} \hat{p}$$

Rearranging,

$$\Rightarrow \left[\left\{\frac{3\alpha}{R_0^2}(1-\alpha)\frac{\left(1+\hat{i}\omega\frac{R_0}{c}\right)}{\left(-\omega^2-\hat{i}\omega\lambda+\omega_B^2\right)}+\frac{(1-\alpha)}{c^2}\right\}\omega^2-\left\{k_x^2+k_y^2\right\}\right]\hat{P}=0$$

Hence the dispersion relation is,

$$\left\{\frac{3\alpha}{R_0^2}(1-\alpha)\frac{\left(1+\hat{i}\omega\frac{R_0}{c}\right)}{\left(-\omega^2-\hat{i}\omega\lambda+\omega_B^2\right)}+\frac{(1-\alpha)}{c^2}\right\}\omega^2=\left\{k_x^2+k_y^2\right\}$$

where

$$\frac{1}{C_M^2}=\frac{3\alpha}{R_0^2}(1-\alpha)\frac{\left(1+\hat{i}\omega\frac{R_0}{c}\right)}{\left(-\omega^2-\hat{i}\omega\lambda+\omega_B^2\right)}+\frac{(1-\alpha)}{c^2}$$

Speed of propagation of harmonic disturbance of frequency ω.

Rewriting,

$$\boxed{\frac{1}{C_M^2}=\frac{\omega_{b0}^2\left(1+\hat{i}\omega\frac{R_0}{c}\right)}{C_{M0}^2\left(-\omega^2-\hat{i}\omega\lambda+\omega_B^2\right)}+\frac{(1-\alpha)}{c^2}}$$

where $\omega_{b0}^2=\frac{3P_{L0}}{\rho R_0^2}-\frac{2\sigma}{\rho R_0^3}$ and $C_{M0}^2=\frac{\omega_{b0}^2R_0^2}{3\alpha(1-\alpha)}$

natural frequency for isothermal conditions

low frequency sound speed of mixture of bubbles in incompressible liquid ($\omega\to 0$, $c\to\infty$)

Appendix F :- Derivation of equivalent Rayleigh System

Disturbance relation for bubble dynamic response as

$$\hat{R} = \frac{-\left(1 + \hat{i}\omega_L \frac{R_0}{c}\right)\frac{\hat{p}}{\rho R_0}}{\left(-\omega_L^2 - \hat{i}\omega_L\lambda + \omega_B^2\right)}$$

Substituting in disturbance continuity equation,

$$\hat{i}k\hat{u} + \hat{v}' = \hat{i}\frac{3\alpha}{R_0}\omega_L\left\{\frac{\left(1 + \hat{i}\omega_L \frac{R_0}{c}\right)\frac{\hat{p}}{\rho R_0}}{\left(-\omega_L^2 - \hat{i}\omega_L\lambda + \omega_B^2\right)}\right\} + \frac{\hat{i}\omega_L}{\rho c^2}\hat{p}$$

$$\Rightarrow \hat{p} = \frac{\left(\hat{i}k\hat{u} + \hat{v}'\right)}{\left[\frac{\hat{i}\omega_L}{\rho c^2} + \frac{\hat{i}\omega_L(3\alpha)}{\rho R_0^2}\left\{\frac{\left(1 + \hat{i}\omega_L \frac{R_0}{c}\right)}{\left(-\omega_L^2 - \hat{i}\omega_L\lambda + \omega_B^2\right)}\right\}\right]}$$

$$\Rightarrow \hat{p} = \frac{\left(\hat{i}k\hat{u} + \hat{v}'\right)}{\frac{\hat{i}\omega_L}{\rho}\left[\frac{1}{c^2} + \frac{(3\alpha)}{R_0^2}\left\{\frac{\left(1 + \hat{i}\omega_L \frac{R_0}{c}\right)}{\left(-\omega_L^2 - \hat{i}\omega_L\lambda + \omega_B^2\right)}\right\}\right]}$$

Substituting $\hat{p}$ into $\hat{u}$ momentum disturbance equation,

$$\cancel{\rho}(1-\alpha)\left(-\hat{i}\omega_L\hat{u}+U'\hat{v}\right) = -\hat{i}k\cancel{\rho}\left[\underline{\hat{i}k\hat{u}+\hat{v}'}\right]$$

$$\cancel{i}\omega_L\left[\frac{1}{c^2}+\frac{(3\alpha)}{R_0^2}\left\{\frac{\left(1+\hat{i}\omega_L\frac{R_0}{c}\right)}{\left(-\omega_L^2-\hat{i}\omega_L\lambda+\omega_B^2\right)}\right\}\right]$$

Rearranging,

$$\frac{\omega_L(1-\alpha)}{k}\underbrace{\left[\frac{1}{c^2}+\frac{(3\alpha)}{R_0^2}\left\{\frac{\left(1+\hat{i}\omega_L\frac{R_0}{c}\right)}{\left(-\omega_L^2-\hat{i}\omega_L\lambda+\omega_B^2\right)}\right\}\right]}_{1/C_m^2(\omega_L)}\left(\hat{i}\omega_L\hat{u}-U'\hat{v}\right)-\hat{i}k\hat{u}=\hat{v}'$$

Hence using the mixture sound speed relation,

$$\boxed{\frac{\omega_L}{kC_m^2}\left(\hat{i}\omega_L\hat{u}-U'\hat{v}\right)-\hat{i}k\hat{u}=\hat{v}'}$$

(Eq. (3.5) in d'Agostino et al (1997))

Now, taking derivative of $\hat{u}$ mom disturbance equation,

$$\Rightarrow \rho(1-\alpha)\left(-\hat{i}\omega_L\hat{u}'+\hat{i}kU'\hat{u}+U''\hat{v}+U'\hat{v}'\right)=-\hat{i}k\hat{p}'$$

$$\cancel{\rho}\cancel{(1-\alpha)}\left(-\hat{i}\omega_L\hat{u}'+\hat{i}kU'\hat{u}+U''\hat{v}+U'\hat{v}'\right)=-\hat{i}k\,\cancel{\rho}\cancel{(1-\alpha)}\,\hat{i}\omega_L\hat{v}$$

$$\Rightarrow \quad \hat{i}\,\omega_L\,\hat{u}' = \hat{i}\,k\,U'\hat{u} + U''\hat{v} + U'\hat{v}' - k\,\omega_L\,\hat{v}$$

$$\hat{u}' = \frac{k\,U'\hat{u}}{\omega_L} + \hat{i}k\,\hat{v} - \hat{i}\,\frac{U''}{\omega_L}\,\hat{v} - \hat{i}\,\frac{U'}{\omega_L}\,\hat{v}'$$

$$\hat{u}' = \cancel{\frac{k\,U'\hat{u}}{\omega_L}} + \hat{i}k\,\hat{v} - \hat{i}\,\frac{U''}{\omega_L}\,\hat{v} - \hat{i}\,\frac{U'}{\cancel{\omega_L}}\left[\frac{\cancel{\omega_L}}{K C_m^2}\left(\hat{i}\,\omega_L\,\hat{u} - U'\hat{v}\right) \cancel{- \hat{i}\,k\,\hat{u}}\right]$$

$$\boxed{\Rightarrow \quad \hat{u}' = \hat{i}k\,\hat{v} - \hat{i}\,\frac{U''}{\omega_L}\,\hat{v} - \frac{\hat{i}\,U'}{K C_m^2}\left(\hat{i}\,\omega_L\,\hat{u} - U'\hat{v}\right)}$$

(Eq. (3.6) in d, Agostino et al 1997)

Appendix G :- Equivalence of the system with the classical Rayleigh equation

Setting $c_m \to \infty$,

$$\hat{u}' = \hat{i}k\hat{v} - \hat{i}\frac{U''}{\omega_L}\hat{v} - \cancel{\frac{\hat{i}U'}{KC_m^2}\left(\hat{i}\omega_L\hat{u} - U'\hat{v}\right)}$$

$$\hat{v}' = \cancel{\frac{\omega_L}{KC_m^2}\left(\hat{i}\omega_L\hat{u} - U'\hat{v}\right)} - \hat{i}k\hat{u}$$

Hence the system reduces to,

$$\hat{u}' = \hat{i}k\hat{v} - \hat{i}\frac{U''}{\omega_L}\hat{v}$$

$$\hat{v}' = -\hat{i}k\hat{u}$$

Now eliminating $\hat{u}$ from these two,

$$\Rightarrow \hat{u} = \frac{\hat{i}\hat{v}'}{k} \Rightarrow \hat{u}' = \frac{\hat{i}\hat{v}''}{k}$$

$$\therefore \frac{\cancel{\hat{i}}\hat{v}''}{k} = \cancel{\hat{i}}k\hat{v} - \frac{\cancel{\hat{i}}U''\hat{v}}{\omega_L}$$

$$\Rightarrow \hat{v}'' = k^2\hat{v} - \frac{k}{\omega_L}U''\hat{v}$$

$$(D^2 - k^2)\hat{v} + \frac{k}{\omega_L}U''\hat{v} = 0$$

$$\Rightarrow \frac{\omega_L}{k}(D^2-k^2)\hat{v} + U''\hat{v} = 0$$

$$\frac{(\omega - kU)}{k}(D^2-k^2)\hat{v} + U''\hat{v} = 0$$

$$\boxed{\Rightarrow (U-c)(D^2-k^2)\hat{v} - U''\hat{v} = 0}$$

Hence reduces to Rayleigh equation in the limit $C_m \to \infty$.