Wilson loop expectations for finite gauge groups

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Lattice gauge theories are models from physics obtained by discretizing continuous spacetime $\mathbb{R}^4$ by a lattice $\varepsilon\mathbb{Z}^4$.

They are rigorously defined, so we can actually prove things.

The continuum counterparts to lattice gauge theories are Euclidean Yang-Mills theories. Rigorous construction of such theories is of physical importance.

One approach to rigorously defining a Euclidean Yang-Mills theory is to take a scaling limit of lattice gauge theories.

In order to do so, various properties of lattice gauge theories must be very well understood.

This talk is about understanding one particular property.
The key objects of interest in a lattice gauge theory are the Wilson loops, which are certain random variables.

Introduced by Wilson (1974) to give a theoretical explanation of an observed phenomenon.

The explanation was in terms of Wilson loop expectations.

Since then, there have been a number of results analyzing Wilson loop expectations; see Chatterjee’s survey “Yang-Mills for probabilists” for a detailed overview.

One approach to taking a scaling limit involves being able to calculate Wilson loop expectations. More on this later.
Notation

- Let $G$ be a group, whose elements are $d \times d$ unitary matrices. The identity is denoted $I_d$. We will often refer to $G$ as the gauge group.
- Let $\Lambda := [-N, N]^4 \cap \mathbb{Z}^4$ be a large box.
- Let $\Lambda_1$ be the set of positively oriented edges in $\Lambda$. An edge $(x, y)$ is positively oriented if $y = x + e_i$, for some standard basis vector $e_i$.
- Let $\Lambda_2$ denote the set of plaquettes in $\Lambda$. A plaquette is a unit square whose four boundary edges are in $\Lambda$. Pictorially:

![Pictorial representation of a plaquette](image)

- Given an edge configuration $\sigma : \Lambda_1 \to G$, and a plaquette $p$ as above, define $\sigma_p := \sigma_{e_1} \sigma_{e_2} \sigma_{e_3}^{-1} \sigma_{e_4}^{-1}$.
Definition of lattice gauge theories

- Define
  \[ S_\Lambda(\sigma) := \sum_{p \in \Lambda_2} \text{Re(Tr}(I_d) - \text{Tr}(\sigma_p)). \]

- Let \( G^{\Lambda_1} := \{\sigma : \Lambda_1 \to G\}. \) Let \( \mu_\Lambda \) be the product uniform measure on \( G^{\Lambda_1} \).

- For \( \beta \geq 0 \), define the probability measure \( \mu_{\Lambda,\beta} \) on \( G^{\Lambda_1} \) by
  \[ d\mu_{\Lambda,\beta}(\sigma) := Z_{\Lambda,\beta}^{-1} \exp(-\beta S_\Lambda(\sigma)) \, d\mu_\Lambda(\sigma). \]

- We say that \( \mu_{\Lambda,\beta} \) is the lattice gauge theory with gauge group \( G \), on \( \Lambda \), with inverse coupling constant \( \beta \).

- Examples: \( G = U(1), SU(2), SU(3) \).

- For \( U(1) \), the continuum theory may be constructed directly - Gross (1986). Convergence of the lattice \( U(1) \) theory was established by Driver (1987).
Definition of Wilson loops

- Let $\gamma$ be a closed loop in $\Lambda$, with directed edges $e_1, \ldots, e_n$.
- The Wilson loop variable $W_\gamma$ is defined as

$$W_\gamma(\sigma) := \text{Tr}(\sigma_{e_1} \cdots \sigma_{e_n}).$$

[If $e$ is negatively oriented, then $\sigma_e := \sigma_{-e}^{-1}$.]

- Let $\langle W_\gamma \rangle_{\Lambda, \beta}$ be the expectation of $W_\gamma$ under $\mu_{\Lambda, \beta}$. Define

$$\langle W_\gamma \rangle_{\beta} := \lim_{\Lambda \uparrow \mathbb{Z}^4} \langle W_\gamma \rangle_{\Lambda, \beta}.$$  

[This limit may only exist after taking a subsequence, but I will pretend that this technical point is not present.]
Recently, Chatterjee (2018) computed Wilson loop expectations to leading order at large $\beta$, when $G = \mathbb{Z}_2 = \{\pm 1\}$.

For a loop of length $\ell$, we have

$$\langle W_\gamma \rangle_\beta \approx e^{-2\ell \epsilon^{-12\beta}}.$$ 

Suppose we have a loop $\gamma$ of length $\ell$ in $\mathbb{R}^4$. For $\epsilon > 0$, we can obtain a discretization $\gamma_\epsilon$ in $\epsilon\mathbb{Z}^4$ of length $\epsilon^{-1}\ell$.

If we set $\beta_\epsilon := -\frac{1}{12} \log \epsilon$, then as $\epsilon \downarrow 0$, we have

$$\langle W_{\gamma_\epsilon} \rangle_{\beta_\epsilon} \longrightarrow e^{-2\ell}.$$ 

This is the first step in one approach to taking a scaling limit.

We thus want to understand the leading order of $\langle W_\gamma \rangle_\beta$ at large $\beta$, for general gauge groups.
In recent work, I’ve computed the leading order for finite gauge groups. First, some notation for the formula.

Define

\[ \Delta_G := \min_{g \neq I} \text{Re}(\text{Tr}(I_d) - \text{Tr}(g)). \]

\[ G_0 := \{ g \in G : \text{Re}(\text{Tr}(I_d) - \text{Tr}(g)) = \Delta_G \}. \]

\[ A := \frac{1}{|G_0|} \sum_{g \in G_0} g. \]

**Theorem (C. 2020)**

Let \( \beta \geq \Delta_G^{-1} (1000 + 14 \log|G|) \). Let \( \gamma \) be a loop of length \( \ell \). Let \( X \sim \text{Poisson}(\ell|G_0|e^{-6\beta\Delta_G}) \). Then

\[ |\langle W_\gamma \rangle_\beta - \text{Tr}(EA^X)| \leq 10de^{-c(G)\beta}. \]
Let $-1 \leq \lambda_1, \ldots, \lambda_d \leq 1$ be the eigenvalues of $A$. Then

$$\text{Tr}(E A^X) = \sum_{i=1}^{d} e^{-(1-\lambda_i)\ell|G_0|e^{-6\beta\Delta G}}.$$ 

There is a recent article by Forsström, Lenells, and Viklund (2020), which handles finite Abelian gauge groups. They are able to obtain a much better $\beta$ threshold in this case.
Example: take $K \geq 2$. Let $G = \{e^{i2\pi k/K}, 0 \leq k \leq K - 1\}$.

Then $\Delta_G = 1 - \cos(2\pi/K)$, $G_0 = \{e^{i2\pi/K}, e^{-i2\pi/K}\}$, and $A = \cos(2\pi/K)$. Letting $\lambda := \ell |G_0| e^{-6\beta (1-\cos(2\pi/K))}$, then

$$\mathbb{E}_{\text{Poisson}(\lambda)} A = e^{-\lambda (1-A)}.$$

If $K = 2$, then $\Delta_G = 2$, $G_0 = \{-1\}$, $A = -1$, $\lambda = \ell e^{-12\beta}$. 
I will first outline the proof of the theorem in the Abelian case. The main probabilistic insights are already all present.

In essence, the proof has two main steps.

- Use a Peierls-type argument to show $\langle W_\gamma \rangle_\beta \approx \text{Tr}(E_A^{N_\gamma})$, where $N_\gamma$ is a count of weakly dependent rare events.
- Show $N_\gamma \approx \text{Poisson}$.

This two step outline was already present in Chatterjee (2018).

When the gauge group is non-Abelian, this still works. I will describe the main ideas behind showing this.
Suppose $d = 1$. Take some large box $\Lambda$. Recall $\mu_{\Lambda, \beta}$ is a probability measure on $G^{\Lambda_1}$ with the form

$$\mu_{\Lambda, \beta}(\sigma) = Z_{\Lambda, \beta}^{-1} \exp \left( -\beta \sum_{p \in \Lambda_2} (1 - \text{Re}(\sigma_p)) \right).$$

Define $\text{supp}(\sigma) := \{ p \in \Lambda_2 : \sigma_p \neq 1 \}$. Let $\Sigma \sim \mu_{\Lambda, \beta}$. Let $\mathbf{S} := \text{supp}(\Sigma)$.

When $\beta$ is large, $\Sigma_p = 1$ for most $p$, and so $\mathbf{S}$ is typically composed of sparsely distributed clumps.

We will see that $\mathbf{S}$ is easier to work with than $\Sigma$. 

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Here’s a 2D cartoon of S:
Decomposing $S$

- Since $S$ is typically made of sparsely distributed clumps, let us try to decompose $S$ into more elementary components.
- In general, any $P \subseteq \Lambda_2$ has a unique decomposition $P = V_1 \cup \cdots \cup V_n$ into “connected components”.

**Definition**

A set $V \subseteq \Lambda_2$ is called a **vortex** if it cannot be decomposed further.

- It turns out that if $P = V_1 \cup \cdots \cup V_n$, then

\[
\mathbb{E}[ W_{\gamma}(\Sigma) \mid S = P ] = \prod_{i=1}^{n} \mathbb{E}[ W_{\gamma}(\Sigma) \mid S = V_i ].
\]

- So we want to understand $\mathbb{E}[ W_{\gamma}(\Sigma) \mid S = V ]$, for vortices $V$. 

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Understanding vortex contributions

- For a vortex $V$, we say that $V$ appears in $S$ if $V$ is in the vortex decomposition of $S$.
- For an edge $e$, define $P(e)$ to be the set of plaquettes which contain $e$. Note $|P(e)|= 6$.

- In 3D, $P(e)$ looks like this.
- The smallest vortex which can appear in $S$ must be $P(e)$, for some edge $e$. All other vortices must have size $\geq 7$. 
For a vortex \( P(e) \), we have

\[
\mathbb{E}[W_\gamma(\Sigma) \mid S = P(e)] = \begin{cases} 
1 & e \notin \gamma \\
A & e \in \gamma 
\end{cases}.
\]

Let \( N_\gamma \) be the number of edges \( e \) in \( \gamma \) such that \( P(e) \) appears in \( S \). We then have

\[
\mathbb{E}[W_\gamma(\Sigma) \mid S] = A^{N_\gamma} Y,
\]

where \( Y \) is the contribution from vortices of size \( \geq 7 \).

Next, we show that with high probability, we can ignore vortices of size \( \geq 7 \).
In order for a vortex $V$ to be such that $\mathbb{E}[W_\gamma(\Sigma) \mid S = V] \neq 1$, it must be close to the loop $\gamma$.

Larger vortices are much less likely to appear in $S$.

So if we look in a neighborhood of $\gamma$, only $P(e)$ vortices are likely to appear.

We thus have

$$P(Y \neq 1) = \text{lower order.}$$

Thus on an event of high probability,

$$\mathbb{E}[W_\gamma(\Sigma) \mid S] = A^{N_\gamma}.$$
It remains to show $N_\gamma \approx \text{Poisson}$.

We apply the dependency graph approach to Stein’s method.

Given vortices $V_1 = P(e_1), \ldots, V_n = P(e_n)$ which are not too close to each other, we need to have

$$\mathbb{P}(V_1, \ldots, V_n \text{ appear in } S) \approx \prod_{i=1}^{n} \mathbb{P}(V_i \text{ appears in } S).$$

This is done by cluster expansion. Cluster expansion is a fairly well known tool; for example it appears in Seiler’s 1982 monograph on lattice gauge theories.

Thus we see why $S$ is nice to work with: it has “a lot of independence”.
The general case

- There are some technical difficulties that appear in the non-Abelian case.
- In the remaining time, I will present the key idea needed to handle these difficulties.
- I will then give a toy example showing why the key idea is useful.
The key idea

- Let us now think of $\Lambda$ as a graph.
- The fundamental group $\pi_1(\Lambda)$ is made of (equivalence classes of) closed loops in $\Lambda$ which begin and end at some fixed basepoint. Every closed loop is given by some sequence of edges $e_1 \cdots e_n$.

Observation (Szlachányi and Vecsernyès (1989))

Any $\sigma \in G^{\Lambda_1}$ induces a homomorphism $\psi_\sigma : \pi_1(\Lambda) \to G$, defined by

$$\psi_\sigma(e_1 \cdots e_n) := \sigma e_1 \cdots \sigma e_n.$$
Let $T$ be a spanning tree of $\Lambda$. Suppose $\sigma \in G^{\Lambda_1}$ is such that $\sigma_e = l_d$ for all $e \in T$.

Suppose additionally that $\sigma_p = l_d$ for all $p \in \Lambda_2$.

I then claim that in fact, $\sigma_e = l_d$ for all $e \in \Lambda_1$.

Initial attempt:
The crucial topological fact: any loop in $\pi_1(\Lambda)$ is a product of “Lasso type” loops:

For any Lasso type loop $L \in \pi_1(\Lambda)$, we have $\psi_{\sigma}(L) = l_d$.

Thus $\psi_{\sigma}$ is trivial.

Given $e = (x, y) \in \Lambda_1 \setminus T$, take a loop $a_e \in \pi_1(\Lambda)$ which uses $e$, and such that every other edge of $a_e$ is in $T$.

Since $\sigma = l_d$ on $T$, we have $\psi_{\sigma}(a_e) = \sigma_e$.

Because $\psi_{\sigma}$ is trivial, $\psi_{\sigma}(a_e) = l_d$. 
In summary

- Computing the leading order of Wilson loop expectations is a first step in taking a scaling limit.

- The key probabilistic insight:
  \[ \mathbb{E}[W_\gamma(\Sigma) \mid S] = \text{Tr}(A^{N_\gamma}) \text{ w.h.p.} \]

- The key technical tool: the components of \( S \) appear essentially independently.

- In the non-Abelian case, algebraic topology is a natural language to use.

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