1 Math 145 Thanksgiving Notes

Let me summarize how we construct a product variety $X \times Y$ of two varieties over a field k.

It is possible to do this directly for arbitrary varieties but another approach which is perhaps easier to follow consists of the following steps.

1. Construct the product variety $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$

2. When X, Y are projective varieties construct the product variety $X \times Y \subset \mathbb{P}^n(k) \times \mathbb{P}^m(k)$

3. For any varieties $U \subset X$ and $V \subset Y$ open construct the product $U \times V \subset X \times Y$.

Step 1: We topologize the set $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ by taking closed sets to be given by the vanishing of bihomogeneous ideals $I \subset k[x_0, ..., x_n, y_0, ..., y_m]$. Then we use the Segre embeding

$$\text{(Segre Embedding)} \qquad \begin{array}{rcl} \varphi\colon \mathbb{P}^n(k) \times \mathbb{P}^m(k) & \longrightarrow & \mathbb{P}^{(n+1)(m+1)-1} \\ [x_0\colon \cdots : x_n] \times [y_0\colon \cdots : y_m] & \rightarrow & [x_0 y_0\colon \cdots : x_i y_j\colon \cdots : x_n y_m] \end{array}$$

Then the image is closed (exercise) and the φ is a homeomorphism onto its image (this we can see because we can construct easily a local inverse. Suppose $p = [x_0 y_0\colon \cdots : x_i y_j\colon \cdots : x_n y_m]$ is a point in the image with $x_i y_j \neq 0$ then part of the coordinates of p will look like

$$p = [\cdots : x_i\, y_0\colon \cdots\, x_i y_m\colon \cdots : x_0 y_j\colon \cdots\, x_n y_j]$$

$$\text{divide by } x_i y_j \qquad p = \left[\cdots : \frac{y_0}{y_j}\colon \cdots \frac{y_m}{y_j}\colon \cdots : \frac{x_0}{x_i}\colon \cdots : \frac{x_n}{x_i}\right]$$

then it follows that $\left[\frac{x_0}{x_i}\colon \cdots : 1\colon \cdots : \frac{x_n}{x_i}\right] \times \left[\frac{y_0}{y_j}\colon \cdots : 1\colon \cdots : \frac{y_m}{y_j}\right]$ maps to p where the 1s appear in the ith and jth position.

Also the image is irreducible:

Proposition 1. *$\varphi(\mathbb{P}^n(k) \times \mathbb{P}^m(k))$ is irreducible*

Proof. Suppose $\varphi(\mathbb{P}^n(k) \times \mathbb{P}^m(k)) = Z_1 \cup Z_2$. Let $U_1 = \{y \in \mathbb{P}^m(k) | \mathbb{P}^n(k) \times y \not\subset Z_1\}$ and define U_2 similarly. Then if $y \in U_1 \cap U_2$ then $\mathbb{P}^n(k) \times y \not\subset Z_i$ which contradicts that $\mathbb{P}^n(k) \times y$ is irreducible. Therefore $U_1 \cap U_2 = \emptyset$.

I claim that U_i are open. Let $I(Z_1) = (f_i(x, y))$ (a bihomogeneous ideal). So if $y_0 \in U_1$ then some $(x_0, y_0) \notin Z_1$ and some $f_i(x_0, y_0) \neq 0$ so $\mathbb{P}^m(k)_g = \{g \neq 0\} \subset U_1$ with $g(y) = f(x_0, y)$. This shows U_1 is open. Similarly U_2 is open. If U_i were non empty they would intersect hence one of U_i is empty. Say $U_1 = \emptyset$ hene $\varphi(\mathbb{P}^n(k) \times \mathbb{P}^m(k)) \subset Z_1$. □

We define $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ to be the projective variety $\varphi(\mathbb{P}^n(k) \times \mathbb{P}^m(k))$ which tells us what the regular functions and rational functions are on $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$. This completes step 1.

We can repeatedly use this same arguement to show the product of any two varieties will again be irreducible.

Step 2: Let $X \subset \mathbb{P}^n(k)$ and $Y \subset \mathbb{P}^m(k)$ be projective varieties. Then $X \times Y$ is closed in $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ (exercise). Therefore $\varphi(X \times Y)$ is closed in the variety $\varphi(\mathbb{P}^n(k) \times \mathbb{P}^m(k))$. The image will again be irreducible by the same argument. We define $X \times Y$ to be the variety $\varphi(X \times Y) \subset \varphi(\mathbb{P}^n(k) \times \mathbb{P}^m(k))$

Step 3: If U is any variety then U is open in some projective variety X. Similary another variety V is open is some projective variety Y. Then $U \times V$ is open in $X \times Y$. Therefore $\varphi(U \times V)$ is an open subset of $\varphi(X \times Y)$ and the image will be irreducible by the same argument we've already used twice. We define $U \times V$ to be the variety $\varphi(U \times V)$.

Let's consider one example of this definition at work: $\mathbb{A}^2(k) \times \mathbb{P}^1(k)$.

First we take $\mathbb{A}^2(k) \times \mathbb{P}^1(k) \subset \mathbb{P}^2(k) \times \mathbb{P}^1(k)$ and realize the latter as a projective variety via

$$\begin{array}{rcc} \varphi\colon \mathbb{P}^2(k) \times \mathbb{P}^1(k) & \longrightarrow & \mathbb{P}^5(k) \\ [a\colon b\colon c] \times [x\colon y] & \to & [ax\colon ay\colon bx\colon by\colon cx\colon cy] \end{array}$$

Let $[z_0\colon \cdots\colon z_5]$ be coordinates on $\mathbb{P}^5(k)$. Then the image of φ is $\mathbb{V}(I)$ where

$$I = (z_1 z_2 - z_0 z_3, z_1 z_4 - z_0 z_5, z_3 z_4 - z_2 z_5)$$

Considering $\mathbb{A}^2(k) = \{c \neq 0\} \subset \mathbb{P}^2$ then $\varphi(\mathbb{A}^2(k) \times \mathbb{P}^1(k))$ is the open set $\mathbb{V}(I)_{z_4} \bigcup \mathbb{V}(I)_{z_5}$ where $\mathbb{V}(I)_{z_i} = \mathbb{V}(I) \cap \{z_i \neq 0\}$. Note $\mathbb{V}(I)_{z_i}$ is an affine vareity! $\mathbb{V}(I)_{z_4} \bigcup \mathbb{V}(I)_{z_5}$ correspond to the two open sets $\mathbb{A}^2(k) \times \{x \neq 0\}$ and $\mathbb{A}^2(k) \times \{y \neq 0\}$ each of which is isomorphic to $\mathbb{A}^3(k)$. We can ass this explicitly because we can determine what an affine variety is by computing its regular functions.

$$\begin{array}{rcc} \mathcal{O}_{\mathbb{A}^2(k) \times \mathbb{P}^1(k)}(\mathbb{V}(I)_{z_4}) & = & k[Z_0, Z_1, Z_2, Z_3, Z_5] / (Z_1 Z_2 - Z_0 Z_3, Z_1 - Z_0 Z_5, Z_3 - Z_2 Z_5),\ Z_i = \frac{z_i}{z_4} \\ & & Z_1 - Z_0 Z_5 \text{ means we can get rid of } Z_1 \\ & & Z_3 - Z_2 Z_5 \text{ means we can rid of } Z_3 \\ & & \text{plugging these into } Z_1 Z_2 - Z_0 Z_3 \text{ gives} \\ & & Z_0 Z_5 Z_2 - Z_0 Z_2 Z_5 = 0 \\ \Rightarrow & & \mathcal{O}_{\mathbb{A}^2(k) \times \mathbb{P}^1(k)}(\mathbb{V}(I)_{z_4}) \approx k[Z_0, Z_2, Z_5] \end{array}$$

Of course we've made many choices in this construction and this is ok because different choices lead to isomorphic varieties.

We used the Segre embedding to give a definition of product but in practice we don't need to use the Segre embedding to actuall work with products. For example we can understand $\mathbb{A}^2(k) \times \mathbb{P}^1(k)$ directly. Its closed sets are given by ideals $I \subset k[a, b, x_0, x_1]$ where the generators of I are homogeneous in the variables x_0, x_1. For example $\mathbb{V}(a x_0^2 + (a - b + 1) x_1 x_0)$ is a closed set in $\mathbb{A}^2(k) \times \mathbb{P}^1(k)$. It contains the subset $(0, 1) \times \mathbb{P}^1$ and it contains the subset $\mathbb{A}^2(k) \times [0\colon 1]$. Here is another closed set:

$$B = \mathbb{V}(a x_1 - b x_0) \subset \mathbb{A}^2(k) \times \mathbb{P}^1(k)$$

This is a very interesting closed set, in fact its a variety. But before studying it a bit let's state some facts about products.

Proposition 2. *Let X, Y be varieties and $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ be the two projections.*

1. *π_i are morphisms of varities*

2. $\text{Hom}(Z, X \times Y) = \text{Hom}(Z, X) \times \text{Hom}(Z, Y)$

3. *$f: X' \to X$ and $g: Y' \to Y$ then $f \times g: X' \times Y' \to X \times Y$*

4. *$\Delta(X) = \{(x, x) \in X \times X\}$ is a closed subvariety and $\delta: X \to \Delta(X)$ given by $x \to (x, x)$ is an isomorphism*

Proof. I'll only prove the last one. The rest are exercises to think through.

Its enough to show $\Delta(\mathbb{P}^n(k))$ is closed and this is true becuase

$$\Delta(\mathbb{P}^n(k)) = \mathbb{V}(I), I = (x_i y_j - x_j y_i)_{i,j}$$

where $[x_0: \cdots : x_n] \times [y_0: \cdots : y_n]$ are coordinates on $\mathbb{P}^n(k) \times \mathbb{P}^n(k)$. For $X \subset \mathbb{P}^n(k)$ closed we have

$$\Delta(X) = X \times X \cap \Delta(\mathbb{P}^n(k))$$

which is a closed subset of $X \times X$. Finally for an open $U \subset X$ we have

$$\Delta(U) = U \times U \cap \Delta(X)$$

Which is a closed subset of $U \times U$.

For the last statement note that the composition $X \xrightarrow{\Delta} X \times X \xrightarrow{\pi_i} X$ is the identity.

□

Here is a useful corollary

Corollary 3. *If $f, g: X \to Y$ are two morphisms then $\{x | f(x) = g(x)\}$ is closed. If f, g agree on a dense set then $f = g$.*

Proof. We have a morphism $f \times g: X \times X \to Y \times Y$ and $\{x | f(x) = g(x)\} = (f \times g)^{-1}(\Delta(Y))$ which is closed. If f, g agree on a dense set then $\{x | f(x) = g(x)\}$ is closed and dense so $f = g$. □

In the special case $X = \mathbb{A}^1(k)$ this says if two maps $f, g: \mathbb{A}^1(k) \to Y$ agree on $\mathbb{A}^1(k) - 0$ then they are the same map. This is a result that is used in your current homework.

Now let's return to $B = \mathbb{V}(a x_1 - b x_0) \subset \mathbb{A}^2(k) \times \mathbb{P}^1(k)$. I'll discuss this more after the break but here is some food for thought

there is a morphism	B	$\xrightarrow{\pi_2}$	$\mathbb{P}^1(k)$
there is a morphism	B	$\xrightarrow{\pi_1}$	$\mathbb{A}^2(k)$
we can restrict π_1 and this is an isomorphism!	$\pi_1^{-1}(\mathbb{A}^2(k) - 0)$	$\xrightarrow{\pi_1}$	$\mathbb{A}^2(k) - (0,0)$

Both maps are surjective (check) and if $U = \pi_1^{-1}(\mathbb{A}^2(k) - (0,0))$ then

$$\pi_1 : U \to \mathbb{A}^2(k) - (0,0)$$

is an isomorphism (exercise). Therefore set theoretically $B = \mathbb{A}^2(k) - (0,0) \cup \pi_1^{-1}((0,0))$. What is $\pi_1^{-1}(0,0)$? This fixes the problem that that we saw before: there is a morphism $\mathbb{A}^2 - (0,0) \to \mathbb{P}^1(k)$ but it doesn't extend to all of $\mathbb{A}^2(k)$. By replacing $\mathbb{A}^2(k)$ with the variety B we can extend the morphism.

In algebraic geometry this is called the *Blow up of the Plane*. Be careful talking algebraic geometry when you're in an airport!