Math 145. Homework Problems

1. Homework 1

Ch 1: 1.4, 1.5, 1.6, 1.15, 1.20, 1.22

The following problem will not be graded but you are encouraged to think about it:

1. Let $f_1 = xy - 1$ and $f_2 = x^2 + y^2 - 1$. For any finite field $\mathbb{F}_q$ count the cardinality of the sets $\mathbb{V}(f_i) \subset \mathbb{A}^2(\mathbb{F}_q)$

2. Homework 2

(1) Let $J = ((x - z)(x - y)(x - 2z), x^2 - y^2z) \subset \mathbb{C}[x, y, z]$. Describe the irreducible components of $\mathbb{V}(J)$. Prove or disprove. There exists $f \in I(\mathbb{V}(J))$ such that $f \not\in J$.

(2) Ch.1: 1.36, 1.39, 1.40

2.1. Weak Nullstellensatz. We will use some classical results from field theory to prove

Theorem 2.2 (Weak Nullstellensatz). Assume $k$ is infinite and perfect. Let $m \subset A = k[x_1, \ldots, x_n]$ be a maximal ideal then $L = A/m$ is a finite extension of $k$.

Note that if moreover $k$ is algebraically closed then we must have $L = k$.

2.3. Review of Field theory.

Theorem 2.4. Any field $k$ is has an algebraic closure $\overline{k}$. In particular, $k \rightarrow \overline{k}$ is an algebraic extension and $\overline{k}$ is an algebraically closed field. Any $f \in \overline{k}[x]$ splits into linear factors.

A polynomial $f \in k[x]$ is separable if $f(x)$ has no multiple roots in $\overline{k}$. If $k \rightarrow L$ is an algebraic extension and $\alpha \in L$ then then there is a surjection $\pi_\alpha: k[x] \rightarrow k(\alpha)$ given by $\pi_\alpha(x) = \alpha$. Then a generator of $\ker \pi_\alpha$ is called a minimal polynomial $m_{k,\alpha}$ of $\alpha$; typically $m_{k,\alpha}$ is chosen to be monic. An algebraic extension $k \rightarrow L$ is separable if for every $\alpha \in L$ we have $m_{k,\alpha}$ is separable. Finally, a field $k$ is perfect if every finite extension of $k$ is separable. There are many equivalent characterizations of separable fields (check Wikipedia). Another equivalent definition is $k$ is perfect if $\text{char}(k) = 0$ or if $\text{char}(k) = p > 0$ then the Frobenious map $x \mapsto x^p$ is an automorphism of $k$.

Theorem 2.5 (primitive element theorem). If $k$ is separable then every finite extension of $k$ is generated by one element: if $k \rightarrow L$ is finite then $L = k(\alpha)$ for some $\alpha \in L$.

If $k \rightarrow L$ is a field extension then a subset $S \subset L$ is a transcendence basis of $L$ over $k$ if $\alpha$ is transcendental over $k$ for every $\alpha \in S$ and moreover $L$ is algebraic over $k(\{\alpha\}_{\alpha \in S})$.

Theorem 2.6. Any field extension $k \rightarrow L$ has a transcendence basis. The cardinality of any two transcendence bases are the same.

The cardinality of any transcendence basis is called the transcendence degree of $L$ over $k$.

(1) Suppose $k \rightarrow L$ is an algebraic extension. If $L$ is finitely generated as a $k$-algebra then prove that $L$ is finite over $k$.
(2) Assume $k$ is perfect and let $k \to L$ be a field extension such that $L$ is a finitely generated as a $k$-algebra.

(a) Show there are elements $\alpha_1, \ldots, \alpha_n \in L$ such that
\[ L = k(\alpha_1, \ldots, \alpha_n)[t]/f(t) \]
Moreover if $L \neq k$ then $f(t) \neq 0$.

(b) By choosing $m$ possibly bigger then $n$ show there is a surjective map $\pi : k[x_1, \ldots, x_m, t] \to L$ such that $\pi(x_i) = \alpha_i$ and $\pi(t) = t$.

(c) Show there is a ring homomorphism
\[ \iota : L \to k[x_1, \ldots, x_m, 1\over g, t]/(f(t)) \]
with $\iota(\alpha_i) = x_i$ and $\iota(t) = t$. Hint: take $g = \prod_{i=1}^d b_i$

(3) Now assume $k$ is perfect and infinite. Show there exists $(p_1, \ldots, p_m) \in \mathbb{A}^m(k)$ such that $g(p_1, \ldots, p_m) \neq 0$. Conclude there is a ring homomorphism
\[ k[x_1, \ldots, x_m, 1\over g, t]/(f(t)) \to \overline{k} \]

(4) Use parts (2),(3) to show that if $k$ is perfect and infinite then $L$ is algebraic. Conclude that with (1) you have a proof of the weak Nullstellensatz.