Math 145. Homework Problems

1. Homework 8

Ch 6: 6.35, 6.36, 6.41, 6.42

(1) Suppose X, Y are varieties such that $K(X) \cong K(Y)$. Show that there are nonempty open subsets $U \subset X$ and $V \subset Y$ such that $U \cong V$. Note you may want to do this exercise before doing 6.42.

(2) (construction of MSpec). Let k be an algebraically closed field and let A be a finitely generated k-algebra. Assume that A is a domain. Let MSpec(A) denote the set of maximal ideals in A. For any ideal $I \subset A$ let

$$\mathbf{V}(I) = \{m \in \text{MSpec}(A) | m \supset I\}.$$

The Zariski topology on MSpec(A) has as closed sets the sets $\mathbf{V}(I)$. Let A_m be the localization of A at m. Define rings of functions on MSpec(A) by setting $\mathcal{O}_{\text{MSpec}(A)}(U) = \cap_{m \in U} A_m$ where U is any open set in MSpec(A).

(a) Elements of $f \in A$ define functions on MSpec(A) by setting $f(m)$ to be the image of f in $A/m \cong k$. Let $K = Frac(A)$ be the fraction field of A. Show A_m can be identified with the subset $\{\frac{a}{b} \in K | b(m) \neq 0\}$.

(b) Suppose X is a variety over k. Let $A = \mathcal{O}_X(X)$. Using the Nullstellensatz show that there is a homeomorphism $\phi\colon X \to \text{MSpec}(A)$ and for every open set $U \subset \text{MSpec}(A)$ that $\mathcal{O}_X(\phi^{-1}(U)) \cong \mathcal{O}_{\text{MSpec}(A)}(U)$.

2. Homework 7

Ch 3: 3.12

Ch 5: 5.34, 5.36, 5.38, 5.39(a)

This series of problems will explore properties of smooth curves and also demonstrate that the last problem on homework 4 is incorrect. Throughout these problems assume k is algebraically closed.

(1) Projective space has a property similar to sequential compactness in topology. Let $D^* = \mathbf{A}^1 - 0$. You can take as given the following (although you should are encouraged to verify it): (a) $\mathcal{O}(D^*) = k[t, t^{-1}]$ and (b) any morphism $\phi\colon D^* \to \mathbf{P}^n(k)$ comes from $n+1$ polynomials in $k[t, t^{-1}]$ so that $\phi(t) = [f_0(t) : \cdots : f_n(t)]$ where $f_i \in k[t, t^{-1}]$. Two sets of polynomials $(f_0, \ldots, f_n)$, $(g_0, \ldots, g_n)$ determine the same morphism if there is a unit $u \in k[t, t^{-1}]$ such that $g_i = uf_i$. Now show given any morphism $\phi\colon D^* \to \mathbf{P}^n(k)$ there is a unique extension to a morphism $\overline{\phi}\colon \mathbf{A}^1(k) \to \mathbf{P}^n(k)$ such that $\overline{\phi}|_{D^*} = \phi$. In particular ϕ determines a *unique* limit point $\overline{\phi}(0) \in \mathbf{P}^n(k)$. In fact any projective variety has this property. For affine varieties it is not always possible to extend ϕ. For example if $\phi\colon D^* \to \mathbf{A}^1(k)$ is given by $\phi(t) = t^{-1}$ then ϕ is not extendable. Varieties have an additional property that seems reasonable but requires some effor to prove: If $\psi\colon X \to Y$ is a morphism between varieties and $\phi\colon D^* \to X$ is extendable and $\psi \circ \phi$ is extendable then $\psi(\overline{\phi}(0)) = \overline{\psi \circ \phi}(0)$. Use this for the problem below.

(2) Let $U = \mathbf{P}^2(k) - [0:0:1]$. Then there is a morphism $\psi\colon U \to \mathbf{P}^1(k)$ given by $[x:y:z] \mapsto [x:y]$. Then ψ induces a field extension $K(\mathbf{P}^1(k)) \to K(U) = K(\mathbf{P}^2(k))$ given by $t \mapsto \frac{x}{y}$. Fix any point $p \in \mathbf{P}^1(k)$. Show there is a map $\phi\colon D^* \to U$ such that $\overline{\phi}(0) = [0:0:1]$ and $\overline{\psi \circ \phi}(0) = p$. Conclude that there cannot be an extension $\psi\colon \mathbf{P}^2(k) \to \mathbf{P}^1(k)$. This shows that the last problem on homework 4 is incorrect as stated. A correct statement is: if X is a variety then a nonconstant rational function $f \in K(X)$ is equivalent to a dominant *rational map* from $X \to \mathbf{P}^1(k)$; here rational means the morphism is only defined on an open set and we identify to rational maps if they agree on a nonempty open set. This is something we will discuss more in class.

(3) A version of the last problem on homework 4 is true for projective curves. We can prove a special case. Suppose $C = \mathbf{V}(F) \subset \mathbf{P}^2(k)$ is a smooth projective curve. Let $f \in K(\mathbf{P}^2(k))$ be a nonconstant rational function. Write $f = \frac{A}{B}$ for unique (up to scalar) homogeneous polynomials A, B such that A, B have no common factor and F does not divide either of them. Show that any such f restricts to C to give a morphism $\phi\colon C \to \mathbf{P}^1(k)$ such that $\phi([x:y:z]) = [A(x,y,z) : B(x,y,z)]$. Hint: show ϕ is well defined away from $\mathbf{V}(F, A, B)$. Show if $I(p, F \cap A) > I(p, F \cap B)$ then $\phi(p) = [0:1]$ and if $I(p, F \cap B) > I(p, F \cap A)$ then $\phi(p) = [1:0]$. In fact it is possible to extend this result for any f: if F divides A show there is an open set $U \subset C$ such that $f|_U = 0$. In this case we can associate to f the constant map $\phi(x) = [0:1]$. If F divides B show that the pole set of f intersects C in an open set. In this case we associate f the constant map $\phi(x) = [1:0]$.

(4) Another important fact we will discuss later is that the image of a projective variety under a morphism is closed. Use this fact to prove if $\phi\colon C \to \mathbf{P}^1(k)$ is the morphism associated to $f \in K(C)$ and f is nonconstant then ϕ is surjective.

(5) Let $R = k[x_0, \ldots, x_n]$ and consider it a graded ring by writing it as $R = \oplus_{d \geq 0} R_d$ where R_d is the vector space of homogeneous polynomials of degree d. A standard combinatorial argument shows that $\dim_k R_d = \binom{n+d}{d}$. The d-uple Veronese embedding is an important morphism for projective varieties. It is a morphism $\phi\colon \mathbf{P}^n(k) \to \mathbf{P}^N(k)$ where $N = \binom{n+d}{n} - 1$ and is given by $\phi([x_0 : \ldots : x_n]) = [f_0 : \cdots : f_N]$ where f_i are a basis for R_d. For example the 2-uple embedding of $\mathbf{P}^2(k)$ is given by

$$[x:y:z] \mapsto [x^2 : xy : xz : y^2 : yz : z^2].$$

You can take as given that the d-uple Veronese map has an image which is a projective vareity and ϕ is a homeomorphism onto its image. Let $F \in R_d$ and let ϕ be the d-uple Veronese map. Show that $\phi(\mathbf{V}(F)) = \phi(\mathbf{P}^n(k)) \cap \mathbf{V}(L)$ where L is linear and homogeneous. Conclude that $\mathbf{P}^n(k) - \mathbf{V}(F)$ is affine. Hint: show that $\mathbf{P}^N(k) - \mathbf{V}(L) \cong \mathbf{A}^N(k)$ and a closed subvariety of $\mathbf{A}^N(k)$ is affine.

(6) Let $C = \mathbf{V}(F) \subset \mathbf{P}^2(k)$ be an irreducible plane curve. Show that if F and G have no common factor then $C - \mathbf{V}(F, G)$ is isomorphic to an affine variety.

(7) Let $F, G \in k[x, y, z]$ be homogeneous with no common factor. Suppose H is another homogeneous polynomial such that the difference of intersection cycles is effective

$$H \cdot F - G \cdot F \geq 0.$$

Then a sufficient condition for there to exist a $B \in k[x,y,z]$ such that $H \cdot F - G \cdot F = B \cdot F$ is that

$$H = AF + BG$$

for some $A \in k[x,y,z]$. In this problem you will show this condition is also necessary. We will assume F is irreducible.

(a) Suppose $H \cdot F - G \cdot F = B \cdot F$. Show that $f = \frac{H}{BG} \in K(C)$ does not vanish anywhere on C. Conclude using (3),(4) that f is constant on C; say $f = c$.

(b) Show then $H - cGB$ is a homogeneous polynomial that must vanish on all of C. Therefore $H - cGB = AF$ for some A.

3. Homework 6

Ch 5: 5.1, 5.5, 5.8, 5.15

(1) Suppose we have an exact seqence of vector spaces

$$0 \to V_0 \to V_1 \to \cdots \to V_n \to 0$$

Show that $\sum_i (-1)^i \dim V_i = 0$.

(2) (a) Let $f = y^3 - 4x^3 + 3xy$ and $g = y^2 - x^3$. List all the intersection points $p \in \mathbf{V}(f,g)$ and compute, using properties (1)-(8) of intersection numbers all the values $I(p, f \cap g)$.

(b) Let $F = y^3 - 4x^3 + 3zxy$ and $G = zy^2 - x^3$. Check that your answer to (a) is consistent with Bezout's theorem.

(3) Suppose k is algebraically closed. Let $F, G \in k[x,y,z]$ be homogeneous polynomials of degrees n, m with no common component. Assume $\mathbf{V}(F,G,z) = \emptyset$. Let $X = \mathbf{V}(F,G)$ and $f = F(x,y,1)$, $g = G(x,y,1)$. Show that if $d \geq m+n$ then there is an isomorphism

$$\rho \colon \Gamma(X,d) \to k[x,y]/(f,g)$$

such that $\rho(H) = H(x,y,1)$. Note a proof of this is given in the text (step 3 on page 58). The point is to understand the argument and write it in your own words; you may try to prove it before looking at the text. You will not get full credit if you simply copy verbatim what is in the text.

4. Homework 5

Ch 2: 2.44. Also use corollary 2 of section 2.9 to prove the following. Let k be algebraically closed. If $I = (x,y) \subset k[x,y]$ and $f \in k[x,y]$ and irreducible element such that $p = (0,0) \in \mathbf{V}(f)$. Let $m \subset \mathcal{O}_p(C)$ be the maximal ideal. Show that $\mathcal{O}_p(C)/m^n \cong k[x,y]/(I^n, f)$.

Ch 3: 3.2, 3.11, 3.13, 3.14

4.1. $\mathbf{P}^1(k) \times \mathbf{P}^1(k)$**.** Let k be any field and $R = k[x_0, x_1, y_0, y_1]$. Define a $\mathbf{Z}^2$ grading on R by setting $deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. Let $R(n, m)$ be the vector space of homogeneous polynomials of degree (n, m). For example

$$R(1, 2) = \text{span}\{x_0y_0^2, x_0y_0y_1, x_0y_1^2, x_1y_0^2, x_1y_0y_1, x_1y_1^2\}$$

We say $r \in R$ is bi-homogeneous if $r \in R(n, m)$ for some n, m. An ideal $I \subset R$ is bi-homogeneous if it is generated by bi-homogenous elements.

(1) If I is bi-homogeneous show that $\mathbf{V}(I) = \{(x, y) \in \mathbf{P}^1(k) \times \mathbf{P}^1(k) | f(x, y) = 0 \forall f \in I\}$ is well defined and show that there is a topology on $\mathbf{P}^1(k) \times \mathbf{P}^1(k)$ determined by taking the closed sets to be $\mathbf{V}(I)$ for I a bi-homogeneous ideal.
(2) Let $[x_0 : x_1], [y_0 : y_1]$ be coordinates on $\mathbf{P}^1(k) \times \mathbf{P}^1(k)$ and set $K = k(\frac{x_0}{x_1}, \frac{y_0}{y_1})$. Show that if $f \in K$ then there are bi-homogeneous polynomials a, b of the same degree which are unique up to scalar such that $f = \frac{a}{b}$. If $p \in \mathbf{P}^1(k) \times \mathbf{P}^1(k)$ and $f \in K$ we say f is defined at p if $b(p) \neq 0$.
(3) Assume from now on that k is algebraically closed. Let $Z_{ij} = \mathbf{V}(x_iy_j)$ and let U_{ij} be the complement of Z_{ij}. Show there is an isomorphism $\phi_{ij} \colon \mathbf{A}^2(k) \to U_{ij}$ in the sense that ϕ is a homeomorphism and ϕ induces an isomorphism $\phi^* \colon \mathcal{O}(U_{ij}) \to \mathcal{O}(\mathbf{A}^2(k))$.
(4) Consider the map of sets $\psi \colon \mathbf{P}^1(k) \times \mathbf{P}^1(k) \to \mathbf{P}^3(k)$ given by $[x_0 : x_1] \times [y_0 : y_1] \mapsto [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1]$. Let $[a : b : c : d]$ be coordinates on $\mathbf{P}^3(k)$. Show that $X := \text{im}(\psi) = \mathbf{V}(ad - bc)$. Show that $ad - bc \in k[a, b, c, d]$ is an irreducible element so X is a variety. Show that ψ defines a homeomorphism $\psi \colon \mathbf{P}^1(k) \times \mathbf{P}^1(k) \to X$.
(5) Let U_a be the complement of $\mathbf{V}(a) \subset \mathbf{P}^3(k)$ and set $X_a = X \cap U_a$. Define X_b, X_c, X_d similarly. Show that ψ induces an isomorphism $\mathcal{O}(X_a) \to \mathcal{O}(\psi^{-1}(X_a))$ as well as for the open sets X_b, X_c, X_d

This shows that the abstractly defined variety $\mathbf{P}^1(k) \times \mathbf{P}^1(k)$ is a projective variety in the sense we defined before: it's topology and rings of function can be considered to be inherited from the projective variety X.

5. Homework 4

Ch 4: 4.10, 4.17, 4.25
For all of these problem assume the field k is algebraically closed.

(1) Let $[x : y : z]$ be homogeneous coordinates on $\mathbf{P}^2(k)$. Show that

$$\mathcal{O}(\mathbf{P}^2(k) - [0 : 0 : 1]) = k.$$

Hint: if $f \in K(\mathbf{P}^2(k))$ is any nonconstant rational function then show that the pole set of f is a hypersurface.
(2) (a) Let $\phi \colon X \to Y$ be a morphism of varieties. Show that if ϕ is dense, that is $\overline{\phi(X)} = Y$ then ϕ induces a field extension from rational function on Y to rational functions on X: $\phi^* \colon K(Y) \to K(X)$.
(b) Show that the hypothesis $\overline{\phi(X)} = Y$ is necessary by giving an example of a morphism $\phi \colon X \to Y$ such that there exists no field extentions from $K(Y) \to K(X)$.

(3) Show that if X is a variety then a nonconstant rational funciton $f \in K(X)$ is equivalent to a morphism $X \to \mathbf{P}^1(k)$. Use (2) and the fact that $K(\mathbf{P}^1(k)) = k(t)$.

6. Homework 3

Note what we have denoted $\mathcal{O}(X)$ is reffered to as $\Gamma(X, k)$ in the text. In all the problems you can assume the field k is algebraically closed unless otherwise specified.

In 2.7 you can skip showing that $\phi^{-1}(X)$ is algebraic; just show the second statement.

Ch 2: 2.4,2.7,2.12(b), 2.17, 2.18, 2.21

Ch 4: 4.2, 4.4

7. Homework 2

(1) Let $J = ((x-z)(x-y)(x-2z), x^2 - y^2 z) \subset \mathbf{C}[x, y, z]$. Describe the irreducible components of $\mathbf{V}(J)$. Prove or disprove. There exists $f \in I(\mathbf{V}(J))$ such that $f \notin J$.

(2) Ch 1: 1.36, 1.39, 1.40

7.1. **Weak Nullstellensatz.** We will use some classical results from field theory to prove

Theorem 7.2 (Weak Nullstellensatz)**.** *Assume k is infinite and perfect. Let $m \subset A = k[x_1, \dots, x_n]$ be a maximal ideal then $L = A/m$ is a finite extension of k.*

Note that if moreover k is algebraically closed then we must have $L = k$.

7.3. **Review of Field theory.**

Theorem 7.4. *Any field k is has an algebraic closure $\overline{k}$. In particular, $k \to \overline{k}$ is an algebraic extension and $\overline{k}$ is an algebraically closed field. Any $f \in \overline{k}[x]$ splits into linear factors.*

A polynomial $f \in k[x]$ is *separable* if $f(x)$ has no multiple roots in $\overline{k}$. If $k \to L$ is an algebraic extension and $\alpha \in L$ then then there is a surjection $\pi_\alpha \colon k[x] \to k(\alpha)$ given by $\pi_\alpha(x) = \alpha$. Then a generator of $\ker \pi_\alpha$ is called a minimal polynomial $m_{k,\alpha}$ of α; typically $m_{k,\alpha}$ is chosen to be monic. An algebraic extension $k \to L$ is *separable* if for every $\alpha \in L$ we have $m_{k,\alpha}$ is separable. Finally, a field k is *perfect* if every finite extension of k is separable.

There are many equivalent characterizations of separable fields (check Wikipedia). Another equivalent definition is k is perfect if $char(k) = 0$ or if $char(k) = p > 0$ then the Frobenious map $x \mapsto x^p$ is an automorphism of k.

Theorem 7.5 (primitive element theorem)**.** *If k is separable then every finite extension of k is generated by one element: if $k \to L$ is finite then $L = k(\alpha)$ for some $\alpha \in L$.*

If $k \to L$ is a field extension then a subset $S \subset L$ is a *transcendence basis* of L over k if α is transcendental over k for every $\alpha \in S$ and moreover L is algebraic over $k(\{\alpha\}_{\alpha \in S})$.

Theorem 7.6. *Any field extension $k \to L$ has a transcendence basis. The cardinality of any two transcendence bases are the same.*

The cardinality of any transcendence basis is called the transcendence degree of L over k.

(1) Suppose $k \to L$ is an algebraic extension. If L is finitely generated as a k-algebra then prove that L is finite over k.

(2) Assume k is perfect and let $k \to L$ be a field extension such that L is a finitely generated as a k-algebra.
   (a) Show there are elements $\alpha_1, \dots, \alpha_n \in L$ such that
$$L = k(\alpha_1, \dots, \alpha_n)[t]/f(t)$$
   Moreover if $L \neq k$ then $f(t) \neq 0$.
   (b) By choosing m possibly bigger then n show there is a surjective map $\pi\colon k[x_1, \dots, x_m, t] \to L$ such that $\pi(x_i) = \alpha_i$ and $\pi(t) = t$.
   (c) Show there is a ring homomorphism
$$\iota\colon L \to k[x_1, \dots, x_m, \frac{1}{g}, t]/(f(t))$$
   with $\iota(\alpha_i) = x_i$ and $\iota(t) = t$. Hint: write $f(t) = \sum_{i=1}^{d} \frac{a_i}{b_i} t^i$ where $a_i, b_i \in k[\alpha_1, \dots, \alpha_n]$ and take $g = \prod_{i=1}^{d} b_i(x_1, \dots, x_n)$.

(3) Now assume k is perfect and infinite. Show there exists $(p_1, \dots, p_m) \in \mathbf{A}^m(k)$ such that $g(p_1, \dots, p_n) \neq 0$. Conclude there is a ring homomorphism
$$k[x_1, \dots, x_m, \frac{1}{g}, t]/(f(t)) \to \overline{k}$$

(4) Use parts (2),(3) to show that if k is perfect and infinite then L is algebraic. Conclude that with (1) you have a proof of the weak Nullstellensatz.

8. Homework 1

Ch 1: 1.4, 1.5,1.6, 1.15, 1.20, 1.22

The following problem will not be graded but you are encouraged to think about it:

1. Let $f_1 = xy - 1$ and $f_2 = x^2 + y^2 - 1$. For any finite field $\mathbb{F}_q$ count the cardinality of the sets $\mathbb{V}(f_i) \subset \mathbb{A}^2(\mathbb{F}_q)$