Math 145 Practice Final

The final will be between 5-8 problems. So this is longer than the actual final. Some problems will have multiple parts. The exam should take you 2-4 hours. The field k is always algebraically closed.

1. Prove that the non-constant morphisms $\mathbb{A}^1(k) \to \mathbb{A}^1(k)$ are precisely the maps $x \mapsto f(x)$ for a non-constant polynomial. Deduce that the only automorphisms of $\mathbb{A}^1(k)$ are $x \mapsto ux + v$ where $u \in k^\times$ and $v \in k$. An automorphism is a morphism $X \to X$ which is an isomorphism.

2. For any $[a: b] \in \mathbb{P}^1(k)$ show there is a linear change of coordinates that gives a morphism g: $\mathbb{P}^1(k) \to \mathbb{P}^1(k)$ such that $g([a: b]) = [1: 0]$.

3. Show if $\varphi: \mathbb{P}^1(k) \to \mathbb{P}^1(k)$ is an automorphsim and $\varphi([1: 0]) = [1: 0]$ then the restriction of φ to $\mathbb{P}^1(k) - \{[1: 0]\} = \mathbb{A}^1(k)$ gives an automorphism of $\mathbb{A}^1(k)$. Use (1) to deduce that φ comes from a linear change of coordinates:

$$\varphi([x: y]) = [\varphi_{11}x + \varphi_{12}\, y: \varphi_{21}x + \varphi_{22}y]$$

and show moreover that $\varphi_{21} = 0$.

4. Let $\mathrm{PGL}_2(k) = \mathrm{GL}_2(k)/\sim$ where two matrices are equivalent $a \sim b$ if $a = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} b$ for some $\lambda \in k^\times$. Any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(k)$ gives an automorphism φ of $\mathbb{P}^1(k)$ where

$$\varphi([x: y]) = [ax + by: cx + dy].$$

Use problems (1) - (3) to show any automorphism of $\mathbb{P}^1(k)$ is of the this form and therefore the group of automorphisms of $\mathbb{P}^1(k)$ is $\mathrm{PGL}_2(k)$.

5. Give examples of each of the following

   a) An irreducible plane cuve in $\mathbb{A}^2(\mathbb{C})$ with 5 singularities on the x-axis. Indicate where the singularities occur.

   b) A discrete valuation ring R containing k with maximal ideal m such that $R/m = k$ but $\mathrm{Frac}(R) \neq k(x)$.

   c) An irreducible plane curve C in $\mathbb{A}^2(\mathbb{C})$ such that $(0, 0) \in C$ and C has 3 distinct tangent lines at $(0, 0)$.

   d) An irreducible closed curve in $\mathbb{A}^3(k)$ that is not contained in a plane. Note a variety $X \subset \mathbb{A}^3(k)$ is a curve if the transcendence degree of the function field is 1: $\mathrm{tr.degk}(X) = 1$.

6. For each of the following compute the intersection cycle $F \bullet F'$ and verify Bezout's theorem holds. Assume $\mathrm{char}(k) = 0$

   a) $F = y^2 z - x(x - 2z)(x + z), F' = y^2 + x^2 - 2xz$

   b) $F = (x^2 + y^2)z + x^3 + y^3, F' = x^3 + y^3 - 2xyz$

7. Let $R = k[x_0, \ldots, x_n]$ and R_d the vector space of homogeneous polynomials of degree d. Prove that $\dim R_d = \binom{d+n}{n} = \binom{d+n}{d}$.

8. Let $\psi: \mathbb{P}^n(k) \to \mathbb{P}^N(k)$ be the degree d Veronese embedding where $N = \binom{d+n}{n} - 1$. If $M_0, \ldots, M_N$ is a basis of R_d from the previous problem then $\psi([x_0: \cdots : x_n]) = [M_0: \cdots : M_N]$. Let us choose a basis such that $M_j = x_j^d$ for $0 \leq j \leq n$. Let $[y_0: \cdots : y_N]$ be coordinates on $\mathbb{P}^N(k)$ and let $U_j = \{y_j \neq 0\}$ for $0 \leqslant j \leqslant n$. Compute the ring map $\psi^*: \mathcal{O}_{\mathbb{P}^N}(U_j) \to \mathcal{O}_{\mathbb{P}^n}(\psi^{-1}(U_j))$.

9. Let ψ: $\mathbb{P}^1(k) \to \mathbb{P}^3(k)$ be the degree 3 Veronese embedding. The image of ψ is a closed subvariety of $\mathbb{P}^3(k)$ given by a homogeneous ideal I. Compute generators for I.

10. Let F, G be two irreducible homogeneous degree 2 polynomials in $\mathbb{C}[x, y, z]$. Show there is a linear change of coordinates on $\mathbb{P}^2(k)$ that sends $\mathbb{V}(F)$ to $\mathbb{V}(G)$. Hint: this is a problem about linear algebra; an irreducible degree 2 polynomial defines a nondegenerate quadratic form on $\mathbb{C}^3$.

11. Show that $\mathbb{V}(xy - 1) \subset \mathbb{A}^2(k)$ and $\mathbb{V}(y - x^2) \subset \mathbb{A}^2(k)$ are not isomorphic affine varieties. Show that $\mathbb{V}(x) \subset \mathbb{A}^3(k)$ and $\mathbb{V}(x - y^4 - z^4) \subset \mathbb{A}^3(k)$ are isomorphic.

12. Show that if $X \subset \mathbb{A}^n(k)$ is affine variety then the projective closure $\overline{X} \subset \mathbb{P}^n(k)$ is irreducible.