# Practice Final Solutions

1. A morphism $\varphi: \mathbb{A}^1(k) \to \mathbb{A}^1(k)$ is equivalent to ring homomorphism $\varphi^*: k[x] \to k[x]$ and $\varphi^*$ is determined by the image of $x$, $\varphi^*(x) \in k[x]$ and $\varphi$ is nonconstant if and only if $\varphi^*(x)$ is nonconstant. Let $f(x) = \varphi^*(x)$. If $\varphi$ is an automorphism then there is an inverse $\psi$. Let $\psi^*(x) = g(x)$. The composition of morphisms corresponds to composition of polynomials. Then $f(g(x)) = x$ but on the other hand $\deg f(g(x)) = \deg(f)\deg(g)$ hence $\deg(f) = 1$.

2. If $b = 0$ we there is nothing to prove. Otherise the linear change of coordinates gives by $m = \left(\begin{smallmatrix} 0 & 1 \\ -b & a \end{smallmatrix}\right)$ maps $[a: b]$ to $[1: 0]$.

3. Set $U = \mathbb{P}^1 - [1: 0]$. Then $U$ is isomorphic to $\mathbb{A}^1$ and $\varphi$: $\varphi^{-1}(U) \to U$ is an isomorphism. Also $\varphi^{-1}(U) = \mathbb{P}^1 - \varphi^{-1}([1: 0]) = \mathbb{P}^1 - [1: 0] = U$ hence $\varphi$ defines an automorphism of $\mathbb{A}^1$. By problem 1 we have $\varphi([x: 1]) = [ux + v: 1]$. It must be that $\varphi$ is the linear change of coordinates given by $\left(\begin{smallmatrix} u & v \\ 0 & 1 \end{smallmatrix}\right)$ since both of these morphisms agree on a dense open set.

4. Let $\varphi$ be any automorphism of $\mathbb{P}^1(k)$ and let $[a: b] = \varphi([1: 0])$. Let $m = \left(\begin{smallmatrix} 0 & 1 \\ -b & a \end{smallmatrix}\right)$ then $m \circ \varphi$ is an automorphism that sends $[1: 0]$ to $[1: 0]$ so by part 3 we have $m \circ \varphi = \left(\begin{smallmatrix} u & v \\ 0 & 1 \end{smallmatrix}\right)$ and hence $\varphi = \left(\begin{smallmatrix} 0 & 1 \\ -b & a \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} u & v \\ 0 & 1 \end{smallmatrix}\right)$.

5.
   a) $\mathbb{V}(y^2 - x^2(x-1)^2(x-2)^2(x-3)^2(x-4)^2)$. Note that $a$ is a root of $(x-a)^2$ and its derivative so partial derivatives both vanish at the points $(0,0), (0,1), (0,2), (0,3), (0,4)$

   b) Set $A = k[x, y] / (y^2 = x^3 - x)$ and $R = A_{(x,y)}$ then $\mathrm{Frac}(R)$ is a degree 2 extension of $k(x)$.

   c) $\mathbb{V}(xy(x + 2y) + x^4 + y^4)$, this is irrreducible since the homogeous peices have no common factors.

   d) Let $C$ be the curve in $\mathbb{A}^3(t)$ parametrized by $(t, t^2, t^3)$. The image doesn't lie in any plane since $t, t^2, t^3$ are linearly independent. It is irreducible because it is the image of $\mathbb{A}^1$ which is irreducible.

6. The first case is ommitted. For $F = (x^2 + y^2)z + x^3 + y^3$ and $G = x^3 + y^3 - 2xyz$ we first consider $\mathbb{V}(F, G, z)$ which is $\{[-1: 1: 0], [\zeta: 1: 0], [\zeta^2: 1: 0]\}$ where $\zeta$ is a primitive cube root of $-1$. The remaining points must have $z \neq 0$ so we can dehomogenize $f = x^2 + y^2 + x^3 + y^3$ and $x^3 + y^3 - 2xy$ and find the only solution is $(x, y) = (0, 0)$, or $[0: 0: 1]$. We note that there are no common tangent lines at $[0: 0: 1]$ and the multiplicity on each curve is 2 so this intersection number is 4. There are also no common tangent lines at $[\zeta: 1: 0]$, $[\zeta^2: 1: 0]$ so these each have an intersection number of 1. There is a common tangent line at $[-1: 1: 0]$ and we can calculate using that this is a smooth point of the curve or use Bezout's theorem to see that this number must be 3.

7. omitted

8. $U_j \approx \mathbb{A}^N$ and $\mathcal{O}_{\mathbb{P}^N}(U_j) \approx k[Y_1, \ldots, Y_N]$ where $Y_i = \frac{y_i}{y_j}$. Similarly, $\psi^{-1}(U_j) = \{x_j^d \neq 0\} = \{x_j \neq 0\}$ thus $\psi^{-1}(U_j) \approx \mathbb{A}^n$ and $\mathcal{O}_{\mathbb{P}^n}(\psi^{-1}(U_j)) \approx k[X_1, \ldots, X_n]$ where $X_i = \frac{x_i}{x_j}$. Then $\psi^*(Y_i) = \frac{M_i}{x_j^d}$ which is a polynomial in the variables $X_i$.

9. We have $\mathrm{im}\,\psi = \mathbb{V}(I)$ where $I = \ker(\psi^*)$ where $\psi^*$ is the map of graded rings

$$\begin{array}{rcl} \psi^*: k[y_0, \ldots, y_4] & \to & k[x_0, x_1] \\ (y_0, y_1, y_2, y_3) & \to & (x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3) \end{array}$$

   and $I = (y_0 y_3 - y_1 y_2, y_1^2 - y_0 y_2, y_2^2 - y_1 y_3)$.

10. omitted

11. Notice that $k[x, y]/(y-x^2) \approx k[x]$ and any nonconstant element is not a unit. On the other hand $k[x, y]/xy-1 \approx k[x, x^{-1}]$ and $x$ is a nonconstan unit. For the second case we note

$$k[x, y, z]/(x - y^4 - z^4) \approx k[y, z] \approx k[x, y, z]/x$$

12. Decompose $\overline{X}$ into a union of irreducible algebraic sets. One irreducible component contains $X$. On the other hand $\overline{X}$ is the smallest projective algebraic set containing $X$ so there can only be one irreducbile component.