Practice Final Solutions

1. A morphism \( \varphi : \mathbb{A}^1(k) \to \mathbb{A}^1(k) \) is equivalent to ring homomorphism \( \varphi^* : k[x] \to k[x] \) and \( \varphi^* \) is determined by the image of \( x \), \( \varphi^*(x) \in k[x] \) and \( \varphi \) is nonconstant if and only if \( \varphi^*(x) \) is nonconstant. Let \( f(x) = \varphi^*(x) \). If \( \varphi \) is an automorphism then there is an inverse \( \psi \). Let \( \psi^*(x) = g(x) \). The composition of morphisms corresponds to composition of polynomials. Then \( f(g(x)) = x \) but on the other hand \( \deg f(g(x)) = \deg(f)\deg(g) \) hence \( \deg(f) = 1 \).

2. If \( b = 0 \) we there is nothing to prove. Otherwise the linear change of coordinates gives by \( m = \left( \begin{array}{c} 0 & 1 \\ -b & a \end{array} \right) \) maps \( [a:b] \) to \( [1:0] \).

3. Set \( U = \mathbb{P}^1 - [1:0] \). Then \( U \) is isomorphic to \( \mathbb{A}^1 \) and \( \varphi : \mathbb{A}^1 \to \mathbb{A}^1 \) is an automorphism. Also \( \varphi^{-1}(U) = \mathbb{P}^1 - \varphi^{-1}([1:0]) = \mathbb{P}^1 - [1:0] = U \) hence \( \varphi \) defines an automorphism of \( \mathbb{A}^1 \). By problem 1 we have \( \varphi([x:1]) = [ux + v:1] \). It must be that \( \varphi \) is the linear change of coordinates given by \( \left( \begin{array}{c} u \\ v \end{array} \right) \) since both of these morphisms agree on a dense open set.

4. Let \( \varphi \) be any automorphism of \( \mathbb{P}^1(k) \) and let \( [a:b] = \varphi([1:0]) \). Let \( m = \left( \begin{array}{c} 0 & 1 \\ -b & a \end{array} \right) \) then \( m \circ \varphi \) is an automorphism that sends \( [1:0] \) to \( [1:0] \) so by part 3 we have \( m \circ \varphi = \left( \begin{array}{c} u \\ v \end{array} \right) \) and hence \( \varphi = \left( \begin{array}{c} u \\ v \end{array} \right)^{-1} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \).

5. a) \( \mathbb{V}(y^2 - x^2(x - 1)^2(x - 2)^2(x - 3)^2(x - 4)^2) \). Note that \( a \) is a root of \( (x - a)^2 \) and its derivative so partial derivatives both vanish at the points \( (0,0), (0,1), (0,2), (0,3), (0,4) \)
   b) Set \( A = k[x,y]/(y^2 = x^3 - x) \) and \( R = A_{(x,y)} \) then \( \text{Frac}(R) \) is a degree 2 extension of \( k(x) \).
   c) \( \mathbb{V}(x(y(x + 2)y + x^4 + y^4)) \), this is irreducible since the homogeneous pieces have no common factors.
   d) Let \( C \) be the curve in \( \mathbb{A}^3(t) \) parametrized by \( (t, t^2, t^3) \). The image doesn’t lie in any plane since \( t, t^2, t^3 \) are linearly independent. It is irreducible because it is the image of \( \mathbb{A}^3 \) which is irreducible.

6. The first case is omitted. For \( F = (x^2 + y^2)z + x^3 + y^3 \) and \( G = x^3 + y^3 - 2xyz \) we first consider \( \mathbb{V}(F, G, z) \) which is \( \{ [-1:1:0], [\zeta:1:0], [\zeta^2:1:0] \} \) where \( \zeta \) is a primitive cube root of \( -1 \). The remaining points must have \( z \neq 0 \) so we can dehomogenize \( f = x^2 + y^2 + x^3 + y^3 \) and \( x^3 + y^3 - 2xy \) and find the only solution is \( (x, y) = (0,0) \), or \( [0:0:1] \). We note that there are no common tangent lines at \( [0:0:1] \) and the multiplicity on each curve is 2 so this intersection number is 4. There are also no common tangent lines at \( [\zeta:1:0], [\zeta^2:1:0] \) so these each have an intersection number of 1. There is a common tangent line at \( [-1:1:0] \) and we can add calculate using this that this is a smooth point of the curve or use Bezout’s theorem to see that this number must be 3.

7. omitted

8. \( U_j \approx \mathbb{A}^N \) and \( \mathcal{O}_{\mathbb{P}^N}(U_j) \approx k[Y_1, \ldots, Y_N] \) where \( Y_i = \frac{m}{y_j} \). Similarly, \( \psi^{-1}(U_j) = \{ x_j \neq 0 \} = \{ x_j \neq 0 \} \)
   thus \( \psi^{-1}(U_j) \approx \mathbb{A}^n \) and \( \mathcal{O}_{\mathbb{P}^n}(\psi^{-1}(U_j)) \approx k[X_1, \ldots, X_n] \) where \( X_i = \frac{x_i}{x_j} \). Then \( \psi^*(Y_i) = \frac{M_i}{x_j} \) which is a polynomial in the variables \( X_i \).

9. We have \( \text{im } \psi = \mathbb{V}(I) \) where \( I = \ker(\psi^*) \) where \( \psi^* \) is the map of graded rings
   \[
   \begin{array}{c}
   \psi^* : k[y_0, \ldots, y_4] & \to & k[x_0, x_1] \\
   (y_0, y_1, y_2, y_3) & \mapsto & (x_0^2, x_0x_1, x_0x_1^2, x_1^3)
   \end{array}
   \]
   and \( I = \langle y_0y_1 - y_1y_2, y_1^2 - y_0y_2, y_2^2 - y_1y_3 \rangle \).

10. omitted
11. Notice that $k[x, y] / (y - x^2) \approx k[x]$ and any nonconstant element is not a unit. On the other hand $k[x, y] / xy - 1 \approx k[x, x^{-1}]$ and $x$ is a nonconstant unit. For the second case we note

$$k[x, y, z] / (x - y^2 - z^4) \approx k[y, z] = k[x, y, z] / x$$

12. Decompose $\overline{X}$ into a union of irreducible algebraic sets. One irreducible component contains $X$. On the other hand $\overline{X}$ is the smallest projective algebraic set containing $X$ so there can only be one irreducible component.