Categorical Trace

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List of Questions

1 Introduction

This note is (expanded) from Xinwen Zhu's class at Stanford, 2023 Spring.

1.1 Overview

A fundamental invariant of a linear endomorphism $f : V \to V$ of a finite dimensional vector space is its trace $\mathrm{Tr}(f)$. The construction $f \mapsto \mathrm{Tr}(f)$ can be performed in any symmetric monoidal (higher)-category. In recent years, people realize that several seemingly unrelated theorems/constructions, including but not limited to

- Grothendieck-Riemann-Roch formula
- Atiyah-Bott fixed point formula
- Lefschetz-Verdier formula
- Deligne-Lusztig theory
- V. Lafforgue's construction of excursion algebras

can all be understood in the framework of such trace construction. This course aims to give an introduction of such ideas.

1.1.1 Part I. Higher linear algebra

In this part we discuss the general trace formalism in (higher) symmetric monoidal categories. After explaining some basic examples, we will in particular perform the construction in

(1) symmetric monoidal (2-)category LinCat_k of k-linear (infinity)-categories;

(2) Its Morita category (objects are k-linear monoidal categories and morphisms are bimodule categories).

This part is closely related extended TQFT in dimension 1,2,3, and also related to ideas of non-commutative algebraic geometry.

1.1.2 Part II. Formalism of sheaf theories (a.k.a. six functor formalisms)

To make the trace formalism useful, one needs to supply many interesting examples of (monoidal) k-linear categories. They usually come from category of sheaves (e.g. coherent, constructible) on some space. We discuss how to organize various sheaf theories as (lax) symmetric monoidal functors from the category of correspondences to LinCat_k mentioned above, and specialize to coherent and étale sheaves.

1.1.3 Part III. Applications in algebraic geometry and representation theory

We will apply the methods developed in the course to some of the concrete theories as mentioned above.

1.2 Acknowledgement

I deeply thank Prof. Zhu for the beautiful classes and inspiring discussions. I also thank Daniel Kim and Sean Cotner for helpful conversations and their patience for my many dumb questions.

Lecture 1: 4/4/2023

2 Linear Algebras and Category Theory

2.1 Motivation: Defining the Trace

The goal: to generalize and to study the following constructions:

(1) Given $f : V \to V$ endomorphism of finite-dimensional k-vector spaces, we can define its trace $\mathrm{Tr}(f|V)$ satisfying

   (i) additive: $\mathrm{Tr}(f+g|V) = \mathrm{Tr}(f|V) + \mathrm{Tr}(g|V)$;

   (ii) compatibile with tensor products: $\mathrm{Tr}(f \otimes g|V \otimes W) = \mathrm{Tr}(f|V)\mathrm{Tr}(g|W)$;

   (iii) compatible with duals: $\mathrm{Tr}(f|V) = \mathrm{Tr}(f^\vee|V^\vee)$;

   (iv) Given $f : V \to W$, $g : W \to V$, we have $\mathrm{Tr}(f \circ g|W) = \mathrm{Tr}(g \circ f|V)$).

(2) Let A be an associative k-algebra, define its co-center $\overline{A} := A/[A,A]$, where $[A,A] := \mathrm{span}_k\{ab - ba|a, b \in A\}$.

Remark 2.1. Why $\overline{A}$?

(i) for certain A-module π (e.g. π is finite-dimensional), we can define

$$\begin{aligned} \chi_\pi : A &\to k \\ a &\mapsto \mathrm{Tr}(a|\pi) \end{aligned} \tag{2.1}$$

which will factor through $\overline{A}$. E.g. take $A = k[G]$ group algebra of a finite group G, and then $\overline{A}$ can be identified as the k-span of conjugacy classes of G.

(ii) If π is a finite projective A-module, we can define Chern character $\mathrm{ch}(\pi) \in \overline{A}$, where

$$\mathrm{ch} : K_0(A) \to \overline{A}.$$

E.g. Take $A = \mathscr{H}(G)$ Hecke algebra of a p-adic group and then ch and χ_π give a paring.

Recall the definition of the trace of a linear map: pick a basis $\{v_i\}$ of V and $\{\delta^i\}$ the dual basis of V^*, then

$$\mathrm{Tr}(f|V) = \sum_i \delta^i(f(v_i)). \tag{2.2}$$

Alternatively, define unit map (also called co-evaluation map)

$$\begin{aligned} u_V : k &\to V \otimes V^\vee \\ \mathbf{1} &\mapsto \sum_i v_i \otimes \delta^i \end{aligned} \tag{2.3}$$

and evaluation map

$$\begin{aligned} e_V : V^\vee \otimes V &\to k \\ \lambda \otimes v &\mapsto \lambda(v) \end{aligned} \tag{2.4}$$

and then the trace map can be defined via

$$k \xrightarrow{u_V} V \otimes V^\vee \xrightarrow{f \otimes \mathrm{id}_{V^\vee}} V \otimes V^\vee \to V^\vee \otimes V \xrightarrow{e_V} k. \tag{2.5}$$

The upshort: the trace $\mathrm{Tr}(f|V)$ can be defined in a category $\mathscr{C}$, equipped with $(\otimes, \mathbf{1}, \dots)$ and $X \in \mathscr{C}$ equipped with

$$u_X : \mathbf{1} \to X \otimes X^\vee, \tag{2.6}$$

and

$$e_X : X^\vee \otimes X \to \mathbf{1}. \tag{2.7}$$

Informal definition:

Definition 2.2. A symmetric monoidal category is $(\mathscr{C}, \otimes, a, c, \mathbf{1}, u)$, where

- $\mathscr{C}$ is a category;
- $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$;
- associativity: isomorphism $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$ functorial in X, Y, Z;
- commutativity: isomorphism $c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$ functorial in X, Y.
- $\mathbf{1} \in \mathscr{C}$ and $u : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\cong} \mathbf{1}$,

satisfying certain properties (to guarantee when comparing two ways of multiplying multiple objects, there exists a functorial isomorphism, compatible with different orders of applying a or c), such as

$$\mathbf{1} \otimes (-) : \mathscr{C} \to \mathscr{C} \tag{2.8}$$

is an auto-equivalence (i.e. there exists functorial isomorphism $\mathbf{1} \otimes X \xrightarrow{\cong} X$).

Example 2.3 (Vector spaces). Take $\mathscr{C} = \mathrm{Vect}_k^{\heartsuit}$, $\otimes$ to be the usual tensor products of vector spaces, unit $\mathbf{1} = k$, commutativity $V \otimes W \cong W \otimes V$ given by $v \otimes w \mapsto w \otimes v$.

Example 2.4 (Graded vector spaces). Take $\mathscr{C} = (\mathrm{Vect}_k^{\mathrm{gr}})^{\heartsuit}$, unital object $\mathbf{1} = k$ placed in degree 0, and $\otimes$ tensor products given by $(V \otimes W)^n := \oplus_{i+j=n} V^i \otimes W^j$, and isomorphism $V^i \otimes W^j \cong W^j \otimes V^i$ given by $v \otimes w \mapsto (-1)^{ij} w \otimes v$. The sign here matters (see Example 2.18).

Example 2.5. Take $\mathscr{C}$ to be the categoty of chain complexes of vector spaces $\mathrm{Ch}(\mathrm{Vect}_k^{\heartsuit})$ and derived category $D(\mathrm{Vect}_k^{\heartsuit})$.

Remark 2.6. We can replace k by any commutative ring in previous examples.

Example 2.7. Let X be a topological space (or generally a site), and take $\mathscr{C} = \mathrm{Shv}(X, A)^{\heartsuit}$ sheaves of A-modules on X.

Example 2.8. Let X be a quasi-compact quasi-separated scheme, and $\mathscr{C} = \mathrm{QCoh}(X)^{\heartsuit}$.

We can see that symmetric monoidal category occurs.

Example 2.9. Take $\mathscr{C} = \mathrm{Morita}(\mathrm{Vect}_k^{\heartsuit})$, defined by

- Objects: associative k-algebras $A, B, \ldots$
- Morphisms: $\mathrm{Mor}(B, A) = \{(A, B)\text{-bimodules}\}$ (if we use isomorphism classes, then 1-category; if we don't take isomorphism classes, then 2-category),
- Composition of morphisms given by
$$_A M_B \circ {_B N_C} := M \otimes_B N. \tag{2.9}$$

Then $\mathscr{C}$ is equipped with $\otimes$ given by
$$\otimes : (A, B) \mapsto A \otimes B$$
and $\mathbf{1} = k$.

Example 2.10. Take $\mathscr{C} = \mathrm{Corr}(\mathrm{Sch}_S)$, defined by

- Objects: quasi-compact quasi-separated schemes over S;
- Morphisms:

$$\mathrm{Mor}(Y, X) = \left\{ \begin{tikzcd} & Z \arrow[dl, "L"'] \arrow[dd] \arrow[dr, "R"] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd} \right\} \tag{2.10}$$

  and composition given by

$$\begin{tikzcd} & & Z \times_Y W \arrow[dl] \arrow[dr] & & \\ & Z \arrow[dl] \arrow[dr] & & W \arrow[dl] \arrow[dr] & \\ X & & Y & & R \end{tikzcd}$$

Example 2.11 (Cohomological Correspondence). Take $\mathscr{C} = \mathrm{CohCorr}(\mathrm{Sch}_S^{\mathrm{ft}})$, where S is a scheme of finite type over k, defined by

- Objects: $(X, \mathscr{F})$, where X is a scheme of finite type over S and $\mathscr{F} \in \mathrm{Shv}(X_{\text{ét}}, \mathbb{F}_\ell)$;
- Morphisms: $\mathrm{Mor}((Y, \mathscr{G}), (X, \mathscr{F}))$ is the set of correspondences

$$\begin{tikzcd} & C \arrow[dl, "L"'] \arrow[dr, "R"] & \\ X & & Y \end{tikzcd}$$

  with $u : L_! R^* \mathscr{G} \to \mathscr{F}$.

with $\otimes : ((X, \mathscr{F}), (Y, \mathscr{G})) \mapsto (X \times_S Y, \mathscr{F} \boxtimes \mathscr{G})$.

Example 2.12. Take $\mathscr{C}$ to be LinCat_k the category of k-linear presentable (cocomplete+accessible) stable ∞-categoy, with $\mathrm{Mor}(\mathscr{C}, \mathscr{D})$ consists of k-linear colimit preserving functors, equipped with $\otimes$ called Deligne-Lurie tensor product. Examples of this example:

- $(A\text{-mod}) \otimes (B\text{-mod}) = (A \otimes_k B)\text{-mod}$.
- Let X, Y be qcqs schemes over k, then

$$D(\mathrm{QCoh}(X)^\heartsuit) \otimes D(\mathrm{QCoh}(Y)^\heartsuit) \cong D(\mathrm{QCoh}(X \times Y)^\heartsuit).$$

Now let $(\mathscr{C}, \otimes, a, c, \mathbf{1}, u)$ be a stmmetric monoidal category and $X \in \mathscr{C}$.

Definition 2.13. A dual of X consists of $(X^\vee, u_X, e_X)$,

- $X^\vee \in \mathscr{C}$,
- $u_X : \mathbf{1} \to X \otimes X^\vee$,
- $e_X : X^\vee \otimes X \to \mathbf{1}$,

such that the composition $X \xrightarrow{\cong} \mathbf{1} \otimes X \to (X \otimes X^\vee) \otimes X \underset{a}{\xrightarrow{\cong}} X \otimes (X^\vee \otimes X) \xrightarrow{\mathrm{id}_X \otimes e_X} X \otimes \mathbf{1} \xrightarrow{\cong} X$ is exactly id_X, and a similar diagram for $X^\vee$.

Remark 2.14. A dual of X (if it exists) is unique up to a unique isomorphism.

Definition 2.15. An object X is called dualizable if a dual of X exists.

Definition 2.16. Let X be a dualizable object in $\mathscr{C}$, and $f : X \to X$ an endomorphism. Then we define

$$\mathrm{Tr}(f|X) : \mathbf{1} \xrightarrow{u_x} X \otimes X^\vee \xrightarrow{f \otimes \mathrm{id}_{X^\vee}} X \otimes X^\vee \underset{a}{\xrightarrow{\cong}} X^\vee \otimes X \xrightarrow{e_X} \mathbf{1}. \tag{2.11}$$

Example 2.17. Take $\mathscr{C} = \mathrm{Vect}_k^\heartsuit$. Then $V \in \mathscr{C}$ is dualizable if and only if V is finite-dimensional over k. In this case $\mathrm{Tr}(f|V)$ is the usual trace.

Example 2.18. Take $\mathscr{C} = (\mathrm{Vect}_k^{\mathrm{gr}})^\heartsuit$. Then $V^\bullet$ is dualizable if and only if $\sum \dim V^i < \infty$. In this case

$$((V^\bullet)^\vee)^i = (V^{-i})^\vee$$

and

$$\mathrm{Tr}(f|V) = \sum_i (-1)^i \mathrm{Tr}(f^i|V^i).$$

The sign is due to $V \otimes V^\vee \cong V^\vee \otimes V$.

Example 2.19. Take $\mathscr{C}$ to be $\mathrm{Ch}(\mathrm{Vect}_k^\heartsuit)$ the chain complexes or $D(\mathrm{Vect}_k^\heartsuit)$. Then $V^\bullet \in \mathscr{C}$ is dualizable if and only if $V^\bullet$ is a perfect complex (quasi-isomorphic to a bounded complex of finite projective modules). In this case,

$$((V^\bullet)^\vee)^i = (V^{-i})^\vee$$

and

$$\mathrm{Tr}(f^\bullet|V^\bullet) = \sum_i (-1)^i \mathrm{Tr}(f^i|V^i).$$

In particular, take $f^\bullet = \mathrm{id}$ and then

$$\mathrm{Tr}(\mathrm{id}|V) = \chi(V^\bullet).$$

Example 2.20. If $\mathscr{C} = D(A\text{-mod}^\heartsuit)$ or $\mathscr{C} = D(\mathrm{QCoh}(X)^\heartsuit)$, then V is dualizable if and only if V is perfect.

Example 2.21. If $\mathscr{C} = \mathrm{Morita}(D(\mathrm{Vect}_k^\heartsuit))$, then every object is dualizable. The dual of A is A^{rev} the opposite algebra. The unit is given by the $A \otimes A^{rev}$-module A and similar for the counit (also called evaluation). Then

$$\mathrm{Tr}(\mathrm{id}_A|A) = A \otimes^{\mathbb{L}}_{A \otimes A^{rev}} A = \mathrm{HH}(A, A)$$

is Hoschild homologies. Similarly, given an A-bimodule M, $\mathrm{Tr}(f_M|A) = A \otimes^{\mathbb{L}}_{A \otimes A^{rev}} M = \mathrm{HH}(A, M)$.

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Example 2.22. Consider $\mathscr{C} = \mathrm{Morita}(\mathrm{Vect}_k^\heartsuit)$, any object A is dualizable, with A^{rev} a dual. The unit and counit is given by A as $(k, A \otimes A^{rev})$ and $(A \otimes A^{rev}, k)$-bimodule. Then for any ${}_A M_A$,

$$\mathrm{Tr}({}_A M_A|A) = A \otimes_{A \otimes A^{rev}} (M \otimes A^{rev}) \otimes_{A \otimes A^{rev}} A \in \mathrm{Vect}_k^\heartsuit$$

visualized as "circular tensor product over A"

$$\begin{tikzcd} & A \arrow[dr, dash, bend left] & \\ M \arrow[ur, dash, bend left] & & A \arrow[dl, dash, bend left] \\ & A \arrow[ul, dash, bend left] & \end{tikzcd} \tag{2.12}$$

Therefore, $\mathrm{Tr}({}_A M_A|A) \cong A \otimes_{A \otimes A^{rev}} M = M/\mathrm{span}\{am - ma | a \in A, m \in M\}$.

Example 2.23. Consider $\mathscr{C} = \mathrm{Corr}(\mathrm{Sch}_{/S})$. Every X is dualizable with dual X, the evaluation and unit are given by

$$\begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr] & \\ X \times_S X & & S \end{tikzcd} \tag{2.13}$$

and

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "\Delta"] & \\ S & & X \times_S X \end{tikzcd} \tag{2.14}$$

Given $f : X \to X$, we obtain a correspondence

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}_X"] & \\ X & & X \end{tikzcd} \tag{2.15}$$

and then $\mathrm{Tr}(f|X) = X^f$ fixed points.

Example 2.24. Consider $\mathscr{C} = \mathrm{CohCorr}(\mathrm{Sch}^{ft}_{/S})$ where S/k is a scheme of finite type where l is invertible in the field k. An object is of the form $(X, \mathscr{F})$, where X is a scheme of finite type over S, and $\mathscr{F} \in D(\mathrm{Shv}(X_{\text{ét}}, \mathscr{F})^{\heartsuit})$.

Theorem 2.25. *An object $(X, \mathscr{F})$ in $\mathrm{CohCorr}(\mathrm{Sch}^{ft}_{/S})$ is dualizable if and only if $\mathscr{F}$ is ULA (universally locally acyclic) with respect to $X \to S$.*

Example 2.26. Consider the category LinCat_k. Then $D(\mathrm{QCoh}(X)^{\heartsuit})$, with X quasi-compact quasi-separated scheme, is dualizable, with dual $D(\mathrm{QCoh}(X)^{\heartsuit})^{\vee} \cong D(\mathrm{QCoh}(X)^{\heartsuit})$. The evaluation map is given by

$$\begin{aligned} e_X : D(\mathrm{QCoh}(X)^{\heartsuit}) \otimes D(\mathrm{QCoh}(X)^{\heartsuit}) &\to \mathrm{Vect}_k \\ \mathscr{F} \boxtimes \mathscr{G} &\mapsto \mathrm{R}\Gamma(X, \mathscr{F} \otimes^{\mathbb{L}} \mathscr{G}). \end{aligned} \tag{2.16}$$

and the unit map is given by

$$\begin{aligned} u_X : \mathrm{Vect}_k &\to D\mathrm{QCoh}(X \times X)^{\heartsuit} \\ k &\mapsto \Delta_* \mathscr{O}_X \end{aligned} \tag{2.17}$$

where we use $D(\mathrm{QCoh}(X)^{\heartsuit}) \otimes D(\mathrm{QCoh}(X)^{\heartsuit}) \cong D(\mathrm{QCoh}(X \times X)^{\heartsuit})$.
Given $f : X \to X$, we have

$$Lf^* : D(\mathrm{QCoh}(X)^{\heartsuit}) \to D(\mathrm{QCoh}(X)^{\heartsuit})$$

and the trace is

$$\mathrm{Tr}(Lf^* | D(\mathrm{QCoh}(X)^{\heartsuit})) = \mathrm{R}\Gamma(X^f, \mathscr{O}) \tag{2.18}$$

where X_f is homotopy pullback (as a derived scheme)[a]

$$\begin{tikzcd} X_f \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "\Gamma_f"] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd} \tag{2.19}$$

[a]This is essential since we want base change theorem for quasicoherent sheaves to hold, without flatness condition.

Remark 2.27. Note that we identify $\mathrm{End}_{\mathrm{LinCat}}(\mathrm{Vect}_k) \cong \mathrm{Vect}_k$ because a colimit preserving functor on Vect_k is determined by its value on the 1-dimensional vector space k.

Proof. We want to

- verify the unit and evaluation maps.
- justify the description of the trace (2.18).

(A):To check e_X and u_X defined above are indeed unit and evaluation maps, we should verify that

$$\mathscr{F} \mapsto (\Delta_{12})_*\mathscr{O}_X \boxtimes \mathscr{F} \to R(\mathrm{id}_X \times \pi)_* L(\mathrm{id}_X \times \Delta_{23})^*((\Delta_{12})_*\mathscr{O}_X \boxtimes \mathscr{F}) \tag{2.20}$$

is the identity map, where $\pi : X \to \operatorname{Spec} k$ the structure morphism and

$$\begin{tikzcd} & X \times X \arrow[dl, "\Delta_{12}\times\mathrm{id}_X"'] \arrow[dr, "\mathrm{id}_X\times\Delta_{23}"] & \\ X\times X\times X & & X\times X\times X \end{tikzcd} \quad \begin{tikzcd} X\times X\times X \arrow[d, "p_{12}"'] \arrow[dr, "p_3"] & \\ X\times X & X \end{tikzcd} \quad \begin{tikzcd} X\times X \arrow[d, "q_1"'] \arrow[dr, "q_2"] & \\ X & X \end{tikzcd}$$

$$\begin{tikzcd} & (x,y) \arrow[dl, mapsto, "\Delta_{12}\times\mathrm{id}_X"'] \arrow[dr, mapsto, "\mathrm{id}_X\times\Delta_{23}"] & \\ (x,x,y) & & (x,y,y) \end{tikzcd} \quad \begin{tikzcd} (x,y,z) \arrow[d, mapsto, "p_{12}"'] \arrow[dr, mapsto, "p_3"] & \\ (x,y) & z \end{tikzcd} \quad \begin{tikzcd} (x,y) \arrow[d, mapsto, "q_1"'] \arrow[dr, mapsto, "q_2"] & \\ x & y \end{tikzcd} \tag{2.21}$$

Note that $L(\mathrm{id}_X \times \Delta_{23})^* \circ Lp_{12}^* \cong \mathrm{id}$, $L(\mathrm{id}_X \times \Delta_{23})^* \circ Lp_3^* \cong Lq_2^*$ and $\mathrm{id}_X \times \pi = q_1$. Then we obtain that

$$R(\mathrm{id}_X \times \pi)_* L(\mathrm{id}_X \times \Delta_{23})^*((\Delta_{12})_*\mathscr{O}_X \boxtimes \mathscr{F}) \cong R(q_1)_*(\Delta_*\mathscr{O}_X \otimes^{\mathbb{L}} Lq_2^*\mathscr{F}) = \Phi_{\Delta_*\mathscr{O}_X}\mathscr{F}. \tag{2.22}$$

Fourier-Mukai transform $\Phi_\bullet(-)$ for quasi-coherent sheaves naturally occurs! Recall that Fourier-Mukai transform with kernel $\Delta_*\mathscr{F}$ is exactly identity map[1]:

$$\begin{aligned} R(q_1)_*(\Delta_*\mathscr{O}_X \otimes^{\mathbb{L}} Lq_2^*\mathscr{F}) &\cong R(q_1)_*\Delta_*(\mathscr{O}_X \otimes^{\mathbb{L}} L\Delta^* Lq_2^*\mathscr{F}) \quad \text{(projection formula)} \\ &\cong R(q_1)_*\Delta_*(\mathscr{O}_X \otimes^{\mathbb{L}} \mathscr{F}) \quad (\text{use } q_2 \circ \Delta = \mathrm{id} \text{ and } q_1 \circ \Delta = \mathrm{id}) \\ &\cong \mathscr{F}. \end{aligned} \tag{2.23}$$

(B): To proof (2.18). By definition the trace

$$\operatorname{Tr}(Lf^*|D(\mathrm{QCoh}(X)^\heartsuit)) \in \operatorname{End}_{\mathrm{LinCat}}(\mathrm{Vect}_k) \cong \mathrm{Vect}_k, \tag{2.24}$$

is determined by

$$k \mapsto \Delta_*\mathscr{O}_X \mapsto L(f \times \mathrm{id}_X)^*\Delta_*\mathscr{O}_X \mapsto \mathrm{R}\Gamma(X, L\Delta^* L(f \times \mathrm{id}_X)^*\Delta_*\mathscr{O}_X). \tag{2.25}$$

Note that $(f \times \mathrm{id}_X) \circ \Delta = \Gamma_f$ is graph embedding. Then we see the image of k is

$$\begin{aligned} \mathrm{R}\Gamma(X, L\Delta^* L(f \times \mathrm{id}_X)^*\Delta_*\mathscr{O}_X) &\cong \mathrm{RHom}_X(\mathscr{O}_X, L\Gamma_f^*\Delta_*\mathscr{O}_X) \\ &\cong \mathrm{RHom}_X(\mathscr{O}_X, Rj_*Lj^*\mathscr{O}_X), \quad \text{(base change formula)} \\ &\cong \mathrm{RHom}_{X^f}(Lj^*\mathscr{O}_X, Lj^*\mathscr{O}_X), \\ &\cong \mathrm{RHom}_{X^f}(\mathscr{O}_{X^f}, \mathscr{O}_{X^f}), \\ &\cong \mathrm{R}\Gamma(X^f, \mathscr{O}_{X^f}), \end{aligned} \tag{2.26}$$

where we have to understand things in term of derived geometry so that we have base change theorem for quasi-coherent sheaves (without flatness assumption in general case). □

Remark 2.28. Later we will see that any sheaf theory will give a symmetric monoidal functor $\mathrm{Corr}(\mathrm{Sch}/S) \to \mathrm{LinCat}_k$, and we can do computation by functoriality of trace.

[1] For more similar results, see my notes named *Fourier Transforms in Algebraic World*.

2.2 Monoidal Category

Definition 2.29. A semi-group (non-unital monoidal) category consists of $(\mathscr{C}, \otimes, a)$, where

- $\mathscr{C}$ is a category,
- $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$,
- $$\begin{tikzcd} \mathscr{C} \times \mathscr{C} \times \mathscr{C} \arrow[r, "\otimes\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\otimes"'] & \mathscr{C} \times \mathscr{C} \arrow[d, "\otimes"] \arrow[dl, Rightarrow, "a\cong"] \\ \mathscr{C} \times \mathscr{C} \arrow[r, "\otimes"] & \mathscr{C} \end{tikzcd}$$

such that the diagram

$$\begin{tikzcd} & (x \otimes y) \otimes (z \otimes w) \arrow[dr] & \\ ((x \otimes y) \otimes z) \otimes w \arrow[ur] \arrow[d] & & x \otimes (y \otimes (z \otimes w)) \\ (x \otimes (y \otimes z)) \otimes w \arrow[rr] & & x \otimes ((y \otimes z) \otimes w) \arrow[u] \end{tikzcd} \tag{2.27}$$

commutes (called pentagon identity).

Definition 2.30. A unit of a semi-group category consists of $(\mathbf{1}, u)$, where

- $\mathbf{1} \in \mathscr{C}$,
- $u : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\cong} \mathbf{1}$ is an isomorphism,

such that both functors

- $\mathbf{1} \otimes - : \mathscr{C} \to \mathscr{C}$
- $- \otimes \mathbf{1} : \mathscr{C} \to \mathscr{C}$

are fully faithful.

Proposition 2.31.

$$\begin{aligned} l : \mathbf{1} \otimes - &\to id \\ \mathbf{1} \otimes X &\mapsto X \end{aligned} \tag{2.28}$$

$$\begin{aligned} r : - \otimes \mathbf{1} &\to id \\ X \otimes \mathbf{1} &\mapsto X \end{aligned} \tag{2.29}$$

are isomorphic of functors.

Proof. Note that we have functorial isomorphisms $\mathbf{1} \otimes (\mathbf{1} \otimes X) \cong \mathbf{1} \otimes X$ using asscoiativity a and u. Note that $\mathbf{1} \otimes -$ is fully faithful (in particular, conservative), and hence we obtain functorial isomorphisms $\mathbf{1} \otimes X \cong X$. □

Proposition 2.32. *The following diagrams are commuatative,*

$$\begin{tikzcd} (X \otimes \mathbf{1}) \otimes Y \arrow[rr, "a", "\cong"'] \arrow[dr, "r_x \otimes id_Y"'] & & X \otimes (\mathbf{1} \otimes Y) \arrow[dl, "id_X \otimes l_Y"] \\ & X \otimes Y & \end{tikzcd} \tag{2.30}$$

$$\begin{tikzcd} (\mathbf{1} \otimes X) \otimes Y \arrow[rr, "a"] \arrow[dr, "l_x \otimes id_Y"'] & & \mathbf{1} \otimes (X \otimes Y) \arrow[dl, "l_{X \otimes Y}"] \\ & X \otimes Y & \end{tikzcd} \tag{2.31}$$

and

$$\begin{tikzcd}(X \otimes Y) \otimes \mathbf{1} \arrow[rr, "a"] \arrow[dr, "r_{X\otimes Y}"'] & & X \otimes (Y \otimes \mathbf{1}) \arrow[dl, "id_X \otimes r_Y"] \\ & X \otimes Y & \end{tikzcd} \tag{2.32}$$

Proof. The proof amounts to pentagon identity 2.27 and diagram chasing. I would rather omit the proof. □

Proposition 2.33. *We have canonical isomorphism $l_{\mathbf{1}} \cong r_{\mathbf{1}} \cong u$.*

Proposition 2.34. $\mathrm{End}_{\mathscr{C}}\, \mathbf{1}$ *is a commutative monoid and* $\mathrm{Aut}_{\mathscr{C}}(\mathbf{1}, u) = id_{\mathbf{1}}$.

Proof. I omit the proof of End and sketch the proof of Aut.

$$\begin{tikzcd}\mathbf{1} \otimes \mathbf{1} \arrow[r, "u"] \arrow[d, "f \otimes f"] & \mathbf{1} \arrow[d, "f"] \\ \mathbf{1} \otimes \mathbf{1} \arrow[r, "u"] & \mathbf{1}\end{tikzcd} \tag{2.33}$$

is commutative. But this diagram is the composition of two squares

$$\begin{tikzcd}\mathbf{1} \otimes \mathbf{1} \arrow[r, "u"] \arrow[d, "\mathrm{id} \otimes f"] & \mathbf{1} \arrow[d, "f"] \\ \mathbf{1} \otimes \mathbf{1} \arrow[r, "u"] \arrow[d, "f \otimes \mathrm{id}"] & \mathbf{1} \arrow[d, "\mathrm{id}"] \\ \mathbf{1} \otimes \mathbf{1} \arrow[r, "u"] & \mathbf{1}\end{tikzcd} \tag{2.34}$$

Note the upper square also commutes by Prop. 2.33. Therefore, the lower diagram commutes and we can cancel f using fully faithfulness of $- \otimes \mathbf{1}$ and hence we obtain $f = \mathrm{id}_{\mathbf{1}}$. □

Proposition 2.35. *If $(\mathbf{1}', u')$ is another unit, then there exists a unique*

$$\epsilon : (\mathbf{1}, u) \xrightarrow{\cong} (\mathbf{1}', u').$$

Proof. Existence is proved using the following diagram

$$\begin{tikzcd}\mathbf{1} \otimes \mathbf{1} \arrow[r, "l_{\mathbf{1}}"] \arrow[d] & \mathbf{1} \arrow[d] \\ \mathbf{1} \otimes \mathbf{1}' \arrow[r, "l_{\mathbf{1}'}"] \arrow[d, "\epsilon \otimes \mathrm{id}"] & \mathbf{1}' \arrow[d, "\mathrm{id}_{\mathbf{1}'}"] \\ \mathbf{1}' \otimes \mathbf{1}' \arrow[r, "u'"] & \mathbf{1}'\end{tikzcd} \tag{2.35}$$

and fully faithfulness of $- \otimes \mathbf{1}'$. Then use Prop. 2.33 $l_{\mathbf{1}} = u$.

Uniqueness is by Prop. 2.34. □

Remark 2.36. Therefore, a unit is an inner structure rather than an additional data.

Definition 2.37. A monoidal category is a semi-group category $(\mathscr{C}, \otimes, a)$ such that the unit exists.

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Example 2.38. All previous examples are actually monoidal categories.

Example 2.39. Given M a commutative monoid, then we define a catogory BM with object consisting of a point $*$ and $\mathrm{End}(*) = M$ and $* \otimes * = *$.

Example 2.40. Let $\mathscr{C}$ be a category with finite products. Then $\mathscr{C}$ admits a natural monoidal structure, usually called Cartesian monoidal structure, given by $X \otimes Y := X \times Y$ and $\mathbf{1}$:=final object.

Example 2.41. Let $\mathscr{C}$ be a category. Then $\mathrm{End}(\mathscr{C}) = \mathrm{Fun}(\mathscr{C}, \mathscr{C})$ is a natural monoidal category, with $\otimes$ given by composition of functors. Exercise: define module categories of a monoidal category.

Example 2.42. Let Γ be a (finite) group and $\mathscr{C} := \mathrm{Vect}^\Gamma$ be the category of Γ-graded vector spaces with

$$\{V_\gamma\}_\gamma \otimes \{W_{\gamma'}\}_{\gamma'} = \{(V \otimes W)_{\gamma''}\}_{\gamma''},$$

where $(V \otimes W)_{\gamma''} := \oplus_{\gamma \cdot \gamma' = \gamma''} V_\gamma \otimes W_{\gamma'}$ with unit given by k put in degree 0.

Remark 2.43. Actually, Vect^Γ can be viewed as $\mathrm{Shv}(\Gamma, k)$, i.e. sheaves of k-vector spaces on the discrete space Γ. If we denote the multiplication by $m : \Gamma \times \Gamma \to \Gamma$, then $\mathscr{V} * \mathscr{W} = m_*(\mathscr{V} \boxtimes \mathscr{W})$. This explanation works in more general situation.

Example 2.44. Suppose that $\mathscr{C}$ is a category of spaces with finite product and $\mathscr{D}$ is a "sheaf theory" on $\mathscr{C}$. Let $f : X \to Y$ be a morphism in $\mathscr{C}$. If $Z := X \times_Y X$, then under certain assumptions on $\mathscr{D}$, $\mathscr{D}(Z)$ is a monoidal category, with

$$\begin{tikzcd} X \times_Y X \times_Y X \arrow[r, "\mathrm{id}_X \times \Delta_X \times \mathrm{id}_X", "q"'] \arrow[d, "\mathrm{id}_X \times f \times \mathrm{id}_X"', "p"] & X \times_Y X \times X \times_Y X = Z \times Z \\ X \times_Y X & \end{tikzcd} \tag{2.36}$$

and

$$\begin{aligned} * : \mathscr{D}(Z) \times \mathscr{D}(Z) &\to \mathscr{D}(Z), \\ (\mathscr{F}, \mathscr{G}) &\mapsto p_! q^*(\mathscr{F} \boxtimes \mathscr{G}). \end{aligned} \tag{2.37}$$

For example, Hecke category: $Y = BG$ and $X = BH$, and then [a]

$$Z = X \times_Y X \cong H \backslash G / H,$$

where G is a finite group and H is a subgroup of G. Exercise: spread the meaning of the monoidal structure.

[a] Note that $BH \times_{BG} BH(X)$ is the moduli of (P_1, P_2, α), where P_1, P_2 are left H-torsors over X, $\alpha : P_1 \times^H G \xrightarrow{\cong} P_2 \times^H G$ (quotient by diagonal action of H). Therefore, the stack classfying $(P_1, P_2, \alpha, \epsilon_1, \epsilon_2)$ is a $H \times H$-torsor over $BH \times_{BG} BH$, where ϵ_1, ϵ_2 are trivializations of P_1 and P_2 respectively. Note that $(X \times H) \times^H G \cong X \times G$. But then the data is equivalent to $(X \times G) \xrightarrow{\cong} (X \times G)$, which is further equivalent to $G(X)$. Therefore, G is a $H \times H$-torsor on $BH \times_{BG} BH$, where one H acts on the left and the other one acts on the right (because change of ϵ_1, ϵ_2 has effect on the source and the target of α respectively). Therefore, $[H \backslash G / H] \cong BH \times_{BG} BH$. Secretly, we are mimicing the proof in topology: we are proving $EG \times_{EG/G} EG$ is the same as G, which is $H \times H$ equivarant and gives the desired isomorphism after quotient by $H \times H$.

2.3 Dual

Let $\mathscr{C}$ be a monoidal category throughout this subsection.

Definition 2.45. Let $\mathscr{C}$ be a monoidal category and $x \in \mathscr{C}$ be an object. Then a right dual of x consists of $(x^\vee, e_x, u_x)$ with

- $x^\vee \in \mathscr{C}$,
- $u_x : 1 \to x \otimes x^\vee$,
- $e_x : x^\vee \otimes x \to 1$,

such that

- the composition of $x \cong \mathbf{1} \otimes x \to (x \otimes x^\vee) \otimes x \xrightarrow{\cong} x \otimes (x^\vee \otimes x) \to x \otimes \mathbf{1} \xrightarrow{\cong} x$ is id_x and
- the composition of $x^\vee \cong x \otimes \mathbf{1} \to x \otimes (x^\vee \otimes x) \xrightarrow{\cong} (x \otimes x^\vee) \otimes x \to \mathbf{1} \otimes x \xrightarrow{\cong} x$ is id_x

Definition 2.46. A left dual of x consists of $({}^\vee x, {}^\vee u_x, {}^\vee e_x)$, such that $(x, {}^\vee u_x, {}^\vee e_x)$ is a right dual of ${}^\vee x$.

Remark 2.47. In general, ${}^\vee x$ and $x^\vee$ are different.

Proposition 2.48. *Let $(x^\vee, u_x, e_x)$ be a right dual of x. Then there exist canonical isomorphisms*

$$\mathrm{Hom}(y \otimes x, z) \xrightarrow{\cong} \mathrm{Hom}(y, z \otimes x^\vee) \tag{2.38}$$

functorial in $y, z \in \mathscr{C}$ and

$$\mathrm{Hom}(y, x \otimes z) \xrightarrow{\cong} \mathrm{Hom}(x^\vee \otimes y, z), \tag{2.39}$$

functorial in $y, z \in \mathscr{C}$.

Corollary 2.49. *For a dualizable object $X \in \mathscr{C}$, $- \otimes X$ admits a right adjoint $- \otimes X^\vee$ and $X \otimes -$ admits a left adjoint $X^\vee \otimes -$. In particular, e_X and u_X is given by the unit and counit of this adjunction.*

Corollary 2.50. *Right (resp. left) dual of $X \in \mathscr{C}$, if exists, is unique up to a unique isomorphism.*

Corollary 2.51. *For any dualizable $X \in \mathscr{C}$, $- \otimes X$ commutes with any colimits and $X \otimes -$ commutes with any limits.*

Proof of Proposition 2.48. We have that

$$\mathrm{Hom}(y \otimes x, z) \xrightarrow{\otimes x^\vee} \mathrm{Hom}(y \otimes x \otimes x^\vee, z \otimes x^\vee) \xrightarrow{u_x} \mathrm{Hom}(y, z \otimes x^\vee), \tag{2.40}$$

and

$$\mathrm{Hom}(y, z \otimes x^\vee) \xrightarrow{\otimes x} \mathrm{Hom}(y \otimes x, z \otimes x^\vee \otimes x) \xrightarrow{e_x} \mathrm{Hom}(y \otimes x, z). \tag{2.41}$$

We want to check that the maps above are inverse to each other. Given any $f \in \mathrm{Hom}(y \otimes x, z)$, we have that

$$\begin{tikzcd}
y \otimes x \arrow[r, "\mathrm{id}_y \otimes u_x \otimes \mathrm{id}_x"] \arrow[dr, "\mathrm{id}_{y \otimes z}"'] & y \otimes x \otimes x^\vee \otimes x \arrow[r, "f \otimes \mathrm{id}_{x^\vee \otimes x}"] \arrow[d, "\mathrm{id}_y \otimes \mathrm{id}_x \otimes e_x"] & z \otimes x^\vee \otimes x \arrow[r, "\mathrm{id}_z \otimes e_x"] \arrow[d, "\mathrm{id}_z \otimes e_x"] & z \\
& y \otimes x \arrow[r, "f"] & z \arrow[ur, "\mathrm{id}_z"'] &
\end{tikzcd} \tag{2.42}$$

Similar for the remainings. $\square$

Corollary 2.52. *If every object in $\mathscr{C}$ admits a right dual, then $\mathscr{C} \to \mathscr{C}^{op}$, $x \mapsto x^\vee$ is fully faithful, where $f : x \mapsto y$ is mapped to $f^\vee : y^\vee \to y^\vee \otimes x \otimes x^\vee \to y^\vee \otimes y \otimes x^\vee \to x^\vee$.*

Proof. Note that $\mathrm{Hom}(x, y) \cong \mathrm{Hom}(\mathbf{1}, y \otimes x^\vee) \cong \mathrm{Hom}(y^\vee, x^\vee)$. $\square$

Corollary 2.53. *Let $f : x \to y$ and $g : y^\vee \to x^\vee$ be two morphisms, then $g = f^\vee$ if and only if the following diagram is commutative*

$$\begin{tikzcd}
y^\vee \otimes x \arrow[r, "id \otimes f"] \arrow[d, "g \otimes id"] & y^\vee \otimes y \arrow[d, "e_y"] \\
x^\vee \otimes x \arrow[r, "e_x"] & \mathbf{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \arrow[r, "u_y"] \arrow[d, "u_x"] & y \otimes y^\vee \arrow[d, "id \otimes g"] \\
x \otimes x^\vee \arrow[r, "f \otimes id"] & y \otimes x^\vee
\end{tikzcd} \tag{2.43}$$

Definition 2.54. A monoidal category $\mathscr{C}$ is called rigid if every object $x \in \mathscr{C}$ admits both a left dual and a right dual.

Corollary 2.55. *If $\mathscr{C}$ is rigid and $X \in \mathscr{C}$ any object, then $X \otimes -$ and $- \otimes X$ commute with any limits and colimits. Moreover,*

$$\begin{aligned} \sigma_{\mathscr{C}} : \mathscr{C} &\to \mathscr{C} \\ x &\mapsto (x^\vee)^\vee \end{aligned} \tag{2.44}$$

is an equivalence of category, with a quasi inverse given by

$$\begin{aligned} \sigma_{\mathscr{C}}^{-1} : \mathscr{C} &\to \mathscr{C} \\ x &\mapsto {}^\vee({}^\vee x) \end{aligned} \tag{2.45}$$

2.4 Symmetric Monoidal Categories

Definition 2.56. A symmetric monoidal category consists of $(\mathscr{C}, \otimes, a, c)$ where $(\mathscr{C}, \otimes, a)$ is a monoidal category and

$$c : x \otimes y \xrightarrow{\cong} y \otimes x$$

isomorphisms functorial in x and y, such that

- the composition of $x \otimes y \xrightarrow{c} y \otimes x \xrightarrow{c} x \otimes y$ is $\mathrm{id}_{x \otimes y}$,
- Compatibily of a and c: the diagram

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "a"] \arrow[d, "c"] & x \otimes (y \otimes z) \arrow[d, "c"] \\
(y \otimes x) \otimes z \arrow[d, "c"] & x \otimes (z \otimes y) \arrow[d, "c"] \\
z \otimes (y \otimes x) & (z \otimes y) \otimes x \arrow[l, "a"']
\end{tikzcd} \tag{2.46}$$

commutes.

Remark 2.57. By Maclane's coherence theorem, these axioms suffice to show any ways of multiplication (using a, c and multiplication $\otimes$ in different orders) of any tuples of objects will be compatible.

Remark 2.58. This is a subtle definition. If you modify c by a twist of certain matrix, then you obtain something like quantum groups.

Remark 2.59. In a symmetric monoidal category, a left dual is the same thing as a right dual. (Use c, and uniqueness of left/right dual to show this.)

So we can talk dualizable objects.

2.5 Internal Hom

Let $\mathscr{C}$ be a symmetric monoidal category throughout this subsection.

Definition 2.60. Let $x, y \in \mathscr{C}$. Then we define the internal Hom $\underline{\mathrm{Hom}}(x, y)$ to be an object in $\mathscr{C}$ (if it exists), representing the functor

$$z \mapsto \mathrm{Hom}_{\mathscr{C}}(z \otimes x, y),$$

i.e. functorial (in z) isomorphisms

$$\mathrm{Hom}(z, \underline{\mathrm{Hom}}(x, y)) \cong \mathrm{Hom}(z \otimes x, y).$$

We write $X^* := \underline{\mathrm{Hom}}(X, \mathbf{1})$ if it exists.

Lemma 2.61. *Let $x \in \mathscr{C}$.*

(1) If x is dualizable, then internal Hom $\underline{\mathrm{Hom}}(x, y)$ exists for any $y \in \mathscr{C}$, which is isomorphic to $x^\vee \otimes y$.

(2) x is dualizable if $x^*, \underline{\mathrm{Hom}}(x, x)$ exist and the natural map

$$x^* \otimes x \to \underline{\mathrm{Hom}}(x, x) \tag{2.47}$$

is an isomorphism[2].

For proof, one can consult [LZ22, Lem. 1.4].

Definition 2.62. A symmetric monoidal category $\mathscr{C}$ is called closed if $\underline{\mathrm{Hom}}(x, y)$ exists for all $x, y \in \mathscr{C}$.

Example 2.63. The following categories are closed: $\mathrm{Vect}_k^\heartsuit$, $D(\mathrm{Vect}_k^\heartsuit)$ and LinCat_k.

[2]Note that $x^* \otimes x \to \underline{\mathrm{Hom}}(x, x)$ is the same thing as $x^* \otimes x \otimes x \to x$, then the map is $(e_x \otimes \mathrm{id}_x) \circ (\mathrm{id}_{x^*} \otimes c)$.

2.6 Properties of Traces

Definition 2.64. Let $f : X \to X$ be an endomorphism of a dulaizable object in a symmetric monoidal category $\mathscr{C}$. Then we define $\mathrm{Tr}(f|X) \in \mathrm{End}(\mathbf{1})$ to be the composition

$$\mathbf{1} \xrightarrow{u_X} X \otimes X^\vee \xrightarrow{f \otimes \mathrm{id}_{X^\vee}} X \otimes X^\vee \xrightarrow{c} X^\vee \otimes X \xrightarrow{e_X} \mathbf{1}. \tag{2.48}$$

Proposition 2.65. *The followings hold.*

(1) $Tr(f|X) = Tr(f^\vee|X^\vee)$.

(2) $Tr(g \circ f|X) = Tr(f \circ g|Y)$ *for* $f : X \to Y$ *and* $g : Y \to X$.

(3) $Tr(f \otimes g|X \otimes Y) = Tr(f|X) \cdot Tr(g|Y)$ *for* $f : X \to X$ *and* $g : Y \to Y$.

(4) If $\mathscr{C}$ *is an additive monoidal category, then* $Tr(f \oplus g|X \oplus Y) = Tr(f|X) + Tr(g|Y)$.

(5) (relative trace) $f : x \otimes y \to x \otimes y$ *and* x, y *are dualizable, can define* $Tr(f|x \otimes y|y) \in \mathrm{End}_{\mathscr{C}}(x)$ *by*

$$\begin{tikzcd}
x \arrow[rr, dotted, "Tr(f|x\otimes y|y)"] \arrow[d, "id_x\otimes u_y"] & & x \\
x \otimes y \otimes y^\vee \arrow[r, "f\otimes id_{y^\vee}"] & x \otimes y \otimes y^\vee \arrow[r, "id_x\otimes c"] & x \otimes y^\vee \otimes y \arrow[ul, "id_x\otimes e_y"']
\end{tikzcd} \tag{2.49}$$

Then we have $Tr(f|x \otimes y) = Tr(Tr(f|x \otimes y|y)|x)$

Corollary 2.66. *Trace factors through split Grothendieck group, i.e.* $Tr : K^\oplus \to \mathrm{End}_{\mathscr{C}} \mathbf{1}$.

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Recall that we define trace for an endomorphism of a dualizable object in a symmetric monoidal category $\mathscr{C}$.

Definition 2.67. In particular, we define $\dim X := \mathrm{Tr}(\mathrm{id}_X|X) \in \mathrm{End}_{\mathscr{C}}(\mathbf{1}) =: k$.

Definition 2.68. Let $\mathscr{C}$ be a rigid additive symmetric monoidal category. Let

$$\mathscr{N}\mathscr{C} := \{f : X \to Y|\mathrm{Tr}(f \circ g) = 0, \forall g : Y \to X\}. \tag{2.50}$$

Corollary 2.69. $\mathscr{N}\mathscr{C}$ *is a tensor ideal. For any* $(f : X \to Y) \in \mathscr{N}\mathscr{C}$.

(1) for any $g : Y \to Z$, $g \circ f \in \mathscr{N}\mathscr{C}$,

(2) for any $h : W \to X$, $f \circ h \in \mathscr{N}\mathscr{C}$;

(3) for any $g : S \to T$, $(f \otimes g : X \otimes S \to Y \otimes T) \in \mathscr{N}\mathscr{C}$.

So we can define $\overline{\mathscr{C}} := \mathscr{C}/\mathscr{N}\mathscr{C}$, *whose collection of objects is the same as* $\mathscr{C}$ *but morphisms are modified*

$$\mathrm{Hom}_{\overline{\mathscr{C}}}(X, Y) := \mathrm{Hom}_{\mathscr{C}}(X, Y)/(\mathscr{N}\mathscr{C} \cap \mathrm{Hom}_{\mathscr{C}}(X, Y)),$$

which is still an additive rigid symmetric monoidal category.

2.7 (Symmetric) Monoidal Functors

Definition 2.70. Let $\mathscr{C}$ and $\mathscr{D}$ be two (symmetric) monoidal categories. A (symmetric) monoidal functor is a pair (F, λ), where

- $F : \mathscr{C} \to \mathscr{D}$ a functor,
- $\lambda : F(X) \otimes F(Y) \xrightarrow{\cong} F(X \otimes Y)$ functorial in X and Y,

such that

- λ is compatible with a (and c), i.e. for any $A, B, C \in \mathscr{C}$, we have

$$\begin{tikzcd}
 & F((A \otimes B) \otimes C) \arrow[r, "F(a_{\mathscr{C}})"] & F(A \otimes (B \otimes C)) & \\
F(A \otimes B) \otimes F(C) \arrow[ur, "\lambda"] & & & F(A) \otimes F(B \otimes C) \arrow[ul, "\lambda"'] \\
(F(A) \otimes F(B)) \otimes F(C) \arrow[u, "\lambda"] \arrow[rrr, "a_{\mathscr{D}}"] & & & F(A) \otimes (F(B) \otimes F(C)) \arrow[u, "\lambda"']
\end{tikzcd} \tag{2.51}$$

  and similar diagrams for c;

- $F(\mathbf{1}_{\mathscr{C}}) = \mathbf{1}_{\mathscr{D}}$.

Lemma 2.71. *Let $F : \mathscr{C} \to \mathscr{D}$ be a monoidal functor.*

(1) If $X^\vee \in \mathscr{C}$ is a right dual of $X \in \mathscr{C}$, then $F(X^\vee) \in \mathscr{D}$ is a right dual of $F(X) \in \mathscr{D}$;

(2) If $F : \mathscr{C} \to \mathscr{D}$ is a symmetric monoidal functor and $X \in \mathscr{C}$ is dualizable, then the following diagram commutes

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{C}} X \arrow[r, "F"] \arrow[d, "Tr"] & \mathrm{End}_{\mathscr{D}} F(X) \arrow[d, "Tr"] \\
\mathrm{End}_{\mathscr{C}} \mathbf{1}_{\mathscr{C}} \arrow[r, "F"] & \mathrm{End}_{\mathscr{D}} \mathbf{1}_{\mathscr{D}}
\end{tikzcd} \tag{2.52}$$

2.8 Rigid Symmetric Monoidal Abelian Categories

Theorem 2.72. *Let $(\mathscr{C}, \otimes, a, c)$ be a rigid symmetric monoidal abelian category. Suppose that we have*

$$\begin{tikzcd}
0 \arrow[r] & a' \arrow[r] \arrow[d, "f'"] & a \arrow[r] \arrow[d, "f"] & a'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & a' \arrow[r] & a \arrow[r] & a'' \arrow[r] & 0
\end{tikzcd} \tag{2.53}$$

then $Tr(f|a) = Tr(f'|a') + Tr(f''|a'')$.

Corollary 2.73. *Tr factors through $K_0(\mathscr{C}) \to \mathrm{End}_{\mathscr{C}} \mathbf{1}$.*

Lemma 2.74. *The functor $- \otimes X$ preserves short exact sequence, for any $X \in \mathscr{C}$, where $\mathscr{C}$ is a rigid symmetric monoidal abelian category.*

Proof. By Corollary 2.51. □

Lemma 2.75. *For $0 \to a' \to a \to a'' \to 0$ a short exact sequence, the sequence $0 \to (a'')^\vee \to a^\vee \to (a')^\vee \to 0$ is also exact.*

Proof. Note that $a^\vee = \underline{\mathrm{Hom}}_{\mathscr{C}}(a, \mathbf{1})$. Now apply adjunction of internal Hom and tensors, and the exactness of tensor product (Lemma 2.74). □

Let $F^\bullet\mathscr{C}$ be the category of (finite) filtered objects $(X, F^\bullet X)$ of $\mathscr{C}$, where $X \in \mathrm{Ob}(\mathscr{C})$ and $F^\bullet$ is a finite decreasing filtration of X by subobjects $0 = F^n X \hookrightarrow F^{n-1}X \hookrightarrow \cdots \hookrightarrow F^0 X \hookrightarrow F^{-1}X \hookrightarrow \cdots$, such that $F^i X = X$ for some $i < 0$.

Then we obtain

$$\begin{aligned} \otimes : F^\bullet\mathscr{C} \otimes F^\bullet\mathscr{C} &\to F^\bullet\mathscr{C} \\ (x, F^\bullet) \otimes (y, F^\bullet) &\mapsto (x \otimes y, F^\bullet) \end{aligned} \tag{2.54}$$

where $F^h(x \otimes y) := \sum_{i+j=h} F^i x \otimes F^j y \hookrightarrow x \otimes y$.

Note that $F^\bullet\mathscr{C}$ is rigid, additive, since $(x, F^\bullet)^\vee = (x^\vee, F^\bullet)$, where $F^i x^\vee := (x/F^{1-i}x)^\vee$ and unit given $\mathbf{1}$ with $(0 \subseteq \mathbf{1} \subseteq \mathbf{1} \cdots)$ where the first $\mathbf{1}$ is in filtration degree 0[3] .

$$\begin{tikzcd} & F^\bullet\mathscr{C} \arrow[dl, "\text{forgetful functor}"'] \arrow[dr, "\text{gr functor}"] & \\ \mathscr{C} & & \mathscr{C} \end{tikzcd} \qquad \begin{tikzcd} & (x, F) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (x, F) \mapsto x & & \mathrm{gr}_F(x) = \oplus_i F^i x / F^{i+1} x \end{tikzcd} \tag{2.55}$$

both are symmetric monoidal functors.

Corollary 2.76. *Let $\mathscr{C}$ be a rigid symmetric monoidal abelian category. If $f : X \to X$ is nilpotent, then*

$$Tr(f|X) = 0.$$

Let $\mathscr{C}$ be a rigid symmetric monoidal abelian category as above. If $U \hookrightarrow \mathbf{1}$ is a subobject, then $\mathbf{1} = (\mathbf{1}/U)^\vee \oplus U$. Note that $\mathrm{End}_{\mathscr{C}}\, \mathbf{1} = k$ is a commutative ring. We then have a bijection

$$\{\text{idempotents}\} \leftrightarrow \{\text{subobjects of } \mathbf{1}\}. \tag{2.56}$$

In practice, we may often assume that $\mathrm{End}_{\mathscr{C}}\, \mathbf{1} = k$ is a field and hence $\mathbf{1}$ is irreducible[4].

Proposition 2.77. *Let $\mathscr{C}$ be a rigid symmetric monoidal pseudo-abelian (i.e. idempotent complete additive category[5]). Suppose that* $\mathrm{End}_{\mathscr{C}}\, \mathbf{1} =: k$ *is a field. Then*

(1) if $\mathscr{C}$ is semisimple abelian, then $\mathscr{N}\mathscr{C} = 0$;

(2) suppose that

- $\mathscr{N}\mathscr{C} = 0$,
- $Tr(f) = 0$ *for any nilpotent f,*
- $\mathrm{Hom}(x, y)$ *is finite dimensional over k for any x, y,*

then $\mathscr{C}$ is semi-simple abelian.

Proof. (1) for any $f : X \to Y$, $X = \oplus_i X_i$, $Y = \oplus_j Y_j$ decomposition into irreducible objects, $f_{ij} = e_j \circ f \circ e_i : X_i \to Y_j$. Then $f \in \mathscr{N}\mathscr{C}$ if and only if $f_{ij} \in \mathscr{N}\mathscr{C}$ for all i, j. Therefore, we can assume without loss of generality that both $X = Y$ are irreducible.

Now we want $\mathscr{N}\mathscr{C} \cap \mathrm{End}_{\mathscr{C}}(X) = \{0\}$. Note that $\mathscr{N}\mathscr{C} \cap \mathrm{End}_{\mathscr{C}}(X)$ is a two-sided ideal and hence either $\mathrm{End}_{\mathscr{C}}(X) \subseteq \mathscr{N}\mathscr{C}$ or $\mathscr{N}\mathscr{C} \cap \mathrm{End}_{\mathscr{C}}(X) = \{0\}$, since $\mathrm{End}(X)$ is a division algebra by irreducibility of X. If $\mathrm{End}_{\mathscr{C}}(X) \subseteq \mathscr{N}\mathscr{C}$ holds, then $0 = \mathrm{Tr} : \mathrm{End}_{\mathscr{C}}(X) \to k$, i.e. $e_X \circ c \circ \varphi \circ u_X = 0$ for any $\varphi \in \mathrm{End}_{\mathscr{C}}(X)$. By the definition of dualizable (the composition of $X \xrightarrow{u_X} (X \otimes X^\vee) \otimes X \underset{\cong}{\xrightarrow{a}} X \otimes (X^\vee \otimes X) \xrightarrow{e_X} X$ is id_X), we know that u_X, e_X are both non-zero unless $X = 0$ (u_X is injective and e_X is surjective).

(2) first note that $\mathrm{End}_{\mathscr{C}}(X)$ is a semi-simple k-algebra. Consider the Jacobson radical $\mathscr{J}_X \subseteq \mathrm{End}_{\mathscr{C}}(X)$. Then any element of $\mathscr{J}_X$ is nilpotent, because $\mathrm{End}_{\mathscr{C}}(X)$ is a finite-dimensional algebra, and by our assumption that nilpotent elements have trace 0, $\mathscr{J}_X \subseteq \mathscr{N}\mathscr{C} = 0$. Therefore, $\mathrm{End}_{\mathscr{C}}(X)$ is semi-simple k-algebra for any $X \in \mathscr{C}$. Now use the following lemma to complete the proof. □

[3] We want the dual of $\mathbf{1}$ to be $\mathbf{1}$ and that's why the shift 1 occurs in $F^i x^\vee := (x/F^{1-i}x)^\vee$.

[4] Note that $\mathrm{End}_{\mathscr{C}}\, \mathbf{1}$ acts on $X \cong X \otimes \mathbf{1} \cong \mathbf{1} \otimes X$ naturally, so idempotents will induce decomposition of $\mathscr{C}$.

[5] Equivalently speaking, pseudo-abelian category is a category that is preadditive and is such that every idempotent has a kernel. Idempotent complete means every idempotent splits. Preadditive means that category is enriched over abelian groups, i.e. each Hom set is an abelian group and compositions are bilinear (distribution law holds).

Lemma 2.78. *Let $\mathscr{C}$ be a pseudo-abelian, k-linear category (i.e.* $\mathrm{Hom}(X, Y)$ *is a finite dimensional k-vector space) for some field k. Then $\mathscr{C}$ is semi-simple abelian if and only if* $\mathrm{End}_{\mathscr{C}}(X)$ *is a semi-simple k-algebra for all X.*

Example 2.79 (Fake Motives). Let F be a field and SmProj/F be the category of smooth projective varieties over F. Given any reasonable equivalence relation (rational equivalence, numerical equivalence, algebraic equivalence, etc.) on $Z^\bullet(X)$ group of cycles, we can define $A^\bullet(X) = Z^\bullet(X)/\sim$ with a ring structure given by intersection product. There is a well-defined pull-back functor and push-forward functor for $f : X \to Y$, i.e. we have

$$f^* : A^*(Y) \to A^*(X), \tag{2.57}$$

and

$$f_* : A^*(X) \to A^{*-\dim X+\dim Y}(Y). \tag{2.58}$$

One can define $\mathrm{Mot}_F^{\sim} = \{(X, p, m)\}$, where $X \in \mathrm{SmProj}/F$, $p \in A^{\dim X}(X \times X)$ such that $p \star p = p$, $m \in \mathbb{Z}$, where

$$A^*(X \times Y) \times A^*(Y \times Z) \longrightarrow A^*(X \times Z)$$

$$\begin{tikzcd}
 & X \times Y \times Z \arrow[dl, "p_{12}\times p_{23}"'] \arrow[dr, "p_{13}"] & \\
X \times Y \times Y \times Z & & X \times Z
\end{tikzcd} \tag{2.59}$$

and $a \star b := (p_{13})_*(p_{12}^*(a)p_{23}^*(b))$. Then $\mathrm{Hom}((X, p, m), (Y, q, n)) := q \cdot A^{\dim X - m + n}(X \times Y) \cdot p$.
Claim: $\mathrm{Mot}_F^{\sim}$ is a rigid symmetric monoidal pseudo-abelian category with $\mathrm{End}\,\mathbf{1} = k$.
Observe that the dual of $(X, \mathrm{id}, 0)$ is $(X, \mathrm{id}, \dim X)$. **Fact:** for a self-correspondence

$$\begin{tikzcd}
 & Z \arrow[dl] \arrow[dr] & \\
X & & X
\end{tikzcd} \tag{2.60}$$

$\mathrm{Tr}(Z|X) = \deg(Z \cap \Delta_X)$. Then $\mathscr{N}\mathrm{Mot}_F^{\sim} = 0$ if and only if $\sim$ is numerical equivalence.

Corollary 2.80. Mot_F^{num} *is semi-simple abelian.*

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2.9 Lax (Symmetric) Monoidal Functor

Definition 2.81. A lax (symmetric) monoidal functor consists of (F, λ), where

- $F : \mathscr{C} \to \mathscr{D}$ is a functor between two (symmetric) monoidal categories;
- $\lambda : F(X) \otimes F(Y) \to F(X \otimes Y)$ is natural transform,

$$\begin{tikzcd}
\mathscr{C} \times \mathscr{C} \arrow[r, "\otimes"] \arrow[d, "F\times F"'] & \mathscr{C} \arrow[d, "F"] \\
\mathscr{D} \times \mathscr{D} \arrow[r, "\otimes"] \arrow[ur, Rightarrow, "\lambda"] & \mathscr{D}
\end{tikzcd} \tag{2.61}$$

such that

- compatible with a (and also c);
- there exists $\epsilon : \mathbf{1}_{\mathscr{D}} \to F(\mathbf{1}_{\mathscr{C}})$, such that the composition

  $$\mathbf{1}_{\mathscr{D}} \otimes F(X) \to F(\mathbf{1}_{\mathscr{C}}) \otimes F(X) \to F(\mathbf{1}_{\mathscr{C}} \otimes X) \cong F(X)$$

  is $l_{F(X)}$ for any $x \in \mathscr{C}$ and similarly for $F(X) \otimes \mathbf{1}_{\mathscr{D}}$.

Remark 2.82. If such ϵ exists, then it is unique up to unique isomorphism.

Example 2.83. The forgetful functor
$$\mathrm{Vect}_k^{\otimes} \to \mathrm{Sets}^{\times}$$
is a lax symmetric monoidal functor, where λ is given by $V \times W \mapsto V \otimes W$.

Example 2.84. For $\mathscr{C}$ a closed (symmetric) monoidal category , we always have
$$\begin{aligned} G : \mathscr{C} &\to \mathrm{Sets}^{\times} \\ X &\mapsto \mathrm{Hom}_{\mathscr{C}}(X, \mathbf{1}) \end{aligned} \tag{2.62}$$
and this is a lax symmetric monoidal functor with λ given by
$$\mathrm{Hom}_{\mathscr{C}}(X, \mathbf{1}) \times \mathrm{Hom}_{\mathscr{C}}(Y, \mathbf{1}) \to \mathrm{Hom}_{\mathscr{C}}(X \otimes Y, \mathbf{1}). \tag{2.63}$$

Definition 2.85. Let (F, λ) and (G, μ) be two (symmetric) monoidal functors. A natural monoidal transform is $\varphi : F \Rightarrow G$, such that
$$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "\lambda"] \arrow[d, "\varphi"] & F(X \otimes Y) \arrow[d, "\varphi"] \\ G(X) \otimes G(Y) \arrow[r, "\mu"] & G(X \otimes Y) \end{tikzcd} \tag{2.64}$$
commutes.

We can organize symmetric monoidal categories into 2-categories.

- $\mathrm{Fun}^{\mathrm{lax,mon}}(\mathscr{C}, \mathscr{D})$ contains $\mathrm{Fun}^{\mathrm{mon}}(\mathscr{C}, \mathscr{D})$ as a full subcategory.
- $\mathrm{Fun}^{\mathrm{lax,sym}}(\mathscr{C}, \mathscr{D})$ contains $\mathrm{Fun}^{\mathrm{sym}}(\mathscr{C}, \mathscr{D})$ as a full subcategory.
- $\mathrm{Cat}^{\mathrm{lax,mon}}$ contains $\mathrm{Cat}^{\mathrm{mon}}$ as a subcategory (not 2-full).

2.10 Algebra Objects

Definition 2.86. Let $\mathscr{C}$ be a monoidal category. The category of algebra objects in $\mathscr{C}$ is $\mathrm{Alg}(\mathscr{C}) := \mathrm{Fun}^{\mathrm{lax,mon}}(*, \mathscr{C})$. Concretely, this means an object $A \in \mathscr{C}$, equipped with

- $m : A \otimes A \to A$,
- $\epsilon : 1 \to A$,

such that

- the following diagram commutes
$$\begin{tikzcd} & (A \otimes A) \otimes A \arrow[r, "a"] \arrow[dl, "m\otimes\mathrm{id}"] & A \otimes (A \otimes A) \arrow[d, "\mathrm{id}\otimes m"] \\ A \otimes A \arrow[dr, "m"] & & A \otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd} \tag{2.65}$$
- the composition $\mathbf{1} \otimes A \xrightarrow{\epsilon\otimes\mathrm{id}_A} A \otimes A \xrightarrow{m} A$ coincides with l_A.

Similarly, if $\mathscr{C}$ is a symmetric monoidal category, we can define
$$\mathrm{CAlg}(\mathscr{C}) := \mathrm{Fun}^{\mathrm{lax,sym}}(*, \mathscr{C}).$$

Remark 2.87. For $\mathscr{C}$ a symmetric monoidal category,

- both forgetful functors $\mathrm{CAlg}(\mathscr{C}) \to \mathrm{Alg}(\mathscr{C}) \to \mathscr{C}$ are symmetric monoidal functors:

$$(A \otimes A) \otimes (B \otimes B) \xrightarrow{\cong} (A \otimes A) \otimes (B \otimes B) \to A \otimes B.$$

Example 2.88. For $\mathscr{C} = (\mathrm{Vect}_k)^{\heartsuit}$, then

- $\mathrm{Alg}(\mathscr{C})$ =k-algebras;
- $\mathrm{CAlg}(\mathscr{C})$ =commutative k-algebras.

Example 2.89. For $\mathscr{C} = \mathrm{Ch}_k$, then $\mathrm{Alg}(\mathscr{C})$ =DG-algebras.

Example 2.90. For $\mathscr{C} = \mathrm{Cat}^{\times}$, $\mathrm{Alg}(\mathscr{C})$ consists of strict monoidal categories ($(A \otimes B) \otimes C = A \otimes (B \otimes C)$ strictly holds instead of up to a chosen isomorphism), where the strictness is due to the (strict) commutativity of diagram (2.86). By Maclane's strictness theorem, $\mathrm{Alg}(\mathscr{C})$ is equivalent to $\mathrm{Cat}^{\mathrm{mon}}$.

Exercise: Develop the notion of (left) modules in a monoidal category.

2.11 Another Way to Define Monoidal Categories

Now we want to give other (equivalent) definitions of (symmetric) monoidal categoires, which are more suitable to higher category settings.

2.11.1 Some Category Theory

Definition 2.91. Let Δ be the category of finite ordered sets with non-decreasing maps.

Remark 2.92. More concretely, Δ is equivalent the category consisting of

- Objects: $[n] = \{0, 1, \ldots, n\}$;
- Morphisms: $\varphi : [m] \to [n]$, such that $\varphi(i + 1) \geq \varphi(i)$;

which can be visualized as (only face map and dengeneration map need to be depicted)

$$\begin{tikzcd} {[0]} \arrow[r, bend left] \arrow[r, bend right] & {[1]} \arrow[l] \arrow[r, bend left] \arrow[r, bend left=50] \arrow[r, bend left=70] & {[2]} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, bend left] \arrow[r, bend left=50] \arrow[r, bend right] \arrow[r, bend right=50] & {[3]} \arrow[l] \arrow[l, shift left] \arrow[l, shift right] & \cdots \end{tikzcd} \tag{2.66}$$

Definition 2.93. For a category $\mathscr{C}$, we define

$$\mathscr{C}^{\Delta^{op}} = \mathrm{Fun}(\Delta^{op}, \mathscr{C}) \tag{2.67}$$

called the category of simplicial objects in $\mathscr{C}$.

Definition 2.94. Let $\mathscr{C}$ be a category with finite product. Then $\mathrm{Mon}(\mathscr{C})$ is defined to consist of $X_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathscr{C})$, such that

(A) $X_0 = *$(final object in $\mathscr{C}$),

(B) $\{0, 1\} \cong \{i, i + 1\} \hookrightarrow \{0, 1, \ldots, n\}$ induces n maps $X_n \to X_1$, which together give an isomorphism $X_n \xrightarrow{\cong} X_1 \times \cdots \times X_1$.

Lemma 2.95. *The following holds.*

(1) A monoidal object $\{X_\bullet\}$ in $\mathrm{Mon}(\mathscr{C})$ *is uniquely determined by*

$$\begin{tikzcd} X_0 \arrow[r, bend left] & X_1 \arrow[l] \arrow[r, bend left] \arrow[r, bend left=50] \arrow[r, bend left=70] & X_2 \arrow[l] \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

because $\mathscr{C}$ is an ordinary category.

(2) Mon(Sets) = Mon. *Actually*

$$
\begin{tikzcd}
X_1 \times X_1 & X_2 \arrow[l, "\cong"'] \arrow[r, bend left] \arrow[r, bend right] & X_1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{1,2\} \arrow[dr] & \\
\{0,1\} \arrow[ur] \arrow[r] & \{0,1\} \arrow[r] & \{0,1,2\}
\end{tikzcd}
\tag{2.68}
$$

$$
\begin{tikzcd}
X_1 \times X_1 \arrow[rr, bend right, "m"'] & X_2 \arrow[l, "\cong", "d_0, d_2"'] \arrow[r, "d_1"] & X_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\{0,1\} \arrow[r, "\cong"] & \{0,2\} \arrow[r] & \{0,1,2\}
\end{tikzcd}
$$

(3) Alg(Cat$^{\times}$) = Mon(Cat).

Proof. The proof of this lemma is left as an exercise.

Multiplication $X_2 \cong X_1 \times X_1 \to X_1$ is given by $\{0,1,2\} \to \{0,2\}$, and unit $u : * \to X_1$ is given by $\{0,1\} \to \{0\}$.

For (1), by definition, $X_n \cong X_1 \times \cdots X_1$ is determined by X_1. We only need to show the morphisms "in higher degrees" are actually determined by ≤ 2 degrees. Only need to check degeneration maps and face maps, using inductive method and simplicail identities.

(2) Associativity will be witnessed in $X_3 \to X_1$. □

Recall Grothendieck's construction. Given a contravariant functor $F : \mathscr{C}^{op} \to \mathrm{Cat}$, we can construct a functor $\widetilde{F} : \widetilde{\mathscr{C}} \to \mathscr{C}$. Construction of $\widetilde{\mathscr{C}}$:

- Objects: $\{(c, \eta) | c \in \mathscr{C}, \eta \in F(c)\}$.
- Morphisms: a morphism $(c, \eta) \to (d, \lambda)$ is a morphism $\varphi : c \to d$ in $\mathscr{C}$ and $\eta \to F(\varphi)(\lambda)$.

Proposition 2.96. *This funtor $\widetilde{F} : \widetilde{\mathscr{C}} \to \mathscr{C}$ is a Cartesian fibration.*

Let me recall the definition of Cartesian fibration.

Definition 2.97. Let $F : \mathscr{D} \to \mathscr{C}$ be a funtor, a map $(d_1 \to d_2) \in \mathscr{D}$ is called a Cartesian arrow, if for every $d \in \mathscr{D}$,

$$
\mathrm{Hom}_{\mathscr{D}}(d, d_1) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{D}}(d, d_2) \times_{\mathrm{Hom}_{\mathscr{C}}(F(d), F(d_2))} \mathrm{Hom}_{\mathscr{C}}(F(d), F(d_1)). \tag{2.69}
$$

Remark 2.98. More geometric in mind (view F as a morphism of presheaves, i.e. a map of spaces, or more rigorously by viewing Hom set Hom(A, B) as A-points of B), we can write in the following way: $d_1 \to d_2$ in $\mathscr{D}$ is called a Cartesian arrow, if for any $d \in \mathscr{D}$ (tested again any $d \in \mathscr{D}$), we have a Cartesian square of d-points

$$
\begin{tikzcd}
d_1(d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & d_2(d) \arrow[d] \\
F(d_1(d)) \arrow[r] & F(d_2(d))
\end{tikzcd}
\tag{2.70}
$$

Definition 2.99. A functor $F : \mathscr{D} \to \mathscr{C}$ is called a Cartesian fibration if for every

- $(c_1 \to c_2) \in \mathscr{C}$,
- $d_2 \in \mathscr{D}$, $F(d_2) \xrightarrow{\cong} c_2$,

there exists a Cartesian arrow $(d_1 \to d_2)$ over $(c_1 \to c_2)$.

Remark 2.100. Given $c_1 \to c_2$, d_2, $F(d_2) \xrightarrow{\cong} c_2$, then a Cartesian lifting $d_1 \to d_2$ (if exists) is unique up to unique isomorphism by Yoneda's lemma.

Given a category $\mathscr{C}$, define

$$
(\mathrm{Cartesian}_{/\mathscr{C}})_{\mathrm{strict}} \tag{2.71}
$$

to be

- Objects: $\mathscr{D} \to \mathscr{C}$ a Cartesian fibration,

- Morphisms:

$$\begin{tikzcd} \mathscr{D}_1 \arrow[r] \arrow[dr] & \mathscr{D}_2 \arrow[d] \\ & \mathscr{C} \end{tikzcd}$$

 such that Cartesian arrows are mapped to Cartesian arrows.

Theorem 2.101. *Grothendieck's construction gives an equivalence of categories*

$$\mathrm{Fun}(\mathscr{C}^{op}, \mathrm{Cat}) \xrightarrow{\cong} (\mathrm{Cartesian}_{/\mathscr{C}})_{strict}. \tag{2.72}$$

Similarly, we can define cocartesian arrows and cocartesian fibrations.

2.11.2 Second Definition of Monoidal Categories

Definition 2.102. A monoidal category is a coCartesian fibration $\mathscr{C} \to \Delta^{\mathrm{op}}$, and $\mathscr{C}_n$ denotes the fiber over $[n]$, such that

- $\mathscr{C}_0 = *$ is the final object,
- an isomorphism $C_n \xrightarrow{\cong} C_1 \times \cdots \times C_1$ induced by $[n] \supseteq \{i, i+1\} \leftarrow [1] = \{0, 1\}$.

Remark 2.103. By this definition, we see easily that $\mathrm{Alg}(\mathrm{Cat}^{\times}) = \mathrm{Mon}(\mathrm{Cat})$.

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Definition 2.104. Then we define the category $\mathrm{Cat}^{\mathrm{Mon}}$ of monoidal categories to be the full subcategory of $(\mathrm{CoCart}_{/\Delta^{\mathrm{op}}})_{\mathrm{strict}}$ consisting of objects satisfying the two properties in Definition 2.102, where the morphisms are strict diagrams

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[dr] & \mathscr{C} \arrow[d] \\ & \Delta^{\mathrm{op}} \end{tikzcd} \tag{2.73}$$

sending coCartesian arrows to coCartesian arrows.

Definition 2.105. Define $\mathrm{Cat}^{\mathrm{LaxMon}} \subseteq \mathrm{CoCart}_{/\Delta^{\mathrm{op}}}$ by

- Objects: $\mathscr{C} \to \Delta^{\mathrm{op}}$ coCartesian fibrations satisfying the two properties in Definition 2.102;
- Morphisms:

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[dr] & \mathscr{D} \arrow[d] \\ & \Delta^{\mathrm{op}} \end{tikzcd} \tag{2.74}$$

 such that F sends coCartesian arrows over $\rho_i : [1] = \{i-1, i\} \to [n]$ to coCartesian arrows.

Definition 2.106. $\mathrm{Alg}(\mathscr{C}) := \mathrm{Fun}^{\mathrm{lax,mon}}(*, \mathscr{C})$=sections of $\mathscr{C} \to \Delta^{\mathrm{op}}$ sending ρ_i to coCartesian arrows.

Definition 2.107. A symmetric monoidal category is a coCartesian fibration $\mathscr{C} \to \mathrm{Fin}_*$, where Fin_* is the category of all finite pointed sets $(I, *)$ with morphisms given by maps between sets sending the specified point $*$ to $*$, such that $\mathscr{C}_I$ is the fiber over the finite pointed set I, satisfying

- $\mathscr{C}_0 = *$ is the final object,
- an isomorphism $\mathscr{C}_I \xrightarrow{\cong} \prod_{i\in I-\{*\}} \mathscr{C}_i$ induced by

$$\begin{aligned} \rho_i : I &\to \{*, i\} \\ j &\mapsto \begin{cases} i, & \text{if } j = i; \\ *, & \text{if } j \neq i. \end{cases} \end{aligned} \tag{2.75}$$

We can similarly define the categories $\mathrm{Cat}^{\mathrm{sym}}$, $\mathrm{Cat}^{\text{lax sym}}$, $\mathrm{CAlg}(\mathscr{C})$.
Fact:

$$\begin{aligned} \mathrm{Cut} : \Delta^{\mathrm{op}} &\to \mathrm{Fin}_* \\ [n] &\mapsto \text{Set of } \{[n] = S \sqcup T | S \text{ and } T \text{ are convex subsets}\} \end{aligned} \tag{2.76}$$

with $* = (S = \emptyset)$.

Remark 2.108. Compare with Lurie's notation: $< n >=\mathrm{Cut}[n]$ via

$$j \mapsto \{0, 1, \dots, j-1\} \sqcup \{j, j+1, \dots, n\}. \tag{2.77}$$

2.12 Enriched Category

Let $\mathscr{A}$ be a monoidal category.

Definition 2.109. A category $\mathscr{C}$ enriched over $\mathscr{A}$ consists of

(1) A collection of objects;

(2) For $x, y \in \mathrm{Ob}(\mathscr{C})$, a Hom space $\underline{\mathrm{Hom}}(x, y) \in \mathscr{A}$;

(3) For $x, y, z \in \mathscr{C}$, $\underline{\mathrm{Hom}}(z, y) \otimes \underline{\mathrm{Hom}}(x, y) \to \underline{\mathrm{Hom}}(x, z)$ satisfying the pentagon axiom.

(4) There exists $\mathrm{id}_x : \mathbf{1} \to \underline{\mathrm{Hom}}(x, x)$ satisfying the usual axiom for units.

Given an enriched category $\mathscr{C}$ (over $\mathscr{A}$), we can recover an ordinary category $\underline{\mathscr{C}}$ by

- Objects: the same as $\mathscr{C}$;
- Morphisms: $\mathrm{Hom}_{\underline{\mathscr{C}}}(x, y) := \mathrm{Hom}_{\mathscr{A}}(\mathbf{1}, \underline{\mathrm{Hom}}(x, y))$.

Example 2.110. If $\mathscr{A}$ is a closed symmetric monoidal category, then $\mathscr{A}$ is enriched over itself and $\underline{\mathscr{A}} = \mathscr{A}$, by replacing the set $\mathrm{Hom}_{\mathscr{A}}(x, y)$ with the object $\underline{\mathrm{Hom}}(x, y) \in \mathscr{A}$. In this case we easily see by adjunction of tensor product and inner Hom that $\mathrm{Hom}_{\mathscr{A}}(1, \underline{\mathrm{Hom}}(x, y)) \cong \mathrm{Hom}_{\mathscr{A}}(1 \otimes x, y) \cong \mathrm{Hom}_{\mathscr{A}}(x, y)$.

Example 2.111. A category enriched over Ch_k is called a DG-category.

3 Higher (Linear) Algebra

3.1 Introduction to Higher Categories

Due to the limitation of time, we could only sketch the basic ideas and languages[6]. For details, one can always consult [Lur09] and [Lur17].

[6]Due to laziness, I will be even more sketchy than the class.

3.1.1 Vocabularies

Principle: forbid strict equality "=" and replace it with isomorphisms "$\approx$".

Therefore, we upgrade Sets$\to$Groupoids$\to$2-Groupoids$\to \cdots \to \infty$-groupoids. However, it is combinatorially hard to describe them in category theory language. However, we can realize Groupoids as homotopy 1-types, i.e. fundamental groupoids and 2-groupoids as homotopy 2-types and so on. Then ∞-Groupoids should be understood as Spaces up to homotopy. Given a space S, we can construct a simplicial set $\mathrm{Sing}_\bullet(S)$ by considering $\{\Delta^n \to S\}_{n\geq 0}$.

We introduce the notions:

- simplicial sets;
- Δ^n, $\wedge_n^i$;
- Kan complexes;
- geometric realization functor

$$\begin{aligned} |\bullet| : \mathrm{sSet} &\to \mathrm{CG} \subseteq \mathrm{Top} \\ X_\bullet &\mapsto |X_\bullet| \end{aligned} \tag{3.1}$$

  where CG denotes the category of compactly generated spaces. The geometric realization functor is defined to be the left adjoint to $\mathrm{Sing}_\bullet$, or the unqiue (up to unique isomorphism) functor satisfying

  (1) sending Δ^n to $|\Delta^n| := \{(t_0, \ldots, t_n) \subseteq [0,1]^{n+1} | \sum_{i=0}^n t_i = 1\}$;
  (2) preserving colimits.

Facts:

- Since simplicial sets are presheaves on Δ^{op}, all limits and colimits exist (and are computed pointwisely). Therefore, $\mathrm{sSet}^\times$ is a closed symmetric monoidal category. In other words, the Hom spaces can be enriched over simplicial sets. Then by adjunction of internal Hom and tensor products (given by Cartesian products in this case), we must have

$$\underline{\mathrm{Hom}}(S,T)_n \cong \mathrm{Hom}(\Delta^n, \underline{\mathrm{Hom}}(S,T)) \cong \mathrm{Hom}(\Delta^n \times S, T). \tag{3.2}$$

  Then the category Kan of all Kan complexes inherts a closed symmetric monoidal category structure.

- The two functors

$$\begin{tikzcd} \mathrm{Kan} \arrow[r, bend left, "|\bullet|"] & \mathrm{CG} \arrow[l, bend left, "\mathrm{Sing}_\bullet"] \end{tikzcd} \tag{3.3}$$

  both preserve finite products (and are Quillen adjunctions to each other). In particular, $|S \times T| = |S| \times |T|$, where the product on the right hand side is calculated in CG (i.e. endowed with CG topology).

- We can define Homotopy pullback of Kan complexes by

$$R \times_S^h T := R \times_{S^{\{0\}}} \underline{\mathrm{Hom}}(\Delta^1, S) \times_{S^{\{1\}}} T, \tag{3.4}$$

  whose geometric realization is the homotopy fiber product (invariant under homotopy)

$$|R \times_S^h T| = |R| \times_{|S|}^h |T| = \{r \in |R|, t \in |T|, \gamma : [0,1] \to S | \gamma(0) = f(r), \gamma(1) = g(t)\}, \tag{3.5}$$

  where $f : R \to S$ and $g : T \to S$ denote the maps.

- Similarly we can define homotopy pushouts which are compatible with homotopy pullbacks of compactly generated topological spaces, $R \sqcup_S^h T = R \sqcup_{\{0\}\times S} \Delta^1 \times S \sqcup_{\{1\}\times S} T$. For example, homotopy pushout of

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "\mathrm{id}"] & * \\ S^1 & \end{tikzcd} \tag{3.6}$$

  is D^2 the disk.

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Recall that infinity category theory upgrades set theory to ∞-groupoids/spaces/Ani. We have (CG, W) as a model, where CG stands for compactly generated spaces and W denotes the weak equivalence given by weak homotopy equivalence (i.e. the morphisms which induce isomorphism on π_0 and all π_i). Note that we also have (sSet, W). Then we have adjoint functors

$$|\bullet| : \mathrm{sSet} \leftrightarrow (\mathrm{CG}, W) : \mathrm{Sing}_\bullet \tag{3.7}$$

and both sides are closed symmeyric monoidal category and the two functors are symmetric monoidal functors.

Then using localization of categories, we obtain their homotopy category

$$\mathscr{H} := \mathrm{sSet}[W^{-1}] \xrightarrow{\cong} \mathrm{CG}[W^{-1}], \tag{3.8}$$

where $\mathrm{sSet}[W^{-1}] \cong \mathrm{Kan}/\mathrm{homotopy}$ and $\mathrm{CG}[W^{-1}] = \mathrm{CW}/\mathrm{homotopy}$.

Definition 3.1. Two maps $f, g : K \to L$ of Kan complexes are homotopy equivalent, if there exists a map

$$F : \Delta^1 \times K \to L,$$

such that $F|_{\{0\}\times K} = f$ and $F|_{\{1\}\times K} = g$.

There exists good notion of homotopy limits and colimits in sSet and CG, such that

$$\underline{\mathrm{Hom}}(X, \underset{\overleftarrow{I}}{\overset{h}{\lim}} F) \xrightarrow{\cong} \underset{\overleftarrow{I}}{\overset{h}{\lim}} \underline{\mathrm{Hom}}(X, F(i)). \tag{3.9}$$

Moreover,

- we have a final object $*$;
- loop space $\Omega X := * \times^h_X *$;
- homotopy groups $\pi_i(X, *) = \pi_{i-1}(\Omega X, *)$, and $\pi_0 : \mathrm{sSet} \to \mathrm{Set}$;
- fiber sequences give long exact sequences of homotopy groups.

3.1.2 Infinity Categories

Definition 3.2. A topological (resp. simplicial) category is a category enriched over CG (resp. sSet).

Definition 3.3. A functor $F : \mathscr{C} \to \mathscr{D}$ of topological (resp. simplicial) categories is an enriched functor, i.e. any $x, y \in \mathscr{C}$,

$$\mathrm{Maps}(x, y) \to \mathrm{Maps}(F(x), F(y)) \tag{3.10}$$

is a morphism in CG (resp. sSet). A funtor F is called

- fully faithful, if for any $x, y \in \mathscr{C}$,
$$\mathrm{Maps}(x, y) \to \mathrm{Maps}(F(x), F(y)) \tag{3.11}$$
is a weak equivalence;
- essentially surjective, if $hF : h\mathscr{C} \to h\mathscr{D}$ is essentially surjective, where $h\mathscr{C}$ is the homotopy category (an ordinary 1-category) of $\mathscr{C}$, i.e.
$$\mathrm{Ob} h\mathscr{C} = \mathrm{Ob}\mathscr{C}$$
and
$$\mathrm{Hom}_{h\mathscr{C}}(x, y) = \pi_0(\mathrm{Maps}_{\mathscr{C}}(x, y)).$$
- an equivalence, if it is fully faithful and essentially surjective.

Remark 3.4. If a functor F is fully faithful (resp. an equivalence), then hF is fully faithful (resp. an equivalence) on homotopy categories.

Remark 3.5. Sadly, an equivalence may not admit a “quasi-inverse" as in ordinary 1-category theory.

3.1.3 Quasi-categories

We have a pair of adjoint functors

$$G : \mathrm{sSet} \leftrightarrow \mathrm{Cat} : N_\bullet, \tag{3.12}$$

where $G(S_\bullet) = \varinjlim_{[n]\to S_\bullet}[n]$, where the partial ordered set $[n]$ is identified with the corresponding 1-category. The nerve functor $N_\bullet$ can be described explicitly by

$$(N_\bullet\mathscr{C})_n = \{C_0 \to C_1 \to \cdots \to C_n | (C_i \to C_{i+1}) \in \mathrm{Mor}(\mathscr{C})\}. \tag{3.13}$$

Definition 3.6. Let S be a simplicial set, then the followings are equivalent:

- $S \cong N_\bullet\mathscr{C}$ for some 1-category $\mathscr{C}$;
- S satisfies strong lifting property for inner horns: for any $n \geq 2, 0 < i < n$, there exists a unique lifiting

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & S \\ \Delta^n \arrow[ur, dotted, "\exists!"] & \end{tikzcd} \tag{3.14}$$

- S satisfies Segal condition: for any n, an isomorphism $S_n \xrightarrow{\cong} S_1 \times_{S_0} S_1 \times \cdots \times_{S_0} S_1$ induced by $[1] = \{i-1, i\} \hookrightarrow [n]$.

Definition 3.7. A quasi-category is a simplicial set $S_\bullet$, satisfying inner horn lifting property, i.e. for any $n \geq 2$, $0 < i < n$, there exists a lifting

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & S \\ \Delta^n \arrow[ur, dotted, "\exists"] & \end{tikzcd} \tag{3.15}$$

Example 3.8. Kan complexes are quasi-categories.

Example 3.9. Ordinary categories are quasi-categories (identified with their nerves).

Definition 3.10. A functor $F : \mathscr{C} \to \mathscr{D}$ of quasi-categories is a map of simplicial sets.

- We have $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ functor category.
- Objects of $\mathscr{C}$ are elmements in $\mathscr{C}_0$, morphisms are elements in $\mathscr{C}_1$, 2-morphisms are elements in $\mathscr{C}_2$, etc.
- $x, y \in \mathscr{C}_0$,
$$\mathrm{Map}_{\mathscr{C}}(x, y) := \{x\} \times_{\mathscr{C}} \mathrm{Fun}([1], \mathscr{C}) \times_{\mathscr{C}} \{y\} \tag{3.16}$$
is a Kan complex[7].
- $F : \mathscr{C} \to \mathscr{D}$ is called fully faithful, if
$$\mathrm{Maps}_{\mathscr{C}}(x, y) \xrightarrow{\cong} \mathrm{Maps}_{\mathscr{C}}(F(x), F(y)) \tag{3.17}$$
is an equivalence for any $x, y \in \mathscr{C}_0$.
- We can define homotopy category $h\mathscr{C}$.
- F is called essentially surjective if hF is essentially surjective.
- F is called an equivalence if F is fully faithful and essentially surjective.

[7]This is deduced from the exponential laws for inner/left/right/Kan fibrations, see Kerodon or my notes on infinity categories.

- Fact: $F : \mathscr{C} \to \mathscr{D}$ is an equivalence, if and only if there exists $G : \mathscr{D} \to \mathscr{C}$, such that

$$\begin{aligned} G \circ F &\cong \mathrm{id}_{\mathscr{C}}; \\ F \circ G &\cong \mathrm{id}_{\mathscr{D}}. \end{aligned} \tag{3.18}$$

We can define a colimit preserving functor

$$\mathfrak{C} : \mathrm{sSet} \to \{\mathrm{SimplicialCategories}\}, \tag{3.19}$$

by defining $\mathfrak{C}[n]$ be the category

- Objects: 0,1,. . . ,n.
- Mapping spaces:

$$\mathrm{Maps}_{\mathfrak{C}[n]}(i, j) = \begin{cases} \emptyset, & \text{if } i > j; \\ P_{ij}, & \text{if } i \leq j, \end{cases}$$

  where P_{ij} is the partially ordered set consisting of subsets of $\{i, i+1, \ldots, j\}$.

Define homotpy coherent nerve N^{hc} to be the right adjoint of $\mathfrak{C}$, using adjoint functor theorem.

Theorem 3.11. *Let $\mathscr{C}$ be a simplicial category, such that for any $x, y \in \mathscr{C}$, $\mathrm{Maps}_{\mathscr{C}}(x, y) \in \mathrm{Kan}$, then $N^{hc}\mathscr{C}$ is a quasi-category.*

Example 3.12. The category **Kan** consisting of all Kan complexes is enriched over itself, and

$$N^{hc}\mathbf{Kan} =: \mathrm{Spc} \text{ (or denoted by Ani)}. \tag{3.20}$$

$$N^{hc}\mathrm{QCat} = \hat{\mathrm{Cat}}_\infty, \tag{3.21}$$

where QCat is

- Objects: quasi-categories;
- Morphisms: $\mathrm{Fun}(\mathscr{C}, \mathscr{D})^{\cong}$ (taking cores, i.e. the maximal Kan sub-complex).

Note that $\hat{\mathrm{Cat}}_\infty$ is the $(\infty, 1)$-category of $(\infty, 1)$-categories, which is not an $(\infty, 2)$-category.
We also have the following notions:

- Final/Initial object: x is called final (resp. initial) if for any $y \in \mathscr{C}$, $\mathrm{Maps}(y, x)$ (resp. $\mathrm{Maps}(x, y)$) is a contractible Kan complex.
- Over/under category: given a functor $F : I \to \mathscr{C}$, define

$$\mathscr{C}_{/F} := \mathscr{C} \times_{\mathrm{Fun}(I,\mathscr{C})} \mathrm{Fun}([1] \times I, \mathscr{C}) \times_{\mathrm{Fun}(I,\mathscr{C})} \{F\}, \tag{3.22}$$

  where $\mathscr{C} \to \mathrm{Fun}(I, \mathscr{C})$ is the constant map. Similarly define $\mathscr{C}_{F/}$. If I is a singleton, then we identify F with its image $X \in \mathscr{C}_0$, and write $\mathscr{C}_{/X}$ (resp. $\mathscr{C}_{X/}$).
- limit and colimit: given $F : I \to \mathscr{C}$, $\varprojlim_I F$ is a final object of $\mathscr{C}_{/F}$ and $\varinjlim_I F$ is an initial object of $C_{F/}$.
- Fact: all (small) limits and colimits exist in Spc and $\hat{\mathrm{Cat}}_\infty$.
- Homotopy limits and colimits in **Kan** corresponds to limits and colimits in Spc.
- Cartesian fibration and coCartesian fibration: let $F : \mathscr{C} \to \mathscr{D}$ be a functor, an arrow $c_1 \xrightarrow{f} c_2$ is called an F-Cartesian arrow, if for any $c \in \mathscr{C}$

$$\mathrm{Maps}(c, c_1) \xrightarrow{\cong} \mathrm{Maps}(c, c_2) \times_{\mathrm{Maps}(F(c), F(c_2))} \mathrm{Maps}(F(c), F(c_1)) \tag{3.23}$$

  is a weak equivalence. Then F is called a Cartesian fibrations, if for every arrow $d_1 \xrightarrow{g} d_2$ in $\mathscr{D}$, and $c_2 \in \mathscr{C}_0$ such that $F(c_2) = d_2$, then there exists a Cartesian arrow $c_1 \to c_2$ lying over $g : d_1 \to d_2$.

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3.1.4 (1-)Full subcategoires

Definition 3.13. We say $\mathscr{C} \hookrightarrow \mathscr{D}$ is a full subcategory of $\mathscr{D}$ if this functor is fully faithful.

Remark 3.14. This implies $h\mathscr{C} \to h\mathscr{D}$ to be fully faithful.

Definition 3.15. We say $\mathscr{C} \to \mathscr{D}$ is 1-full, if it fits into

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & \mathscr{D} \arrow[d] \\ \mathscr{C}^\circ \arrow[r] & h\mathscr{D} \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \tag{3.24}$$

.

Note that $(\mathrm{Cart}/_{\mathscr{D}})_{\mathrm{str}} \hookrightarrow \mathrm{Fun}([1], \mathscr{D}) \times_{\hat{\mathrm{Cat}}_\infty} \mathscr{D}$ is 1-full:

- Objects: $F : \mathscr{C} \to \mathscr{D}$ a Cartesian fibration;
- 1-morphisms: strict commutative diagram

$$\begin{tikzcd} \mathscr{C}_1 \arrow[r, "F"] \arrow[dr, "F_1"'] & \mathscr{C}_2 \arrow[d, "F_2"] \\ & \mathscr{D} \end{tikzcd}$$

  such that F sends Cartesian arrows to Cartesian arrows.

Theorem 3.16. *We have straighten-unstraighten correspondence*

$$(\mathrm{Cart}/_{\mathscr{D}})_{str} \xrightarrow{\cong} \mathrm{Fun}(\mathscr{D}^{op}, \hat{\mathrm{Cat}}_\infty)$$

and

$$(\mathrm{CoCart}/_{\mathscr{D}})_{str} \xrightarrow{\cong} \mathrm{Fun}(\mathscr{D}, \hat{\mathrm{Cat}}_\infty).$$

3.1.5 Yonenda's Lemma

Theorem 3.17. *Yoneda embedding* $\mathscr{C} \hookrightarrow \mathrm{Fun}(\mathscr{C}^{op}, \mathrm{Spc})$ *given by* $c \mapsto \mathrm{Maps}_{\mathscr{C}}(-, c)$ *is fully faithful.*

Proof. Idea:

- Approach 1 (see Lurie's HA book for details): Using the quasi-category model, we can define twisted arrow category $\mathscr{D} \to \mathscr{C} \times \mathscr{C}^{op}$ is a right fibration, which corresponds to $\mathscr{C}^{op} \times \mathscr{C} \to \mathrm{Spc}$.
- A model free proof:

$$\begin{tikzcd} \mathrm{Fun}([1], \mathscr{C}) \arrow[r] \arrow[dr, "\text{source}"'] & \mathscr{C} \times \mathscr{C} \arrow[d, "\mathrm{pr}_1"] \\ & \mathscr{C} \end{tikzcd} \tag{3.25}$$

  both projections are Catesian fibrations (check this!) and

$$\begin{tikzcd} \hat{\mathrm{Cat}}_\infty \arrow[r] & \hat{\mathrm{Cat}}_\infty \\ & \mathscr{C}^{op} \arrow[ul] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C}_{/c} \arrow[r] & \mathscr{C} \\ & c \arrow[ul] \arrow[u] \end{tikzcd} \tag{3.26}$$

  gives $\mathrm{Fun}(\mathscr{C}^{op}, \mathrm{Fun}([1], \hat{\mathrm{Cat}}_\infty))$, which by straightening gives us $\mathscr{C}^{op} \to \mathrm{CoCart}_{/\mathscr{C}}$.

□

Theorem 3.18 (Enhanced Straightening)**.** *The straightening correspondence is functorial in $\mathscr{D}$, i.e. there exists*

$$\hat{\mathrm{Cat}}_\infty^{op} \times \hat{\mathrm{Cat}}_\infty \to \hat{\mathrm{Cat}}_\infty, \tag{3.27}$$

given by $(\mathscr{C}, \mathscr{D}) \mapsto \mathrm{Fun}(\mathscr{C}, \mathscr{D})$.

3.1.6 Adjoint Functors

Let $\mathscr{C}$ and $\mathscr{D}$ be two ∞-categories, and let $F : \mathscr{C} \leftrightarrows \mathscr{D} : G$ be a pair of functors. Then the followings are equivalent:

(1) there exists $\mathscr{M} \to [1]$ Cartesian and CoCartesian fibration, such that $\mathscr{M}_{\{1\}} \cong \mathscr{C}$ and $\mathscr{M}_{\{2\}} \cong \mathscr{D}$, which is equivalent to say $F \in \mathrm{Fun}([1]^{op}, \hat{\mathrm{Cat}}_\infty)$ and $G \in \mathrm{Fun}([1], \hat{\mathrm{Cat}}_\infty)$.

(2) There exists $e : F \circ G \to \mathrm{id}_{\mathscr{D}}$ and $u : \mathrm{id}_{\mathscr{C}} \to G \circ F$, such that

$$F \to F \circ (G \circ F) \cong (F \circ G) \circ F \to F$$

is homotopy to identity 2-morphism and similar for

$$G \to (G \circ F) \circ G \cong G \circ (F \circ G) \to G,$$

which is equivalent to say (F, G) form an ajoint pair between $\mathscr{H}$-enriched categories $h\mathscr{C} \leftrightarrows h\mathscr{D}$.

(3) there exists

$$e : F \circ G \to \mathrm{id}_{\mathscr{D}}$$

and

$$u : \mathrm{id}_{\mathscr{C}} \to F \circ G$$

such that for any $c \in \mathscr{C}$ and $d \in \mathscr{D}$, we have

$$\begin{tikzcd} \mathrm{Maps}_{\mathscr{D}}(F(c), d) \arrow[r, "G"] \arrow[dr, "\text{equivalence}"'] & \mathrm{Maps}_{\mathscr{C}}(G \circ F(c), G(d)) \arrow[d, "u"] \\ & \mathrm{Maps}_{\mathscr{C}}(c, G(d)) \end{tikzcd} \tag{3.28}$$

(Actually once we fix u, then e is unique up to a contractible Kan complex).

3.1.7 (Symmetric) Monoidal Categories

We can define a symmetric monoidal category $\mathscr{C}^{\otimes}$ to be a coCartesian fibration $\mathscr{C}^{op} \to \mathrm{Fin}_*$, such that the fibers satisfy

- $\mathscr{C}_* = *$,
- $\mathscr{C}_I \cong \prod_{i \in I} \mathscr{C}$, induced by ρ_i.

Similarly, we can define a monoidal category $\mathscr{C}^{\otimes}$ to be a coCartesian fibration $\mathscr{C}^{op} \to \Delta^{op}$, such that the fibers satisfy

- $\mathscr{C}_* = *$,
- $\mathscr{C}_I \cong \prod_{i \in I} \mathscr{C}$, induced by ρ_i.

Therefore,

$$\begin{aligned} \hat{\mathrm{Cat}}^{\mathrm{Sym}} &\hookrightarrow (\mathrm{CoCart}_{/\mathrm{Fin}_*})_{\mathrm{str}} \\ \hat{\mathrm{Cat}}^{\mathrm{Mon}} &\hookrightarrow (\mathrm{CoCart}_{/\Delta^{op}})_{\mathrm{str}} \end{aligned} \tag{3.29}$$

are full subcategories.

Proposition 3.19. *Let $\mathscr{C}$ be an ∞-category with finite products. Then it admits a natural cartesian symmeyric monoidal structure $\mathscr{C}^{\times}$.*

Proof. Idea: want $\mathscr{C}^{\times} \to \mathrm{Fin}_*$, such that the fiber over I is $\mathscr{C}^{I-*}$. Define

$$\begin{aligned} \mathrm{Fin}_*^{op} &\to \hat{\mathrm{Cat}}_\infty, \\ I &\mapsto \mathrm{Fun}(I, \mathscr{C}) \times_{\mathrm{Fun}(*,\mathscr{C})} \star, \end{aligned} \tag{3.30}$$

where $\star$ denotes the final object in $\mathscr{C}$. Then pass to its straightening. The strategy is that you construct things using machinery of infinity categories instead of construction by hand. Then you check properties. □

$$\begin{aligned} \mathrm{Alg}(\hat{\mathrm{Cat}}_\infty^{\times}) &\cong \hat{\mathrm{Cat}}_\infty^{\mathrm{Mon}}, \\ \mathrm{CAlg}(\hat{\mathrm{Cat}}_\infty^{\times}) &\cong \hat{\mathrm{Cat}}_\infty^{\mathrm{Sym}} \end{aligned} \tag{3.31}$$

admit arbitrary limits and colimits.

Remark 3.20. $\hat{\mathrm{Cat}}_\infty^{\times}$ admits all limits and colimits and

$$\mathrm{Alg}(\hat{\mathrm{Cat}}_\infty^{\times}) \to \hat{\mathrm{Cat}}_\infty^{\times}$$

$$\mathrm{CAlg}(\hat{\mathrm{Cat}}_\infty^{\times}) \to \hat{\mathrm{Cat}}_\infty^{\times}$$

preserve all limits and colimits.

Fact: let $F : I \to \mathscr{C}$ be a functor and $c := \underrightarrow{\mathrm{colim}}_I F$, then for any $d \in \mathscr{C}$,

$$\mathrm{Maps}_{\mathscr{C}}(c, d) \xrightarrow{\cong} \varprojlim_{i \in I} \mathrm{Maps}(F(i), d). \tag{3.32}$$

If $c' = \varprojlim_I F$, then

$$\mathrm{Maps}_{\mathscr{C}}(d, c') \cong \varprojlim_{i \in I} \mathrm{Maps}(d, F(i)). \tag{3.33}$$

3.1.8 Compactly Generated Infinity Categories

Definition 3.21. Suppose that $\mathscr{C}$ admits all filtered colimits.

(1) An object $c \in \mathscr{C}$ is called compact, if $\mathrm{Maps}_{\mathscr{C}}(c, \varinjlim_{i \in I} d_i) \cong \varinjlim_{i \in I} \mathrm{Maps}(c, d_i)$ commutes with any filtered colimits.

(2) $\mathscr{C}$ is called compactly generated, if there exists a set of compact objects $\{c_i\}$, such that every $c \in \mathscr{C}$ can be written as a filtered colimit of $\{c_i\}$.

Remark 3.22. Let $\mathscr{C}^\circ \subset \mathscr{C}$ be the full subcategory spanned by $\{c_i\}$. Consider

$$\iota : \mathscr{C} \to \mathscr{P}(\mathscr{C}^\circ) := \mathrm{Fun}((\mathscr{C}^\circ)^{\mathrm{op}}, \mathrm{Spc}) \tag{3.34}$$

given by the restriction of Yondeda's embedding to $\mathscr{C}^\circ$. Then $\mathscr{C}$ is compactly generated if and only if

(i) ι is fully faithful;

(ii) the essential image consists of $F : (\mathscr{C}^\circ)^{\mathrm{op}} \to \mathrm{Spc}$ that preserve finite limits.

In this case, all the compact objects are generated by $\{c_i\}$ under finite colimits and retract.

Example 3.23. Spc and $\mathrm{Mod}_A^{\heartsuit}$ are compactly generated, where A is any commutative ring.

Fix a regular cardinal κ, then we can define

(1) An object $c \in \mathscr{C}$ is called κ-compact, if $\mathrm{Maps}_{\mathscr{C}}(c, \varinjlim_{i \in I} d_i) \cong \varinjlim_{i \in I} \mathrm{Maps}(c, d_i)$ for all κ-filtered I.

(2) $\mathscr{C}$ is called compactly generated, if there exists a set of κ-compact objects $\{c_i\}$, such that every $c \in \mathscr{C}$ can be written as a κ-filtered colimit of $\{c_i\}$.

If we take $\kappa = \aleph_0$, then we recover Definition 3.21.

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3.1.9 Adjoint Functor Theorem

- $\mathrm{Pr}^L \subseteq \hat{\mathrm{Cat}}_\infty$ 1-full subcategory, whose objects are presentable ∞-categories and morphisms are colimit preserving functors.
- $\mathrm{Pr}^L_\omega \subseteq \mathrm{Pr}^L$ 1-full subcategory consisting of compactly generated ∞-categories, 1-morphisms are continuous functors that admit continuous right adjoints (i.e. 1-morphisms are colimit preservering functors which preserve compact objects).

Theorem 3.24. *Let $\mathscr{C}$ be presentable ∞-category. Then*

(1) $F : \mathscr{C}^{op} \to \mathrm{Spc}$ *is representable if and only if* $F(\varinjlim_i c_i) = \varprojlim_i F(c_i)$.

(2) $F : \mathscr{C} \to \mathrm{Spc}$ *is representable if and only if F is accessible*[a] *and* $F(\varprojlim_i c_i) = \varprojlim_i F(c_i)$.

[a] A functor F is called accessible if F commutes with κ-filtered colimits for some κ.

Corollary 3.25 (Adjoint Functor Theorem)**.** *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between presentable ∞-categories. Then*

- *F admits a right adjoint if and only if F preserves colimits, a.k.a. F is a continuous functor;*
- *F admits a left adjoint if and only if F preserves limits and F is accessible.*

Proof. A direct corollary of Theorem 3.24. □

Corollary 3.26. *Limits exist in $\mathscr{C}$.*

Proof. Given diagram $F : I \to \mathscr{C}$, we want

$$\mathrm{Maps}(c, \varprojlim_I F) \cong \varprojlim_{i\in I} \mathrm{Maps}(c, F(i)).$$

Note that the functor $\varprojlim_{i\in I} \mathrm{Maps}(-, F(i))$ preserves colimits and hence by Theorem 3.24, it is representable. □

Remark 3.27. We can also define Pr^R in a similar pattern. Then we have $\mathrm{Pr}^L \cong (\mathrm{Pr}^R)^{\mathrm{op}}$.

Proposition 3.28. *Let $\mathscr{C}$ be a compactly generated ∞-category, and $F : \mathscr{C} \to \mathscr{D}$ be a continuous functor between presentable ∞-categories. Then its right adjoint F^R is a continuous functor if and only if F sends compact objects to compact objects.*

Theorem 3.29. *Arbitrary limits and colimits exist in Pr^L and the inclusion functor $\mathrm{Pr}^L \hookrightarrow \hat{\mathrm{Cat}}_\infty$ preserves limits (but not colimits in general).*

Given a functor

$$\begin{aligned} F : I &\to \mathrm{Pr}^L \\ i &\mapsto c_i \end{aligned} \tag{3.35}$$

we then obtain a functor[8] $G : I^{\mathrm{op}} \to \hat{\mathrm{Cat}}_\infty$. Then we have

$$\varinjlim_I F \cong \varprojlim_{I^{\mathrm{op}}} G. \tag{3.36}$$

Theorem 3.30. Pr^L *has a natural closed symmetric monoidal structure with* $\underline{\mathrm{Hom}}(\mathscr{C}, \mathscr{D}) = \mathrm{Fun}^L(\mathscr{C}, \mathscr{D})$. *The inclusion functor* $\mathrm{Pr}^L \subseteq \hat{\mathrm{Cat}}_\infty$ *is a lax symmetric monoidal functor with* $\mathscr{C} \times \mathscr{D} \to \mathscr{C} \otimes \mathscr{D} \cong Fun^R(\mathscr{C}^{op}, \mathscr{D})$.

Remark 3.31. **Warning:** for $\mathscr{C}$ a presentable ∞-category, in general $\mathscr{C}^{\mathrm{op}}$ is **NOT** presentable.

[8] For $\varphi : i \to j$, we have a continuous functor $F(\varphi) : F(i) \to F(j)$ (by defintion morphisms in Pr^L are continuous functors), then it admits a right adjoint functor $F(j) \to F(i)$.

Example 3.32. Let $\mathscr{C}^0$ be a small category and $\mathscr{C} = \mathscr{P}(\mathscr{C}^0) := \mathrm{Fun}((\mathscr{C}^0)^{\mathrm{op}}, \mathrm{Spc})$. Fact: $\mathscr{C} = \mathscr{P}(\mathscr{C}^0)$ is compactly generated. If $\mathscr{C} = \mathscr{P}(\mathscr{C}^0)$ and $\mathscr{D} = \mathscr{P}(\mathscr{D}^0)$, then $\mathscr{C} \otimes \mathscr{D} = \mathscr{P}(\mathscr{C}^0 \times \mathscr{D}^0)$.

$\mathrm{Pr}^L_\omega \subseteq \mathrm{Pr}^L$ clsoed symmetric monoidal subcategory of Pr^L.

Definition 3.33. Define $\mathrm{Idem}^{\mathrm{Rex}} \subseteq \hat{\mathrm{Cat}}_\infty$ 1-full subcategory by

- Objects: small idempotent complete[a] category with finite colimits;
- 1-morphisms: right exact (a.k.a. finite colimit preserving functors).

[a]By [Lur09][5.4.3.6], a small $(\infty,1)$-category is idempotent complete if and only if it is accessible. In ordinary category theory, idempotent morphism e satisfies $e^2 = e$ but in infinity category it is not a strictly equality but homotopic coherently, i.e. an additional structure instead of a property. Idempotent complete means every idempotent splits.

Example 3.34. For a commutative ring R, $(\mathrm{Mod}_R)_{f.p.}$ finitely presented R-modules in $\mathrm{Mod}_R^{\heartsuit}$.

Theorem 3.35.

$$\begin{aligned} \mathrm{Ind} : \mathrm{Idem}^{Rex} &\xrightarrow{\cong} \mathrm{Pr}^L_\omega : ()^\omega \\ \mathscr{C} &\mapsto \mathrm{Ind}(\mathscr{C}), \\ \mathscr{D}^\omega &\leftarrow \mathscr{D}, \end{aligned} \tag{3.37}$$

where $\mathscr{D}^\omega$ is defined to be the full subcategory of compact objects of $\mathscr{D}$ and $\mathrm{Ind}(\mathscr{C})$ *is obtained by formally adding all filtered colimits, i.e.* $\mathrm{Ind}(\mathscr{C}) = \{X \in \mathscr{P}(\mathscr{C}) | X(\varinjlim_{i\in I} c_i) = \varprojlim_{i \in I} X(c_i)$ *for I finite*$\}$.

Remark 3.36. $\mathrm{Idem}^{\mathrm{Rex}}$ has limits and clomits, and the inclusion $\mathrm{Idem}^{\mathrm{Rex}} \subseteq \hat{\mathrm{Cat}}_\infty$ preserves limits and $\mathrm{Pr}^L_\omega \subseteq \hat{\mathrm{Cat}}_\infty$ does not preserve limits in general.

3.2 Non-abelian Derived Categories

Let $\mathscr{C}$ be a small category with finite coproducts and

$$\mathscr{P}_\Sigma(\mathscr{C}) := \{X \in \mathscr{P}(\mathscr{C}) \,|\, X(\coprod_i c_i) = \prod_i X(c_i), i \in I \text{ finite}\}$$

is a presentable ∞-category. In fact, $\mathscr{P}_\Sigma(\mathscr{C})$ is the subcategory of $\mathscr{P}(\mathscr{C})$ generated by $\mathscr{C}$ under sifted colimts (filtered colimits + geometric realization[9]).

Now let $\mathscr{C}$ be an ordinary 1-category with colimits. We can define $\mathscr{C}^\omega$ to be the full subcategory consisting of compact objects X (a.k.a. $\mathrm{Hom}(X, -)$ commutes with filtered colimits), and define $\mathscr{C}^P$ to be the full subcategory consisting of projective objects ($\mathrm{Hom}(c, -)$ preserves reflective coequalizers[10]). Define compact projective objects $\mathscr{C}^{cp} = \mathscr{C}^c \cap \mathscr{C}^p$. Assume that $\mathscr{C}$ is generated by $\mathscr{C}^{cp}$ compact projective objects under colimits and $\mathscr{C}^{cp}$ is small.

Then define

$$\mathrm{Ani}(\mathscr{C}) := \mathscr{P}_\Sigma(\mathscr{C}^{\mathrm{cp}}) \tag{3.38}$$

the derived category of $\mathscr{C}$.

Example 3.37. Let $\mathscr{C}$ be the category Sets. Then $\mathscr{C}^{\mathrm{cp}}$=finite sets. Then Ani(Sets) = Spc.

Example 3.38. Let $\mathscr{C} := \mathrm{Mod}_R^{\heartsuit}$ for a commutative ring R. Then $\mathscr{C}^c = (\mathrm{Mod}_R^{\heartsuit})_{f.p.}$ and $\mathscr{C}^{cp} = \mathrm{Proj}_R$ finitely presented projective R-modules[a]. Then $\mathrm{Ani}(\mathrm{Mod}_R^{\heartsuit}) =: \mathrm{Mod}_R^{\leq 0}$. Then $h\mathrm{Mod}_R^{\leq 0} \cong D(R)^{\leq 0}$ ordinary derived category of R-modules concentrated at non-positive degrees.

[a]Note that an exact sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} Q \to 0$ with $\mathrm{Ker}(g) = \mathrm{Im}(f)$ is equivalent to say that Q is the reflexive coequalizer of
$\begin{tikzcd} M \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & N \arrow[l, bend left, "g"] \end{tikzcd}$. Hence the assumption $\mathrm{Hom}(X, -)$ preserves reflexive coequalizers implies that $\mathrm{Hom}(X, -)$ preserves all short exact sequences and hence X is a projective module. But for a projective module X, it always preserves finite colimits, and hence preserves reflexive coequalizers.

[9]A geometric realization is a colimit over Δ^{op}.

[10]According to [Lur09][5.5.8.5] The formation of the geometric realizations of simplicial objects should be thought of as the ∞-categorical analogue of the formation of reflexive coequalizers. So for a general quasi-category $\mathscr{C}$, a projective object C is to mean that $\mathrm{Map}(C, -)$ commutes with geometric realizations. This definition is motivated by the fact that the geometric realization of a simplicial space is (homotopy equivalent to) its homotopy colimit

Note that Sets $\hookrightarrow$ Spc preserves limits and filtered colimits but not geometric realization in general.

3.3 Stable Categories

Definition 3.39. We say that $\mathscr{C}$ is a stable ∞-category, if

- $\mathscr{C}$ admits an initial and final object, and they coincide, denoted by 0.
- For every $f : X \to Y$, it admits fiber

$$\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & Y \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \tag{3.39}$$

  and cofiber

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & Y \arrow[d] \\ 0 \arrow[r] & \mathrm{cofib}(f) \end{tikzcd} \tag{3.40}$$

- a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \tag{3.41}$$

  is a pushout diagram if and only if it is a pullback diagram.

Fact:

- Stable categories admits all finite limits and colimits, since the followings are equivalent
  - $\mathscr{C}$ has all finite limits.
  - $\mathscr{C}$ has all equalizers and binary products.
  - $\mathscr{C}$ has all pullbacks and a terminal object.
- $h\mathscr{C}$ has a natural triangulated category structure if $\mathscr{C}$ is a stable ∞-category.
- The subcategory Pr^L_{st} consisting of stable presentable infinity categories is closed under tensor products and the unit is Sp the spectra category.

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Lemma 3.40. *An ∞-category $\mathscr{C}$ is stable if and only if*

(1) any finite limits and finite colimits exist;

(2) a (homotopy coherent) diagram $[1] \times [1] \to \mathscr{C}$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd} \tag{3.42}$$

is a pullback diagram if and only if a pushout diagram.

If $\mathscr{C}$ is pointed (i.e. there exists a zero object), $\mathscr{C}$ admits fiber product

$$\begin{aligned} \Omega : \mathscr{C} &\to \mathscr{C} \\ X &\mapsto \Omega X := 0 \times_X 0 \end{aligned} \tag{3.43}$$

and admits pushout

$$\begin{aligned} \Sigma : \mathscr{C} &\to \mathscr{C} \\ X &\mapsto \Sigma X := 0 \sqcup_X 0 \end{aligned} \tag{3.44}$$

If $\mathscr{C}$ is stale, then Ω, Σ are inverse to each other[11].

Theorem 3.41. *If $\mathscr{C}$ is a stable ∞-category, then $h\mathscr{C}$ admits a triangulated category structure with*

$$T : h\mathscr{C} \to h\mathscr{C} \tag{3.45}$$

given by Σ. Distinguished triangles are given by cofiber sequences.

Example 3.42. If A is a commutative ring, take the DG-category whose objects are finite complexes of finite projective A-modules and morphisms are complexes $\underline{\mathrm{Hom}}(X, Y) = X^\vee \otimes Y$ with suitable differentials. By taking DG-nerve, we obtain Perf_A an infinity category. The ∞-category Perf_A turns out to be stable and idempotent complete. Then we define

$$\mathrm{Mod}_A := \mathrm{Ind}\mathrm{Perf}_A \tag{3.46}$$

which is stable and presentable.

We can organize all stable categories into an ∞-category.

Definition 3.43. A functor $F : \mathscr{C} \to \mathscr{D}$ between stable categories preserves finite colimits (right exact) if and only if it preserves finite limits (left exact). Such a functor is called an exact functor.

Lemma 3.44. *Let $F : \mathscr{C} \to \mathscr{D}$ be an exact functor between stable ∞-categories, then the followings are equivalent:*

(1) F preserves all colimits;

(2) F preserves all coproducts;

(3) F preserves all filtered colimits.

Proof. Any colimit can be written as a coequalizer and coproducts, and a coequalizer is a finite colimit. Any colimit can be written as a filtered colimit and a finite colimit. □

Definition 3.45. Define $\mathrm{LinCat} \subset \mathrm{Pr}^L$ to be the full subcategory consisting of all stable presentable ∞-categories.

Definition 3.46. Define $\mathrm{LinCat}_\omega \subset \mathrm{Pr}^L_\omega$ to be the full subcategory consisting of all stable compactly generated ∞-categories.

Definition 3.47. Define $\mathrm{LinCat}_{\mathrm{sm}} \subset \mathrm{Idem}^{\mathrm{Rex}}$ to be the full subcategory consisting of all small, stable, and idempotent complete ∞-categories.

Theorem 3.48. *The ∞-category* LinCat *is preserved under the tensor product of* Pr^L*, with the unit* Sp*. Therefore,* LinCat *is closed symmetric monoidal category.*

Theorem 3.49. *The ∞-category* $\mathrm{LinCat}_\omega \subseteq \mathrm{LinCar}$ *is a symmetric monoidal subcategory. Moreover,*

$$\mathrm{Ind} : \mathrm{LinCat}_{sm} \xrightarrow{\cong} \mathrm{LinCat}_\omega : (-)^\omega. \tag{3.47}$$

Fact:

- limits and colimits exist in any of them.
- The inclusions $\mathrm{LinCat} \subseteq \mathrm{Pr}^L$ and $\mathrm{LinCat}_\omega \subseteq \mathrm{Pr}^L_\omega$ preserve limits and colimits.

Remark 3.50. Note that

$$\mathrm{Sp} \in \mathrm{CAlg}(\mathrm{Pr}^L)$$

and

$$\mathrm{LinCat} = \mathrm{LMod}_{\mathrm{Pr}^L}(\mathrm{Sp}).$$

Note that Sp itself is a symmetric monoidal category.

[11] Think of homotopy pullback and homotopy pushout for topological spaces. For cofibers, think of homotopy cofibers (mapping cones). Also note that the mapping cone of $Y \to C_f$ where C_f is the mapping cone for $f : X \to Y$ is always the suspension ΣX.

3.4 Algebra and Modules

Recall that for a (symmetric) monoidal category $\mathscr{C}$, we defined

- $\mathrm{Alg}(\mathscr{C}) \subseteq \mathrm{Fun}(\Delta^{op}, \mathscr{C})$ and $\mathrm{CAlg}(\mathscr{C}) \subseteq \mathrm{Fun}(\mathrm{Fin}_*, \mathscr{C})$.

We can define $\mathrm{LMod}(\mathscr{C}) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}} \times [1], \mathscr{C})$ with

- $A_\bullet : \Delta^{\mathrm{op}} \cong \Delta^{\mathrm{op}} \times \{1\} \to \mathscr{C}$,
- $M_\bullet : \Delta^{\mathrm{op}} \cong \Delta^{\mathrm{op}} \times \{0\} \to \mathscr{C}$,
- $M_\bullet \to A_\bullet$, such that
  (1) $A_\bullet$ is an algebra object;
  (2) $M_n \xrightarrow{\cong} A_n \times M_0$.

Then we obtain $\mathrm{LMod}(\mathscr{C}) \to \mathrm{Alg}(\mathscr{C})$. For each $A \in \mathrm{Alg}(\mathscr{C})$, we define

$$\mathrm{Mod}_A(\mathscr{C}) := (\mathrm{Mod}(\mathscr{C})) \times_{\mathrm{Alg}_{\mathscr{C}}} \{A\}. \tag{3.48}$$

Fact: if $\mathscr{C}$ is symmetric monoidal and A is commutative, then $\mathrm{Mod}_A(\mathscr{C})$ has a natural symmetric monoidal category structure and

$$\mathrm{Mod}_A(\mathscr{C}) \to \mathscr{C} \tag{3.49}$$

is a lax symmetric monoidal functor.

- E_1-alg:= Alg(Sp), called associative ring spectra.
- E_∞-alg:= CAlg(Sp) commutative ring spectra.

Given $A \in \mathrm{Alg}(\mathrm{Sp})$ (resp. CAlg(Sp)).

Example 3.51. The unit of Sp is a commtative algebra, called the sphere spectra.

Every (ordinary/classical) commutative ring A gives a commutative algebra in Sp (sometimes denoted by HA and called Elenberg-Maclane spaces).

Fact: If A is ordinary, then $\mathrm{Mod}_{HA}(\mathrm{Sp}) \cong \mathrm{Mod}_A$.

$$\mathrm{LinCat}_A := \mathrm{Mod}_{\mathrm{Mod}_A}(\mathrm{LinCat}). \tag{3.50}$$

Remark 3.52. Informally speaking, $\mathscr{C} \in \mathrm{LinCat}_A$ should be thought as an A-linear stable presentable monoidal category.

Given $\mathscr{C} \in \mathrm{LinCat}_A$, and $x, y \in \mathscr{C}$, we have $\mathrm{Mod}_A \otimes \mathscr{C} \to \mathscr{C}$

$$\mathrm{Maps}(M, \underline{\mathrm{Hom}}(x, y)) = \mathrm{Maps}(M \otimes x, y) \tag{3.51}$$

and

$$\mathrm{Maps}_{\mathscr{C}}(x, y) \cong \mathrm{DK}(\tau^{\leq 0}\underline{\mathrm{Hom}}(x, y)). \tag{3.52}$$

3.5 Derived Algebraic Geometry

3.5.1 Derived Rings and Modules

Fix Λ an ordinary commutative algebra. We have $\mathrm{CAlg}_\Lambda^\heartsuit$ ordinary commutative Λ-algebra, with compact projective objects being polynomial rings of finitely many variables. Then we define the category of simplicial rings (or called derived rings)

$$\mathrm{CAlg}_\Lambda := \mathrm{Ani}(\mathrm{CAlg}_\Lambda^\heartsuit).$$

Fact: There exists a functor $\mathrm{CAlg}_\Lambda \to$ connective E_∞-$\mathrm{Rings}_{/H\Lambda}$, which preserves colimits and limits. It is an equivalence when $\Lambda \supseteq \mathbb{Q}$. In general, it is not even fully-faithful (it doesn't preserve free objects)[12].

For $A \in \mathrm{CAlg}_\Lambda$, we can define $\mathrm{Mod}_A := \mathrm{Mod}_{A^\circ}$.

[12]The problem arises because in positive characteristic taking S_n-invariants or coinvariants is not an exact functor.

Remark 3.53. Define $\mathrm{Mod}^{\heartsuit} := \{(A, M) | A \in \mathrm{CAlg}^{\heartsuit}, M \text{ left } A\text{-module}\}$, and then $\mathrm{Ani}(\mathrm{Mod}^{\heartsuit}) \cong \mathrm{Mod}$. Moreover, $\mathrm{Ani}(\mathrm{Mod}^{\heartsuit}) \times_{\mathrm{CAlg}_{\Lambda}} \{A\}$ is $\mathrm{Ani}(\mathrm{Mod}_A\heartsuit)$, which is a full subcategory of Mod_A. Then we have

$$\begin{tikzcd} h(\mathrm{Ani}(\mathrm{Mod}_A^{\heartsuit})) \arrow[r, hook] \arrow[d, "\cong"] & h\mathrm{Mod}_A \arrow[d, "\cong"] \\ D(A)^{\leq 0} \arrow[r, hook] & D(A) \end{tikzcd} \tag{3.53}$$

3.5.2 Schemes and Stacks

Definition 3.54. A prestack over Λ is an accessible functor (a.k.a. commutes with κ-filtered colimits for some cardinal κ)

$$\mathscr{F} : \mathrm{CAlg}_{\Lambda} \to \mathrm{Spc}.$$

Definition 3.55. We define $\mathrm{AffSch}/_{\Lambda} := \mathrm{CAlg}_{\Lambda}^{\mathrm{op}} \hookrightarrow \mathrm{Fun}^{\mathrm{acc}}(\mathrm{CAlg}_{\Lambda}, \mathrm{Spc})$.

Definition 3.56. Let $\mathscr{F}$ be a prestack over Λ, define

$$\mathrm{QCoh}(\mathscr{F}) := \varprojlim_{\mathrm{Spec}\, A \to \mathscr{F}} \mathrm{Mod}_A \tag{3.54}$$

in $\mathrm{LinCat}_{\Lambda}$.

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Recall that a prestack over Λ is an accessible functor $\mathscr{F} : \mathrm{CAlg}_{\Lambda} \to \mathrm{Spc}$. Note that

$$(\mathrm{CAlg}_{\Lambda})^{\mathrm{op}} \xrightarrow{\text{full}} \mathrm{PreStk}_{\Lambda} \xrightarrow{\text{full}} \mathrm{Fun}(\mathrm{CAlg}_{\Lambda}, \mathrm{Spc}).$$

We define QCoh to be the right Kan extension of Mod

$$\begin{tikzcd} \mathrm{PreStk}_{\Lambda}^{\mathrm{op}} \arrow[r, "\mathrm{QCoh}"] & \mathrm{CAlg}(\mathrm{LinCat}_{\Lambda}) \\ & \mathrm{CAlg}_{\Lambda} \arrow[ul] \arrow[u, "\mathrm{Mod}"'] \end{tikzcd} \tag{3.55}$$

Recall that given functors

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d] & \mathscr{E} \\ \mathscr{D} \arrow[ur, dashed, "\mathrm{Kan}_R F"'] & \end{tikzcd} \tag{3.56}$$

We say $\mathrm{Kan}_R F$ is a right Kan extension of F if

$$\mathrm{Maps}(G, \mathrm{Kan}_R F) \xrightarrow{\cong} \mathrm{Maps}(\mathrm{Res}(G), F), \tag{3.57}$$

where $\mathrm{Res} : \mathrm{Fun}(\mathscr{D}, \mathscr{E}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{E})$ is the restriction functor.
Similarly, we can define left Kan extension.

- Assuming limit exists, $\mathrm{Kan}_R F(d) = \varprojlim_{c \in \mathscr{D}_{d/} \times_{\mathscr{D}} \mathscr{C}} F(c)$.
- If $\mathscr{C} \hookrightarrow \mathscr{D}$ is fully faithful, then the canonical map $\mathrm{Res}\mathrm{Kan}_R F \Rightarrow F$ is an isomorphism.

The previous formula

$$\mathrm{QCoh}(\mathscr{X}) = \varprojlim_{\mathrm{Spec}\, A \to \mathscr{X}} \mathrm{Mod}_A \tag{3.58}$$

is just deduced from this general formalism of Kan extensions.

Remark 3.57. Note that Mod_A is a symmetric monoidal category, and hence $\mathrm{Mod}_A \to \mathrm{CAlg}(\mathrm{LinCat}_\Lambda) \to \mathrm{LinCat}_\Lambda$ and the last forgetful functor commutes with limits, we see $\mathrm{QCoh}(\mathscr{F})$ is actually a Λ-linear symmetric monoidal category.

Definition 3.58. $A \to B$ in CAlg_Λ is called flat if

(1) $\pi_0(A) \to \pi_0(B)$ is flat.

(2) $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \xrightarrow{\cong} \pi_i(B)$.

Definition 3.59. We say $A \to B$ is called Zariski (étale, smooth, ...) if $A \to B$ is flat, and $\pi_0(A) \to \pi_0(B)$ is a Zariski (étale, smooth, ...) localization.

Remark 3.60. $A \to B$ is flat, if and only if $\forall M$ an A-module in degree 0, $M \otimes_A B$ is in degree 0.

Example 3.61. Given $f \in \pi_0(A) \Leftrightarrow f \in \pi_0(\mathrm{Maps}(\Lambda[x], A))$,

$$A_f := A \otimes_{\Lambda[x]} \Lambda[x, x^{-1}]$$

is a Zariski localization.

Let τ = Zariski or étale.

Definition 3.62. A τ-stack is a prestack $\mathscr{X} : \mathrm{CAlg}_\Lambda \to \mathrm{Spc}$ satisfying

(1) $\mathscr{X}(\prod_I A_i) = \prod_I \mathscr{X}(A_i)$ for any I finite.

(2) For $A \to B$ a τ-covering (i.e. $A \to B$ is a τ-map and $\mathrm{Spec}(\pi_0(B)) \to \mathrm{Spec}(\pi_0(A))$ is a surjection), we have

$$\mathscr{X}(A) \xrightarrow{\cong} \varprojlim(\mathscr{X}(B) \rightrightarrows \mathscr{X}(B \otimes_A B) \Rrightarrow \mathscr{X}(B \otimes_A B \otimes_A B) \Rrightarrow \cdots) \tag{3.59}$$

Example 3.63. $\mathrm{Spec}(A)$ and $\mathrm{Maps}(A, -)$ are stacks.

We define $|\cdot| : \mathrm{PreStk} \to \mathrm{Top}$ to be the left Kan extension

$$\begin{tikzcd} \mathrm{CAlg}_\Lambda^{\mathrm{op}} \arrow[d, hook] \arrow[dr, "A \mapsto |\mathrm{Spec}\, A|"] & \\ \mathrm{PreStk} \arrow[r, "|\cdot|", "\text{left Kan}"'] & \mathrm{Top} \end{tikzcd} \tag{3.60}$$

where Top is the ordinary 1-category of topological spaces (not Spc!). In particular,

$$|\mathscr{X}| = \underset{\mathrm{Spec}\, A \to \mathscr{X}}{\varinjlim\mathrm{colim}}\ |\mathrm{Spec}\, A|. \tag{3.61}$$

Then by adjoint functor theorem, $|\cdot| : \mathrm{PreStk} \to \mathrm{Top}$ admits a right adjoint $T \mapsto T^{\mathrm{PreStk}_\Lambda}$. Explicitly, by adjunction $\mathrm{Hom}(\mathrm{Spec}\, A, T^{\mathrm{PreStk}}) \cong \mathrm{Hom}(|\mathrm{Spec}\, \pi_0(A)|, T)$, we see that

$$T^{\mathrm{PreStk}}(A) = \{\text{Continuous maps } \mathrm{Spec}\, \pi_0(A) \to T\}. \tag{3.62}$$

Definition 3.64. Let $\mathscr{X}$ be a prestack and an open substack $\mathscr{Y} \subseteq \mathscr{X}$ is a subfunctor of χ of the form

$$\mathscr{X} \times_{|\mathscr{X}|^{\mathrm{PreStk}}} U^{\mathrm{PreStk}}$$

for some open subset $U \subseteq |\mathscr{X}|$.

More explicitly,

$$\mathscr{Y}(A) = \{\mathrm{Spec}\, A \to \mathscr{X} \,|\, |\mathrm{Spec}\, A| \to |\mathscr{X}| \text{ factors through } U\}. \tag{3.63}$$

Definition 3.65. A (derived) scheme over Λ is a prestack $\mathscr{X}$ satisfying

(i) $\mathscr{X}$ is a Zariski sheaf;

(ii) there exists $\{U_i \subseteq \mathscr{X}$ open substacks $\}$, such that

- $U_i \cong \mathrm{Spec}\, A_i$;
- $\cup |U_i| = |\mathscr{X}|$.

Example 3.66. Every quasi-compact separated classical scheme X gives a derived scheme. Write $X = \cup \operatorname{Spec} A_i$ and $U = \sqcup \operatorname{Spec} A_i$. Take colimit of

$$U \times_X U \rightrightarrows U \tag{3.64}$$

and then Zariski sheafify it, we recover X.

Given a classical prestack $\mathscr{X}$, we can define its derived stack $\mathscr{X}^{\mathrm{der}}$ to be left Kan extension

$$\begin{tikzcd} \mathrm{CAlg}_\Lambda^\heartsuit \arrow[r] \arrow[d, hook, "\mathrm{full}"] \arrow[dr, "\mathscr{X}"] & \mathrm{Grp} \arrow[d] \\ \mathrm{CAlg}_\Lambda \arrow[r, dotted, "\mathscr{X}^{\mathrm{der}}"'] & \mathrm{Spc} \end{tikzcd} \tag{3.65}$$

Remark 3.67. Question: does this construction send classical schemes to derived schemes?

Theorem 3.68 (Flat Descent)**.** *Let $f : X \to Y$ be a faithfully flat morphism of derived schemes, then*

$$\mathrm{QCoh}(Y) \xrightarrow{\cong} \varprojlim(\mathrm{QCoh}(X) \rightrightarrows \mathrm{QCoh}(X \times_Y X) \to \cdots).$$

Remark 3.69. $\cup_I \mathrm{Spec} A_i \cong \mathrm{Spec} \prod_I A_i$ requires I to be finite.

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3.6 Quasi-coherent Sheaves

3.6.1 Dualizability and Duality

Theorem 3.70. *Let X be a quasi-compact quasi-separated derived scheme over Λ. Then*

(1) $\mathrm{QCoh}(X)$ *is compactly generated.*

(2) $\mathrm{QCoh}(X)$ *is dualizable in* LinCat_Λ.

(3) $\mathrm{QCoh}(X)^\omega \cong \mathrm{Perf}(X) \cong$ *dualizable objects.*

(4) For any prestack $\mathscr{Y}$ over Λ

$$\mathrm{QCoh}(X) \otimes_\Lambda \mathrm{QCoh}(Y) \xrightarrow{\cong} \mathrm{QCoh}(X \times_\Lambda Y). \tag{3.66}$$

Remark 3.71. (2) is deduced from (1).

Definition 3.72. Let $\mathscr{X}$ be a prestack. We define $\mathrm{Perf}(\mathscr{X}) \subseteq \mathrm{QCoh}(\mathscr{X})$ consisting of $\mathscr{D} \in \mathrm{QCoh}(\mathscr{X})$ such that $f^*\mathscr{F} \in \mathrm{Mod}_A$ is perfect for every $f : \mathrm{Spec} A \to \mathscr{X}$.

Definition 3.73. $M \in \mathrm{Mod}_A$ is called perfect if M belongs to the smallest idempotent complete stable subcategory that contains A.

Now we begin our proof of Thm. 3.70.

Proposition 3.74. *Thm. 3.70 (1) and (2) holds when $X = \mathrm{Spec} A$ and $\mathrm{QCoh}(X) = \mathrm{Mod}_A$.*

Proof. Dualizable object in Mod_A is contained in $(\mathrm{Mod}_A)^\omega = \mathrm{Perf}_A$.

Lemma 3.75. *Let $\mathscr{C}$ be an idempotent complete symmetric monoidal stable category. Then the full subcategory of dualizable objects is stable under finite colimits and retracts.*

Proof. Direct sums of dulaizable objects is still dualizable (we can write down units and counits directly).
Projecting on factors justify retracts. For taking cones, $x \to y \to z$, consider $z^* \to y^* \to x^*$. □

Since A is obviously dualizable, by this lemma we see that $\mathrm{Perf}_A \subseteq$dualizable objects. □

Lemma 3.76. *Let $\mathscr{C} = \varprojlim_i \mathscr{C}_i$ in $\mathrm{CAlg}(\mathrm{LinCat}_\Lambda)$. Then an object $c \in \mathscr{C}$ is dualziable if and only if its image c_i is dualizable.*

Remark 3.77. By the lemma, we see that for any prestack $\mathscr{X}$, $\mathrm{Perf}(X)$ ≅dualizable objects.

Proof. The natural maps $\mathscr{C} \to \mathscr{C}_i$ are symmetric monoidal functors, which send dualizable objects to dualizable objects.

Definition 3.78. A commutative square

$$[1] \times [1] \to \mathrm{LinCat}_\Lambda$$

$$\begin{tikzcd} \mathscr{C}_{00} \arrow[r, "f'"] \arrow[d, "g'"] & \mathscr{C}_{01} \arrow[d, "g"] \\ \mathscr{C}_{10} \arrow[r, "f"] & \mathscr{C}_{11} \end{tikzcd} \tag{3.67}$$

is called right adjointable if both f, f' admit continuous right adjoint $f^R, (f')^R$ and the induced Beck-Chevalley map (or called base change map)

$$g' \circ (f')^R \to f^R \circ f \circ g' \circ (f')^R \cong f^R \circ g \circ f' \circ (f')^R \to f^R \circ g$$

is an isomorphism.

Let I be a small category. Let $\mathrm{Fun}^{LAd}(I, \mathrm{LinCat}_\Lambda) \subseteq \mathrm{Fun}(I, \mathrm{LinCat})$ be the 1-full subcategory consisting of $C : I \to \mathrm{LinCat}$ such that $C_i \to C_j$ admits continuous right adjoint and $\mathrm{Maps}(C, D)$ are right adjointable for any

$$\begin{tikzcd} C_i \arrow[r] \arrow[d] & C_j \arrow[d] \\ D_i \arrow[r] & D_j \end{tikzcd} \tag{3.68}$$

Similarly one can define $\mathrm{Fun}^{RAd}(I, \mathrm{LinCat}_\Lambda)$.

Proposition 3.79. $\mathrm{Fun}^{RAd}(I, \mathrm{LinCat}_\Lambda)$ *and* $\mathrm{Fun}^{LAd}(I, \mathrm{LinCat}_\Lambda)$ *are presentable and natural inclusion*

$$\mathrm{Fun}^{RAd}(I, \mathrm{LinCat}_\Lambda) \hookrightarrow \mathrm{Fun}(I, \mathrm{LinCat}_\Lambda) \tag{3.69}$$

$$\mathrm{Fun}^{LAd}(I, \mathrm{LinCat}_\Lambda) \hookrightarrow \mathrm{Fun}(I, \mathrm{LinCat}_\Lambda) \tag{3.70}$$

commutes with limit.

Now apply the proposition to

$$\begin{tikzcd} \mathrm{Mod}_A \arrow[r, "\otimes c_A"] \arrow[d, "f^*"] & \mathrm{Mod}_A \arrow[d, "f^*"] \\ \mathrm{QCoh}(X) \arrow[r, "\otimes c"] & \mathrm{QCoh}(X) \end{tikzcd} \tag{3.71}$$

we see that $\otimes c$ is left and right adjointable. The adjunction identities will give the dualizablility data. □

Proposition 3.80. *Let X be a quasi-compact quasi-separated scheme. Then $\mathscr{O}_X$:=the unit of* $\mathrm{QCoh}(X)$ *is compact.*

Proof. If $X = \mathrm{Spec}A$, it is known. If $X = U \cup V$ with U, V affine and $U \cap V$ affine (using the assumption that X is quasi-compact and quasi-separated to reduce the general situation to this specific case), $\mathrm{Hom}(\mathscr{O}_X, -) = \mathrm{fiber}(\mathrm{Hom}(\mathscr{O}_U, -) \oplus \mathrm{Hom}(\mathscr{O}_V, -) \to \mathrm{Hom}(\mathscr{O}_{U\cap V}, -))$ and use fiber=cofiber in stable ∞-categories. □

Remark 3.81. The proposition above implies that dualizable objects in $\mathrm{QCoh}(X)$ are compact.

Lemma 3.82. *Let $f : \mathrm{Spec}A \to X$ be a morphism with X quasi-compact quasi-separated (qcqs). Then $f_* := (f^*)^R$ is continuous.*

Remark 3.83. This lemma implies that f^* preserves compact objects.

Therefore, we proved Thm. 3.70 (3).

Theorem 3.84. *Let $\mathscr{C}$ be a compactly generated category, such that $\mathscr{C}$ is dualziable in* LinCat_Λ. *Then*

$$\mathrm{Ind}(\mathscr{C}^\omega)^{op} \cong \mathscr{C}^\vee \cong \mathscr{C}^* = \mathrm{Fun}^{L,\Lambda}(\mathscr{C}, \mathrm{Mod}_\Lambda). \tag{3.72}$$

Proof.

$$\begin{aligned}(\mathscr{C}^{\omega})^{op}\times\mathscr{C}&\to \mathrm{Mod}_{\Lambda},\\ (c,d)&\mapsto \underline{\mathrm{Hom}}(c,d).\end{aligned}\tag{3.73}$$

Consider $\mathrm{Ind}(\mathscr{C}^{\omega})^{op}\otimes_{\Lambda}\mathscr{C}\to \mathrm{Mod}_A$. □

Theorem above implies that QCoh(X) is dualizable for X quasi-compact quasi-separated.

3.6.2 Fourier-Mukai Transform

Proposition 3.85. *Let $\mathscr{X}$ be a prestack such that* $\mathrm{QCoh}(\mathscr{X})$ *is dualizable. Then for any $\mathscr{Y}$,*

$$\mathrm{QCoh}(\mathscr{X})\otimes\mathrm{QCoh}(\mathscr{Y})\to\mathrm{QCoh}(\mathscr{X}\times_{\Lambda}\mathscr{Y})\tag{3.74}$$

is an equivalence.

Proof.

$$\begin{aligned}\mathrm{QCoh}(\mathscr{X})\otimes\varprojlim_{\mathrm{Spec}A\to\mathscr{Y}}\mathrm{Mod}_A&\cong\varprojlim_{\mathrm{Spec}A\to\mathscr{Y}}\mathrm{QCoh}(\mathscr{X})\otimes\mathrm{Mod}_A\\ &\cong\varprojlim_{\mathrm{Spec}\,A\to\mathscr{Y},\mathrm{Spec}\,B\to\mathscr{X}}\mathrm{Mod}_B\otimes\mathrm{Mod}_A\\ &\cong\varprojlim_{\mathrm{Spec}\,A\to\mathscr{Y},\mathrm{Spec}\,B\to\mathscr{X}}\mathrm{Mod}_{A\otimes B}\\ &\cong\cong\varprojlim_{\mathrm{Spec}\,A\times\mathrm{Spec}\,B\to\mathscr{X}\times_{\Lambda}\mathscr{Y}}\mathrm{Mod}_{A\otimes B}\\ &\cong\mathrm{QCoh}(\mathscr{X}\times_{\Lambda}\mathscr{Y}).\end{aligned}\tag{3.75}$$

Corollary 3.86. *Let $F:\mathrm{QCoh}(X)\to\mathrm{QCoh}(Y)$ be a contiunous functor with X,Y quasi-compact quasi-separated. Suppose that there exists*

$$\mathbb{D}_X:\mathrm{QCoh}(X)^{\vee}\xrightarrow{\cong}\mathrm{QCoh}(X).\tag{3.76}$$

Define $F_{\mathscr{K}}=(pr_y)_*(\mathscr{K}\otimes pr_X^*(-))$. Then

$$\begin{aligned}\mathrm{Fun}^{Lin}(\mathrm{QCoh}(X),\mathrm{QCoh}(Y))&\cong\mathrm{QCoh}(X)^{\vee}\otimes\mathrm{QCoh}(Y)\\ &\xrightarrow[\mathbb{D}_X]{\cong}\mathrm{QCoh}(X)\otimes\mathrm{QCoh}(Y)\\ &\cong\mathrm{QCoh}(X\times Y),\end{aligned}\tag{3.77}$$

i.e. in this case any functor is given by a Fourier-Mukai transform (with a suitable kernel).

□

Let us investigate the self-duality condition. Note that $\mathrm{QCoh}(X)\cong\mathrm{Ind}(\mathrm{Perf}(X))$ and $\mathrm{QCoh}(X)^{\vee}\cong\mathrm{Ind}(\mathrm{Perf}(X)^{\mathrm{op}})$. Thus, we will get

$$\mathbb{D}_X:\mathrm{Perf}(X)^{\mathrm{op}}\xrightarrow{\cong}\mathrm{Perf}(X).\tag{3.78}$$

Consider $\mathrm{Perf}(X)\otimes\mathrm{Perf}(X)\to\mathrm{Mod}_{\Lambda}$ given by $\mathscr{F},\mathscr{G}\mapsto\mathrm{Hom}(\Delta_*\mathscr{O}_X,\mathscr{F}\boxtimes\mathscr{G})$.

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Theorem 3.87. *Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] \arrow[ur, phantom, "\lrcorner", very near end] & Y \end{tikzcd} \tag{3.79}$$

be a Cartesian of quasi-compact quasi-separated schemes. Then

(1) The natural Beck-Chevalley map

$$g^* \circ f_* \xrightarrow{\cong} (f')_* \circ (g')^* \tag{3.80}$$

is an isomorphism.

(2) $f_*(\mathscr{F}) \otimes \mathscr{G} \xrightarrow{\cong} f_*(\mathscr{F} \otimes f^*(\mathscr{G}))$ *for any* $\mathscr{F} \in \mathrm{QCoh}(X)$ *and* $\mathscr{G} \in \mathrm{QCoh}(Y)$.

Theorem 3.88. *Let* $\Phi : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ *be a* Λ*-linear continuous functor with* X *quasi-compact quasi-separated scheme over* Λ*. Then there exists* $K_\Phi \in \mathrm{QCoh}(X \times X)$*, such that*

$$\Phi \cong (p_2)_*(p_1^*(-) \otimes K_\Phi), \tag{3.81}$$

where $p_i : X \times_\Lambda X \to X$ *are projections.*

Let $\mathscr{C}$ be a dualizable Λ-linear category. Then $\mathscr{C}^\vee = \mathrm{Fun}^{L,\Lambda\text{-linear}}(\mathscr{C}, \mathrm{Mod}_\Lambda)$. We have evaluation funtor

$$e_{\mathscr{C}} : \mathscr{C}^\vee \otimes_\Lambda \mathscr{C} \to \mathrm{Mod}_\Lambda \tag{3.82}$$

and also the unit

$$u_{\mathscr{C}} : \mathrm{Mod}_\Lambda \to \mathscr{C} \otimes \mathscr{C}^\vee. \tag{3.83}$$

Note that we can view $u_{\mathscr{C}} \in \mathscr{C} \otimes \mathscr{C}^\vee$.

In general, given $F : \mathscr{C} \to \mathscr{D}$ continuous functor, we construct kernel $K_F \in \mathscr{C}^\vee \otimes \mathscr{D}$ by

$$K_F := (\mathrm{id}_{\mathscr{C}^\vee} \otimes F)(u_{\mathscr{C}}). \tag{3.84}$$

Conversely, given any $K_F \in \mathscr{C}^\vee \otimes \mathscr{D} = \mathrm{Fun}(\mathrm{Mod}_\Lambda, \mathscr{C}^\vee \otimes \mathscr{D})$, we recover the functor by

$$\mathscr{C} \xrightarrow{\cong} \mathscr{C} \otimes_\Lambda \mathrm{Mod}_\Lambda \xrightarrow{\mathrm{id}_{\mathscr{C}} \otimes K_F} \mathscr{C} \otimes \mathscr{C}^\vee \otimes \mathscr{D} \xrightarrow{e_{\mathscr{C}} \otimes \mathrm{id}_{\mathscr{D}}} \mathscr{D}. \tag{3.85}$$

Recall that if $\mathscr{C}$ is compactly generated, then $\mathscr{C}$ is dualizable, and $\mathscr{C}^\vee = \mathrm{Ind}(\mathscr{C}^\omega)^{\mathrm{op}}$. We have

$$(\mathscr{C}^\omega)^{\mathrm{op}} \otimes \mathscr{C}^\omega \to \mathrm{Mod} \tag{3.86}$$

and

$$e_{\mathscr{C}} : \mathrm{Ind}((\mathscr{C}^\omega)^{\mathrm{op}}) \otimes \mathscr{C} \to \mathrm{Mod}_\Lambda. \tag{3.87}$$

But $u_{\mathscr{C}}$ is hard to write down in general.

Take $\mathscr{C} = \mathrm{QCoh}(X)$ with X quasi-compact quasi-separated, then $\mathrm{QCoh}(X)^\omega = \mathrm{Perf}(X)$, and

$$\mathrm{Perf}(X)^{op} \xrightarrow{\cong} \mathrm{Perf}(X) \tag{3.88}$$

by $\mathscr{V} \to \mathscr{V}^\vee$. Therefore,

$$e : \mathrm{QCoh}(X)^\vee \otimes \mathrm{QCoh}(X) \to \mathrm{Mod}_\Lambda \tag{3.89}$$

corresponds to

$$\begin{aligned} e : \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) &\to \mathrm{Mod}_\Lambda \\ (\mathscr{F}, \mathscr{G}) &\mapsto \mathrm{Hom}(\mathscr{O}_X, \mathscr{F} \otimes \mathscr{G}) \end{aligned} \tag{3.90}$$

under $\mathrm{QCoh}(X) \cong \mathrm{QCoh}(X)^\vee$.

Lemma 3.89. *The unit* $u \in \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \cong \mathrm{QCoh}(X \times_\Lambda X)$ *is* $\Delta_* \mathscr{O}_X$.

Proof of Thm.,3.87. By the general process, Φ corresponds bijectively to $K_\Phi \in \mathrm{QCoh(X)} \otimes \mathrm{QCoh}(X)$, where $K_\Phi = (\mathrm{id} \otimes \Phi)(\Delta_*(\mathscr{O}_X))$.

$$\begin{aligned} \mathrm{QCoh}(X) &\to \mathrm{QCoh}(X \times X \times X) \to \mathrm{QCoh}(X) \\ \mathscr{F} &\mapsto F \boxtimes K_\Phi \mapsto (p_2)_* \Delta_{12}^* (\mathscr{F} \boxtimes K_\Phi). \end{aligned} \tag{3.91}$$

Using Δ_{12}^* is a symmeytric monoidal functor, we see that $(p_2)_* \Delta_{12}^* (\mathscr{F} \boxtimes K_\Phi) \cong (p_2)_* (p_1^* \mathscr{F} \otimes K_\Phi)$. □

Example 3.90. Suppose that $\mathscr{C}$ is dualizable. The Serre functor

$$S_\mathscr{C} : \mathscr{C} \to \mathscr{C} \tag{3.92}$$

is a continuous functor corresponding $e_\mathscr{C}^R(\Lambda) := K_{S_\mathscr{C}} \in \mathscr{C}^\vee \otimes \mathscr{C}$, which represents

$$\begin{aligned} e_\mathscr{C} : \mathscr{C}^\vee \otimes \mathscr{C} &\to \mathrm{Mod}, \\ c' \boxtimes c &\mapsto \mathrm{Hom}(e_\mathscr{C}(c' \boxtimes c), \Lambda). \end{aligned} \tag{3.93}$$

Note that $e_\mathscr{C}$ always admits a right adjoint (though its right adjoint might not be continuous).
Take $\mathscr{C} = \mathrm{QCoh}(X)$,

$$e : \mathrm{QCoh}(X \times_\Lambda X) \xrightarrow{\Delta^*} \mathrm{QCoh}(X) \xrightarrow{p_*} \mathrm{Mod}_\Lambda, \tag{3.94}$$

where $p : X \to \mathrm{pt}$. Let

$$\omega_X := (p_*)^R(\Lambda). \tag{3.95}$$

Then $K_{S_\mathscr{C}} = \Delta_X \otimes \omega_X$ and $S_\mathscr{C} = (-) \otimes \omega_X$.

Example 3.91. Suppose that $\mathscr{C}$ is compactly generated. Then

$$\mathrm{Hom}_\mathscr{C}(d, S_\mathscr{C}(c)) = \mathrm{Hom}_\mathscr{C}(c, d)^*, \tag{3.96}$$

for any $c \in \mathscr{C}^\omega, d \in \mathscr{C}$.

Now let $\mathscr{C}$ be dualizable, $\Phi : \mathscr{C} \to \mathscr{C}$, recall that we define $\mathrm{Tr}(\Phi, \mathscr{C})$ by

$$\mathrm{Mod}_\Lambda \xrightarrow{u_\mathscr{C}} \mathscr{C} \otimes \mathscr{C}^\vee \xrightarrow{\Phi \otimes \mathrm{id}_{\mathscr{C}^\vee}} \mathscr{C} \otimes \mathscr{C}^\vee \cong \mathscr{C}^\vee \otimes \mathscr{C} \xrightarrow{e_\mathscr{C}} \mathrm{Mod}_\Lambda. \tag{3.97}$$

We can view $\mathrm{Tr}(\Phi, \mathscr{C}) = e_\mathscr{C}(K_\Phi) \in \mathrm{Mod}_\Lambda$. This is sometimes called the Hochschild homology of $\mathscr{C}$.

Remark 3.92. We can also define the Hochschild cohomology of Φ as $\mathrm{Hom}_{\mathscr{C} \otimes \mathscr{C}^\vee}(u_\mathscr{C}, K_\Phi) \in \mathscr{C} \otimes \mathscr{C}^\vee$.
In particular, take $\Phi = \mathrm{id}_\mathscr{C}$, $Z(\mathscr{C}) = \mathrm{End}_{\mathscr{C}^\vee \otimes \mathscr{C}}(u_\mathscr{C}) = \mathrm{End}_{\mathrm{Fun}^{L,\Lambda}(\mathscr{C}, \mathscr{C})}(\mathrm{id}_\mathscr{C})$ is called the center of $\mathscr{C}$.

$$e_\mathscr{C}(u_\mathscr{C}) = \mathrm{Hom}(\mathrm{Tr}(\mathscr{C}, \mathrm{id}_\mathscr{C}), \Lambda) = \mathrm{Hom}(u_\mathscr{C}, K_{S_\mathscr{C}}), \tag{3.98}$$

where the right hand side is the Hochschild cohomology of the Serre functor. In particular, if $\mathscr{C}$ is 0-Calabi-Yau (i.e. $S_\mathscr{C} = \mathrm{id}_\mathscr{C}$), then

$$\mathrm{Tr}(\mathscr{C}, \mathrm{id}_\mathscr{C})^* = Z(\mathscr{C}). \tag{3.99}$$

Example 3.93. Take $\mathscr{C} = \mathrm{QCoh}(X)$ with X quasi-compact and quasi-separated. Take $\Phi = f^*$ for $f : C \to X$.

$$\begin{tikzcd} & X^f \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \mathrm{pt} \\ \mathrm{pt} & X \arrow[l] \arrow[r, "\Gamma_f"] \arrow[d, "\Delta"] & X \times X & \\ & X \times X \arrow[ur, "f\times \mathrm{id}"'] & & \end{tikzcd} \tag{3.100}$$

For example, take $f = \mathrm{id}_X$, then

$$X^{\mathrm{id}} = X \times_{X\times X} X.$$

If $X = \mathrm{Spec}A$, then $X \times_{X\times X} X$ corresponds to

$$A \otimes^L_{A\otimes A} A = \bigoplus_i \Omega^i_A[i],$$

when A is smooth. In general , if X is a smooth variety over k a field of characteristic 0,

$$H^i \mathscr{O}_{X\times_{X\times X}X} \cong \Omega^{-i}_X. \tag{3.101}$$

Then

$$R\Gamma(X^{\mathrm{id}}, \mathscr{O}_X) = \oplus_i H^i(X, \Omega^i_X).$$

Example 3.94. Let A be an associative ring (E_1-algebra) and $\mathscr{C} = \mathrm{LMod}_A$. Then

$$\mathscr{C}^\vee \cong \mathrm{LMod}_{A^{\mathrm{rev}}} \tag{3.102}$$

and $u_{\mathscr{C}} \in \mathscr{C} \otimes \mathscr{C}^\vee \cong \mathrm{Mod}_{A\otimes A^{\mathrm{rev}}}$, given by

$$\begin{aligned} \mathscr{C}^\vee \otimes \mathscr{C} &\to \mathrm{Mod}_A, \\ M &\mapsto M \otimes_{A\otimes A^{\mathrm{rev}}} A. \end{aligned} \tag{3.103}$$

Then

$$\mathrm{Tr}(\mathscr{C}, \mathrm{id}_{\mathscr{C}}) = A \otimes_{A\otimes A^{\mathrm{rev}}} A = \mathrm{HH}_\bullet(A). \tag{3.104}$$

Remark 3.95. The functor

$$\begin{aligned} \mathrm{Morita}(\mathrm{Vect}_\Lambda) &\to \mathrm{LinCat}_\Lambda, \\ A &\mapsto \mathrm{LMod}_A, \end{aligned} \tag{3.105}$$

which is fully faithful and is a symmetric monoidal functor. Therefore, we can calculate the trace in any of the two categories. We have already seen that Hochschild homologies are traces in Morita.

4 Trace Formula

4.1 Smoothness and Properness

Suppose that $\mathscr{C}$ is dualizable. We have unit

$$u_{\mathscr{C}} : \mathrm{Mod}_\Lambda \to \mathscr{C} \otimes \mathscr{C}^\vee \tag{4.1}$$

and evaluzation map

$$\mathscr{C}^\vee \otimes \mathscr{C} \to \mathrm{Mod}_\Lambda. \tag{4.2}$$

Definition 4.1. We say $\mathscr{C}$ is smooth if $u_{\mathscr{C}}$ admits a continuous right adjoint (which is equivalent to say $u_{\mathscr{C}} \in \mathscr{C}\otimes\mathscr{C}^\vee$ is compact).

Definition 4.2. We say $\mathscr{C}$ is proper if $e_{\mathscr{C}}$ admits a continuous right adjoint.

Remark 4.3. If $\mathscr{C}$ is compactly generated, then $\mathscr{C}$ is proper if and only if $\mathrm{Hom}(c, d) \in \mathrm{Perf}_\Lambda$ for any $c, d \in \mathscr{C}^\omega$.

4.2 Constuctions

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Let $A \in \mathrm{CAlg}(\mathrm{LinCat})$ (most important case: $A = \mathrm{Mod}_\Lambda$).

Let $\mathrm{LinCat} := \mathrm{LMod}_A(\mathrm{LinCat})$.

Note that LinCat_A is a closed symmetric monoidal category with internal hom given by $\mathrm{Fun}^L_A(M, N)$ A-linear continuous functors.

$$\begin{tikzcd} A \otimes M \arrow[r, "\mathrm{Id}_A \otimes F"] \arrow[d, "\mathrm{act}_M"] & A \otimes N \arrow[d, "\mathrm{act}_N"] \\ M \arrow[r, "F"] & N \end{tikzcd} \tag{4.3}$$

Suppose that F^R is continuous . Then we have

$$\mathrm{act}_M \circ (\mathrm{Id}_A \otimes F^R) \to F^R \circ \mathrm{act}_N. \tag{4.4}$$

If the above base change map is an equivalence, then F^R is A-linear.

Definition 4.4. Let C be a dualizable object in LinCat_A. We have

$$u_{C/A} : A \to C \otimes_A C^\vee, \tag{4.5}$$

$$e_{C/A} : C^\vee \otimes_A C \to A. \tag{4.6}$$

We say $\mathscr{C}$ is smooth over A, if $u_{C/A}$ admits an A-linear right adjoint. We say C is proper over A if $e_{C/A}$ admits an A-linear right adjoint. We say C is 2-dualizable over A if C is smooth and proper over A.

Example 4.5. $A = \mathrm{Mod}_\Lambda$, X/Λ smooth and proper, then $C = \mathrm{QCoh}(X)$ is 2-dualizable.

Suppose that $F : M \to N$ admits an A-linear right adjoint F^R. Then we have

$$F^\circ := (F^R)^\vee : M^\vee \to N^\vee. \tag{4.7}$$

Lemma 4.6. *$F^\circ := (F^R)^\vee$ admits an A-linear right adjoint $F^\vee$.*

Now $M = N = C$ is 2-dualizable, $\mathrm{Tr}(C, F) \in A$ is dualizable,

$$A \xrightarrow{u_C} C \otimes_A C^\vee \xrightarrow{F \otimes \mathrm{Id}} C \otimes_A C^\vee \cong C^\vee \otimes_A C \xrightarrow{e_C} A. \tag{4.8}$$

Example 4.7. If $A = \mathrm{Mod}_\Lambda$, then $\mathrm{Tr}(C, F)$ is a perfect complex.

Lemma 4.8. *Let $F : C \to D$ be an A-linear functor between dualizable A-mod, with A-linear right adjoint F^R, then*

$$(F \otimes F^\circ) \circ u_C \to u_D, \tag{4.9}$$

$$e_C \Rightarrow e_D \circ (F \otimes F^\circ). \tag{4.10}$$

Let $\phi_C : C \to C$, $\phi_D : D \to D$ be right adjointable A-linear functors.

Let $\eta : F \circ \phi_C \Rightarrow \phi_D \circ F$.

$$\begin{tikzcd} C \arrow[r, "\phi_C"'] \arrow[d, "F"] & C \arrow[d, "F"] \arrow[dl, Rightarrow] \\ D \arrow[r, "\phi_D"] & D \end{tikzcd} \tag{4.11}$$

Then we have

$$\mathrm{Tr}(F, \eta) : \mathrm{Tr}(C, \phi_C) = e_C((\phi_C \otimes \mathrm{Id}_{C^\vee})(u_C)) \to e_D((F\phi_C \otimes F\circ)(u_C)) \xrightarrow{\eta} e_D((\phi_D \circ F \otimes F\circ)(u_C)) \to e_D(\phi_D \otimes \mathrm{Id}_{D^\vee}(u_D)) = \mathrm{Tr}(D, \phi_D). \tag{4.12}$$

Diagram is as following:

$$\begin{tikzcd} A \arrow[r, "u_C"] \arrow[d] & C \otimes_A C^\vee \arrow[r, "\phi_C \otimes \mathrm{id}_{C^\vee}"] \arrow[d, "F \otimes F^\circ"] \arrow[dl, Rightarrow] & C \otimes_A C^\vee \arrow[r, "\cong"] \arrow[d, "F \otimes F^\circ"] \arrow[dl, Rightarrow, "\eta"'] & C^\vee \otimes_A C \arrow[r, "e_C"] \arrow[d] & A \arrow[d] \arrow[dl, Rightarrow] \\ A \arrow[r, "u_D"] & D \otimes_A D^\vee \arrow[r, "\phi_D \otimes \mathrm{id}_{D^\vee}"] & D \otimes_A D^\vee \arrow[r, "\cong"] & D^\vee \otimes_A D \arrow[r, "e_D"] & A \end{tikzcd} \tag{4.13}$$

4.3 Trace Formula

Suppose that C is a 2-dualizable over A and $\phi_1, \phi_2 : C \to C$ right adjointable A-linear endomorphism. Let

$$\eta : \phi_1 \circ \phi_2 \xrightarrow{\cong} \phi_2 \circ \phi_1 \tag{4.14}$$

Then in $\mathrm{End}1_A$ we have

$$\mathrm{Tr}(\mathrm{Tr}(C/A, \phi_1), \mathrm{Tr}(\phi_2, \eta^{-1})) \cong \mathrm{Tr}(\mathrm{Tr}(C/A, \phi_2), \mathrm{Tr}(\phi_1, \eta)). \tag{4.15}$$

Example 4.9. $A = \mathrm{Mod}_{\mathbb{C}}$ and $C = \mathrm{QCoh}(X)$. Let X be smooth proper over $\mathbb{C}$, $\phi_1 = f^*$, for $f : X \to X$, $\phi_2 = (-)\otimes\mathscr{E}$ where $\mathscr{E}$ is a vector bundle, $\eta : f^{\mathscr{E}} \xrightarrow{\cong} \mathscr{E}$. Then we will recover Bott-Atiyah fixed point theorem.

Definition 4.10. Let C be A-mod, then $C \cong \mathrm{Fun}_A^L(A, C)$ under $c \mapsto (F_c : a \mapsto a \otimes c)$. We say c is A-compact, if F_c admits A linear right adjoint. We say c is A-admissible if F_c admits A-linear left adjoint.

Example 4.11. $A = \mathrm{Mod}_\Lambda$ and C is Λ-linear. Suppose that C is compactly generated and $c \in C$. The map $F_c : M \mapsto M \otimes c$ admits a right adjoint $F_c^R = \mathrm{Hom}(c, -)$, since

$$\mathrm{Hom}(M, \mathrm{Hom}(c, d)) = \mathrm{Hom}(M \otimes c, d). \tag{4.16}$$

Then $F_c^R = \mathrm{Hom}(c, -)$ is continuous if and only if c is compact.

Example 4.12. Note that $\mathrm{Hom}(F_c^L(d), \Lambda) = \mathrm{Hom}(d, c)$. Then d compact implies $F_c^L(d)$ is a perfect Λ-module. Therefore, F_c^L exists if and only if $\mathrm{Hom}_C(d, c)$ is perfect for every $d \in C^\omega$. Consider $C = \mathrm{Rep}(G)$, where G is a p-adic representation. Then we recover the admissibily of p-adic representations.

Let C be dualizable in LinCat_A. Suppose that $c \in C$ is A-compact with $F_c : A \to C$. Let $\Phi : C \to C$ is A-linear. Suppose that we have $\eta : c \mapsto \Phi_c$ ($\eta : F_c \circ \mathrm{Id}_A \to \Phi \circ F_c$).

We have

$$\mathrm{Tr}(F_c, \eta) : 1_A \to \mathrm{Tr}(C, \Phi_C). \tag{4.17}$$

Example 4.13. $A = \mathrm{Mod}_\Lambda$, $\mathrm{ch}(c, \eta) := \mathrm{Tr}(F_c, \eta) \in H^0\mathrm{Tr}(C, \Phi_C)$ is called twisted Chern character. If $\Phi_C = \mathrm{Id}_C$, and $\eta = \mathrm{Id}_c$, then $\mathrm{ch}(c) \in H^0\mathrm{Tr}(C, \mathrm{Id}_C)$ is called Chern character.

Example 4.14. If $C = \mathrm{QCoh}(X)$ with X smooth, $\mathscr{F} \in \mathrm{QCoh}(X)$ is A-compact if and only if $\mathscr{F} \in \mathrm{Perf}(X)$. Then

$$\mathrm{Tr}(C, \mathrm{Id}_C) = \oplus H^\bullet(X, \Omega_X), \tag{4.18}$$

and then

$$H^0(\mathrm{Tr}(C, \mathrm{Id}_C)) = \oplus H^i(X, \Omega_X^i). \tag{4.19}$$

Theorem 4.15 (Riemann-Roch Theorem)**.** $A = \mathrm{Mod}_\Lambda$ *and C compactly generated, A-linear.*

(1) $ch : K_0(C^\omega) \to H^0Tr(C, Id_C)$.

(2) Let $F : C \to D$ be right adjointable. Then

$$\begin{array}{ccc} K_0(C^\omega) & \xrightarrow{ch} & H^0Tr(C, \mathrm{Id}_C) \\ \downarrow & & \downarrow \\ K_0(D^\omega) & \xrightarrow{ch} & H^0Tr(D, \mathrm{Id}_D) \end{array} \tag{4.20}$$

Definition 4.16. Let $A \in \mathrm{Alg}(\mathrm{LinCat})$. A sequence of A-modules $M \xrightarrow{F} C \xrightarrow{G} N$ is called a localization sequence if

(1) F^R, G^R exist as A-linear functors and $\mathrm{Id}_M \xrightarrow{\cong} F^R \circ F$ and $G \circ G^R \xrightarrow{\cong} \mathrm{Id}_N$.

(2) $G \circ F = 0$, and for every $c \in C$,

$$F \circ F^R(c) \to c \to G^R \circ G(c) \tag{4.21}$$

is a cofiber sequence in C.

If in addition G^R is also right adjointable, then $(M, G^R(N))$ forms a semi orthogonal decomposition.

Suppose that $A \in \mathrm{CAlg}(\mathrm{LinCat})$ and M, N, C are dualizable. $u_M \to M \otimes_A M^\vee$.

Proposition 4.17. *The following*

$$(F \otimes F^\circ)u_M \to u_C \to (G^R \otimes (G^\circ)^R)u_N \tag{4.22}$$

is a cofiber sequence in $C \otimes_A C^\vee$.

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Let $A \in \mathrm{Alg}(\mathrm{LinCat})$.

Definition 4.18. A sequence $M \xrightarrow{F} C \xrightarrow{G} N$ of A-;inear categories is called a localization sequence if

(1) both F and G admit A-linear right adjoints F^R and G^R respectively, and

$$\mathrm{Id}_M \xrightarrow{\cong} F^R \circ F, \tag{4.23}$$

$$G \circ G^R \xrightarrow{\cong} \mathrm{Id}_N, \tag{4.24}$$

(2) $G \circ F = 0$ and for any $c \in C$,

$$F(F^R(c)) \to c \to G^R(G(c)) \tag{4.25}$$

is a fiber sequence in C.

If in addition G^R admits an A-linear right adjoint, then we say $(F(M), G^R(N))$ forms a semi-orthogonal decomposition of C.

Remark 4.19. If $A = \mathrm{Mod}_\Lambda$ and M, N, C compact, then the existence of F^R, G^R is equivalent to the requirment that F^R, G^R preserve compact objects, and the existence of $(G^R)^R$ is to say G^R preserves compact objects. Then we get

$$\begin{tikzcd} M^\omega \arrow[r, "F"] & C^\omega \arrow[r, "G"] & N^\omega \arrow[l, bend left, "G^R"] \end{tikzcd} \tag{4.26}$$

So localization sequence

$$K_0(M^\omega) \to K_0(C^\omega) \to K_0(N^\omega). \tag{4.27}$$

Note that $F(m)$ compact implies m compact, since

$$\begin{aligned} \mathrm{Hom}(m, \varinjlim_i m_i) &\cong \mathrm{Hom}(F(m), F(\varinjlim_i m_i)) \\ &\cong \mathrm{Hom}(F(m), \varinjlim_i F(m_i)) \\ &\cong \varinjlim_i \mathrm{Hom}(F(m), F(m_i)) \\ &\cong \varinjlim_i \mathrm{Hom}(m, m_i). \end{aligned} \tag{4.28}$$

Therefore, $K_0(C^\omega) = K_0(M^\omega) \oplus K_0(N^\omega)$.

Proposition 4.20. *Let $M \xrightarrow{F} C \xrightarrow{G} N$ be a localization sequence. Then*

$$(F \otimes F^{\circ})u_M \to u_c \to (G^R \otimes (G^{\circ})^R)u_N \tag{4.29}$$

is a fiber sequence in $C \otimes_A C^{\vee}$.

Proof. Note that we have fiber sequences

$$(F \otimes \mathrm{id})(F^R \otimes \mathrm{id})u_C \to u_C \to (G^R \otimes \mathrm{id})(G \otimes \mathrm{id})u_C, \tag{4.30}$$

and

$$(\mathrm{id} \otimes F^{\circ})(\mathrm{id} \otimes (F^{\circ})^R)(G \otimes \mathrm{id})u_C \to (G \otimes \mathrm{id})u_C \to (\mathrm{id} \otimes (G^{\circ})^R)(\mathrm{id} \otimes G^{\circ})(G \otimes \mathrm{id})u_C, \tag{4.31}$$

and

$$(\mathrm{id} \otimes F^{\circ})(\mathrm{id} \otimes (F^{\circ})^R)(F^R \otimes \mathrm{id})u_C \to (F^R \otimes \mathrm{id})u_C \to (\mathrm{id} \otimes (G^{\circ})^R)(\mathrm{id} \otimes G^{\circ})(F^R \otimes \mathrm{id})u_C. \tag{4.32}$$

We claim that

$$(\mathrm{id} \otimes F^{\circ})(\mathrm{id} \otimes (F^{\circ})^R)(G \otimes \mathrm{id})u_C = 0 \tag{4.33}$$

and

$$(\mathrm{id} \otimes (G^{\circ})^R)(\mathrm{id} \otimes G^{\circ})(F^R \otimes \mathrm{id})u_C = 0. \tag{4.34}$$

To be added here. □

Proposition 4.21. *Let $M \xrightarrow{F} C \xrightarrow{G} N$ be a localization sequence. Suppose that there exists an A-linear functor*

$$\phi_C : C \to C. \tag{4.35}$$

Let $\phi_M = F^R \circ \phi_C \circ F$, $\phi_N = G \circ \phi_C \circ G^R$, $\eta : F \circ \phi_M \Rightarrow \phi_C \circ F$, $\delta : G \circ \phi_C \Rightarrow \phi_N \circ G$. Then

$$Tr(M, \phi_M) \xrightarrow{Tr(F,\eta)} Tr(CC, \phi_C) \xrightarrow{Tr(G,\delta)} Tr(N, \phi_M) \tag{4.36}$$

is a fiber sequence in A. In addition, if $((F(M)), G^R(N))$ forms a semi-orthogonal decomposition, then

$$Tr(C, \phi_C) \cong Tr(M, \phi_M) \oplus Tr(N, \phi_N). \tag{4.37}$$

Proof. Let $e_C : C \otimes_A C^{\vee} \to A$ be the evaluation map. Note that we have natural transfroms

$$e_M \Rightarrow e_C \circ (F \otimes F^{\circ}), \tag{4.38}$$

$$(F \otimes F^{\circ}) \circ u_M \Rightarrow u_C, \tag{4.39}$$

$$e_C \Rightarrow e_N \circ (G \otimes G^{\circ}), \tag{4.40}$$

and

$$(G \otimes G^{\circ})u_C \Rightarrow u_N. \tag{4.41}$$

By fully-faithfulness, $e_M \Rightarrow e_C \circ (F \otimes F^{\circ})$ is an equivalence. Note that we have

$$\begin{tikzcd}
e_C(\phi_C \otimes 1)(F \otimes F^{\circ})u_M \arrow[r] & \mathrm{Tr}(C, \phi_C) \arrow[r] & e_C(\phi_C \otimes 1)(G^R \otimes (G^{\circ})^R)u_N \arrow[d, "="] \\
e_C(\phi_C \circ F \otimes F^{\circ})u_M \arrow[u, "="] & & e_C(\phi_C \circ G^R \otimes G)u_N \arrow[d] \\
e_C(F \circ \phi_M \otimes F^{\circ})u_M \arrow[u] & & e_C(G^R \circ \phi_N \otimes G)u_N \arrow[d] \\
e_M(\phi_M \otimes \mathrm{Id})u_M \arrow[u] & & e_N(\phi_N \otimes \mathrm{Id})u_N \arrow[d] \\
\mathrm{Tr}(M, \phi_M) \arrow[u, "="] & & \mathrm{Tr}(N, \phi_N)
\end{tikzcd} \tag{4.42}$$

To be added. □

Let $A = \mathrm{Mod}_\Lambda$. Consider

$$\mathrm{ch} : C^\omega \to \mathrm{Tr}(C, \mathrm{id}_C). \tag{4.43}$$

Proposition 4.22. *This map ch induces*

$$K_0(C^\omega) \to H^0(Tr(C, id_C)) \in \mathrm{Mod}_\Lambda. \tag{4.44}$$

Proof. Given $c' \to c \to c''$, we want

$$\mathrm{ch}(c) = \mathrm{ch}(c') + \mathrm{ch}(c''). \tag{4.45}$$

Let $S_2C \subseteq \mathrm{Fun}(\wedge^2, C)$ be the category of fiver sequences.

Consider

$$\begin{aligned} F : C &\to S_2C \\ c &\mapsto (c \xrightarrow{\mathrm{id}} c \to 0) \end{aligned} \tag{4.46}$$

which admits a right adjoint

$$F^R(c' \to c \to c'') = c'.$$

Also define

$$\begin{aligned} G : S_2C &\to C \\ (c' \to c \to c'') &\mapsto c'' \end{aligned} \tag{4.47}$$

which admits a right adjoint

$$G^R(c) = (0 \to c \xrightarrow{\mathrm{id}} c \to c). \tag{4.48}$$

Claim: $C \to S_2C \to C$ forms a semi-orthogonal decomposition.

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\mathrm{ch}"] & S_2C \arrow[r, "G"'] \arrow[d, "\mathrm{ch}"] & C \arrow[l, bend right] \arrow[d, "\mathrm{ch}"] \\ \mathrm{Tr}(C) \arrow[r] & \mathrm{Tr}(S_2C) \arrow[r] & \mathrm{Tr}(C) \arrow[l, bend left] \end{tikzcd} \tag{4.49}$$

Consider

$$\begin{tikzcd} & S_2C \arrow[r, "(c' \to c \to c'') \mapsto c"] \arrow[d, "\mathrm{ch}"] & C \arrow[d] \\ \mathrm{Tr}(C) \oplus \mathrm{Tr}(C) \arrow[r, "\cong"] & \mathrm{Tr}(S_2C) \arrow[r, "\mathrm{add}"] & \mathrm{Tr}(C) \end{tikzcd} \tag{4.50}$$

□

Theorem 4.23 (Grothendieck-Riemann-Roch)**.** *C, D are Λ-linear and $F : C \to D$ admits a continuous right adjoint. THen*

$$\begin{tikzcd} K_0(C^\omega) \arrow[r, "ch"] \arrow[d] & H^0Tr(C) \arrow[d] \\ K_0(D^\omega) \arrow[r, "ch"] & H^0Tr(D) \end{tikzcd} \tag{4.51}$$

Example 4.24. Take $C = \mathrm{QCoh}(X)$ where X is smooth over $\mathbb{C}$. Then $H^0\mathrm{Tr}(C) = \oplus H^i(X, \Omega^i_X)$ by KRH. Take $F = f_*$ for some proper morphism $X \to Y$, we recover Grothendieck-Riemann-Roch in algebraic geometry.

Example 4.25. Take $C = \mathrm{Rep}^{\mathrm{sm}}(G)$, where G is a p-adic group. Then $K_0(C^\omega) \to H^0(\mathscr{H} \otimes_{\mathscr{H}\otimes\mathscr{H}^{\mathrm{rev}}} \mathscr{H}) = \mathscr{H}/[\mathscr{H}, \mathscr{H}] = \overline{\mathscr{H}}$, where $\mathscr{H} = C_c^\infty(G)$ is the Hecke algebra. We can apply to the case where F is either parabolic induction or restriction.

MOre on A-admissible algebras. Suppose that C is A-linear. Recall that $c \in C$ is A-compact, if

$$\begin{aligned} F_c : A &\to C \\ a &\mapsto a \otimes c \end{aligned} \tag{4.52}$$

admits A-linear right adjoint and ee say c is A-admissible if F_c admits A-linear left adjoint.

Now suppse that C is dualizable. Let c be A-admissible. Then $F_c^L : C \to A$ gives

$$(\mathrm{Tr}(F_c^L) : \mathrm{Tr}(C) \to 1_A) \in Z(C, S_C). \tag{4.53}$$

Explicitly, let $c^\vee = (F_c^L)^\vee(1_A) \in C^\vee$. Then

$$u_C \to c \boxtimes_A c^\vee, \tag{4.54}$$

and

$$e_C(c^\vee \boxtimes_A c) \to 1_A \tag{4.55}$$

which is equivalent to

$$c^\vee \boxtimes_A c \to K_{S_c}, \tag{4.56}$$

such that

$$c \xrightarrow{\cong} 1_A \boxtimes_A c \xrightarrow{\cong} (\mathrm{id}_C \otimes e_C)(u_C \otimes \mathrm{id}_C)(1_A \boxtimes_A c) \cong (\mathrm{id}_C \otimes e_C)(u_C \boxtimes_A c) \to (\mathrm{Id}_C \otimes e_C)(c \boxtimes_A c^\vee \boxtimes_A c) \to c \boxtimes_A 1_A \cong c. \tag{4.57}$$

is equivalent to identity and similar map for $c^\vee$.

Example 4.26. Take $A = \mathrm{Mod}_\Lambda$ and $C = \mathrm{Rep}^{\mathrm{sm}}(G)$, where G is a p-adic group. Then π is A-admissible if and only if $R\mathrm{Hom}(c\text{-}\mathrm{Ind}_K^G \Lambda, \pi)$ is perfect for every K open compact subgroup. Then $\mathrm{Tr}(F_c^L)$ gives character

$$\Theta_\pi : \overline{\mathscr{H}} = \mathscr{H}/[\mathscr{H}, \mathscr{H}] \to \Lambda.$$

Lecture 15: 6/1/2023

Recall that given a finitely dimensional vector space V equipped with an linear endomorphism $f : V \to V$, we have $\mathrm{Tr}(f|V) \in \mathrm{End}1$.

For A a k-algebra and F an A-bimodule, we have $HH(A, F) = A \otimes_{A \otimes A^{\mathrm{rev}}} F$.

Given $C \in \mathrm{LinCat}_\Lambda$ dualizable and $F : C \to C$, we have $\mathrm{Tr}(F|C) \in \mathrm{End}_{\mathrm{LinCat}_\Lambda} 1 = \mathrm{Mod}_\Lambda$.

Given $A \in \mathrm{Alg}(\mathrm{LinCat}_\Lambda)$, F an A-bimodule, we have $HH(A, F) := A \otimes_{A \otimes A^{\mathrm{rev}}} F \in \mathrm{LinCat}$ called categorical trace.

Relative tensor product: let R be a symmetric monoidal category with geometric realizations and tensor products commute with geometric realizations. Let $A \in \mathrm{Alg}(R)$ and $\mathrm{Mod}_A(R) := \mathrm{LMod}_{A^{\mathrm{rev}}}(R)$. We define

$$\begin{aligned} \mathrm{RMod}_A(R) \times \mathrm{LMod}_A(R) &\to R \\ (N, M) &\mapsto N \otimes_A M, \end{aligned} \tag{4.58}$$

by

$$\mathrm{Maps}_R(N \otimes_A M, L) = A\text{-bilinear maps } N \times M \xrightarrow{\varphi} L, \tag{4.59}$$

i.e. the colimit of

$$\begin{tikzcd} N \times A \times A \times M \arrow[r] \arrow[r, bend left] \arrow[r, bend right] & N \times A \times M \arrow[r, bend left, "\phi \circ \mathrm{act}_M"] \arrow[r, bend right, "\phi \circ \mathrm{act}_N"'] & L \end{tikzcd} \tag{4.60}$$

In general, $N \otimes_A M$:=geometric realization of $N \otimes A^\bullet \otimes M$, where $A^\bullet$ is the simplicial object defining A.

Example 4.27. $F = A$, then $HH(A) := HH(A, A)$.

Example 4.28. $A = \mathrm{Mod}_\Lambda = 1_{\mathrm{LinCat}_\Lambda}$ then bimodules $\mathrm{BMod}(A) = \mathrm{LinCat}_\Lambda$, and $HH(A, F) = F$.

Example 4.29. Suppose that $\sigma : A \to A$ is an algebra endomorphism, then $HH(A, {}^\sigma A)$ is called twisted categorical trace.

Example 4.30. M left A-module and N right A-module, $F = M \otimes_\Lambda N$ is an A-bimodule, and $HH(A, F) = N \otimes M$.

Remark 4.31. Recall Hoschild cohomology is categorical trace in Morita(Vect_Λ). Here we can also construct Morita(LinCat_Λ).

Recall that $\mathrm{Morita}(\mathrm{Vect}_k) \hookrightarrow \mathrm{LinCat}_\Lambda$ given by $A \mapsto \mathrm{LMod}_A(\mathrm{Vect}_\Lambda)$ is a fully-faithful symmetric monoidal functor. Then we can realize categorical trace above as ordinary trace in LinCat_Λ.

Hypothetical, we should have $\mathrm{Morita}_{\mathrm{LinCat}_\Lambda} \hookrightarrow 2\text{-}\mathrm{LinCat}_\Lambda$ by $A \mapsto \mathrm{LinCat}_A$, which is fully faithful and symmetric monoidal functor. LinCat is $(\infty, 2)$-category and $2\text{-}\mathrm{LinCat}_\Lambda$ is $(\infty, 3)$-category.

Some functoriality of categorical trace.

Let A, B be two algebras and ${}_AM_B$ be an (A, B)-bimodule.

Definition 4.32. We say M is left dualizable if there exists a (B, A)-bimodule N together with

$$\begin{aligned} u_M &: B \to {}_B(N \otimes_A M)_B, \\ e_M &: {}_A(M \otimes_B N)_A \to A, \end{aligned} \tag{4.61}$$

satisfying the usual Zorro relations.

Remark 4.33. Given an (A, B)-bimodule M,

$$\begin{aligned} M \otimes_B (-) : \mathrm{LinCat}_B &\to \mathrm{LinCat}_A \\ L &\mapsto M \otimes_B L \end{aligned} \tag{4.62}$$

M is left dualizable if and only if this functor admits a right adjoint.

Now given

- (A, F_A) and (B, F_B),
- M left dualizable (A, B)-bimodule, and
- $\eta : M \otimes_B F_B \to F_A \otimes_A M$,

then we have

$$\begin{aligned} HH(B, F_B) := B \otimes_{B \otimes B^{\mathrm{rev}}} F_B &\to (N \otimes_A M) \otimes_{B \otimes B^{\mathrm{rev}}} F_B \\ &\cong A \otimes_{A \otimes A^{\mathrm{rev}}} (M \otimes_B F_B \otimes_B N) \\ &\to A \otimes_{A \otimes A^{\mathrm{rev}}} (F_A \otimes M \otimes_B N) \\ &\to A \otimes_{A \otimes A^{\mathrm{rev}}} (F_A \otimes A) \\ &= HH(A, F_A). \end{aligned} \tag{4.63}$$

Definition 4.34. Given $A \in \mathrm{Alg}(R)$, where R is a symmetric monoidal category, is

- called smooth, if $A \in {}_{A \otimes A^{\mathrm{rev}}}\mathrm{BMod}_{1_R}$ is left dualizable as $(A \otimes A^{\mathrm{rev}}, 1_R)$-bimodule.
- called proper, if $A \in {}_{1_R}\mathrm{BMod}_{A^{\mathrm{rev}} \otimes A}$ is left dualizable as $(1_R, A^{\mathrm{rev}} \otimes A)$-bimodule.
- called 2-dulazable, if it is smooth and proper.

Remark 4.35. A is smooth, if and only if there exists T_A with

- $T_A \otimes_{A \otimes A^{\mathrm{rev}}} A \to 1_R$,
- $A \otimes T_A \to A \otimes A^{\mathrm{rev}}$,

satisfying certain conditions.

A is proper, if and only if there exists a Serre module S_A, with

- $A \otimes A^{\mathrm{rev}} \to S_A \otimes A$,
- $A \otimes_{A \otimes A^{\mathrm{rev}}} S_A \to 1_R$,

satisfying certain conditions.

There exists similar trace formula for A 2-dualizable. Let F_1, F_2 be left (A, A)-dualizable,

$$\alpha : F_1 \otimes_A F_2 \xrightarrow{\cong} F_2 \otimes_A F_1, \tag{4.64}$$

then

$$\mathrm{Tr}(HH(F_2, \alpha^{-1})|HH(A, F_1)) = \mathrm{Tr}(HH(F_1, \alpha)|HH(A, F_2)). \tag{4.65}$$

Example 4.36. $R = \mathrm{LinCat}_\Lambda$, $A = F_A = \mathrm{Mod}_\Lambda$, $B = F_B = \mathrm{Mod}_\Lambda$, then M is left dualizable (A, B)-bimodule if and only if M is dualizable in Mod_Λ. In this case

$$\mathrm{Tr}(M, \eta) : \mathrm{Mod}_\Lambda = HH(\mathrm{Mod}_\Lambda) \to HH(\mathrm{Mod}_\Lambda) = \mathrm{Mod}_\Lambda.$$

Definition 4.37. Let M be a left dualizable (A, B)-bimodule in LinCat_Λ. We say M is

- left smooth, if $u_M : B \to N \otimes_A M$ admits a B-bilinear right adjoint.
- left proper, if $e_M : M \otimes_B N \to A$ admits an A-linear right adjoint.

Example 4.38. $A = B = \mathrm{Mod}_\Lambda$, this recovers the previous definition of smooth/proper Λ-linear categories.

Example 4.39. $B = \mathrm{Mod}_\Lambda$, $M = A$ as left A-module, then M is always left dualizable as $(A, 1)$-bimodule, with $N = A$,

$$u_M : \mathrm{Mod}_\Lambda \to A \otimes_A A = A, \tag{4.66}$$

and

$$e_M : A \otimes A \to A \tag{4.67}$$

multiplication. $M = A$ is left proper if e_M admits a continuous right adjoint. M is left smooth if 1_A is compact.

Definition 4.40. An Λ-linear monoidal category A is called

- semi-rigid if $m : A \otimes A \to A$ admits continuous right adjoint,
- (locally) rigid, if it is semi-rigid and 1_A is compact.

Lemma 4.41. *If $A \in \mathrm{Alg}(\mathrm{LinCat}_\Lambda)$ is compactly generated, then A is (locally) rigid, if and only if compact objects admit both left and right duals.*

Example 4.42. We can take $A = \mathrm{Mod}_\Lambda$ or $A = \mathrm{QCoh}(X)$ where X is quasi-compact quasi-separated.

Example 4.43. Take $A = \mathrm{Ind}D_{\text{construcitble}}(B\backslash G/B)$.

Now consider

- (A, F_A), (B, F_B),
- M left proper/smooth (A, B)-bimodule,
- $\eta : F_B \otimes_B M \to F_A \otimes_A M$,

and suppose that η admits a continuous right adjoint, then

$$HH(M, \eta) : HH(B, F_B) \to HH(A, F_A) \tag{4.68}$$

admits a continuous right adjoint.

Corollary 4.44. *Let M be a left A-module, and assume that $\eta : M \to F_A \otimes_A M$ admits continuous right adjoint. Then $HH(M, \eta) : \mathrm{Mod}_\Lambda \to HH(A, F_A)$ corresponds to an object $[M, \eta]_{F_A} \in HH(A, F_A)$. If M is left smooth and proper, then $[M, \eta]_{F_A}$ is compact in $HH(A, F_A)$.*

Proof. Take $(B, F_B) = (\mathrm{Mod}_\Lambda, \mathrm{Mod}_\Lambda)$.

A rigid, $M = A$, $\eta : M \to F_A \otimes_A M \Leftrightarrow x \in F_A$. Then $[M, \eta]_{F_A} = \mathrm{Tr}(X)$. If X is compact, then $\mathrm{Tr}(X)$ is compact. □

5 Lecture 16: 6/6/2023

Recall $A \in \mathrm{Alg}(\mathrm{LinCat})_\Lambda$ is called rigid if

- $m : A \otimes A \to A$ admits an $(A \otimes A^{\mathrm{rev}})$-linear right adjoint;
- 1_A is compact.

Example 5.1. If A is compactly generated then A is rigid if and only if

- every compact object admits both left and right duals;
- 1_A is compact.

Proposition 5.2. *Let A be rigid.*

(1) Let $F : M \to N$ be an A-linear functor of left A-modules. Then if F^R is continuous, then it is A-linear.

$$\begin{tikzcd} A \otimes M \arrow[r, "id_A \circ F"] \arrow[d, "act_M"] & A \otimes N \arrow[d, "act_N"] \\ M \arrow[r, "F"] & N \end{tikzcd} \tag{5.1}$$

and base change map

$$act_M \circ (id_A \circ F) \Rightarrow F^R \circ act_N \tag{5.2}$$

is an isomorphism.

(2) Let M be a left A-module. Then M is dualizable in LinCat_Λ *if and only if M is left dualizable as an A-module.*

$$\begin{tikzcd} \mathrm{Mod}_\Lambda \arrow[r] \arrow[d, "="] & N \otimes M \arrow[d, "right\ adjoint"] \\ \mathrm{Mod}_\Lambda \arrow[r] & N \otimes_A M \end{tikzcd} \tag{5.3}$$

$$\begin{tikzcd} M \otimes N \arrow[r] \arrow[d, "="] & \mathrm{Mod}_\Lambda \arrow[d, "\mathrm{Hom}(1_A,-)"] \\ M \otimes N \arrow[r] & A \end{tikzcd} \tag{5.4}$$

(3) A is dualizable as an object is Mod_Λ*,*

$$\begin{gathered} \mathrm{Mod}_\Lambda \xrightarrow{1_A} A \xrightarrow{m^R} A \otimes_\Lambda A, \\ A \otimes A \xrightarrow{m} A \xrightarrow{\mathrm{Hom}(1_A,-)} \mathrm{Mod}_\Lambda. \end{gathered} \tag{5.5}$$

A is 2-dualizable as an algebra in LinCat_Λ*. A is 2-dualizable if and only if*

- ${}_\Lambda A_{A\otimes A^{rev}}$ *is left dualizable as* Mod_Λ*-module.*
- ${}_{A\otimes A^{rev}}A_\Lambda$ *is left dualizable as $A \otimes A^{rev}$-module.*

Corollary 5.3. *Suppose that A is rigid.*

1. Let $F_1 \to F_2$ be an A-bilinear functor, then

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] & \mathrm{HH}(A, F_1) \arrow[d] \\ F_2 \arrow[r] & \mathrm{HH}(A, F_2) \end{tikzcd} \tag{5.6}$$

is right adjointable.

2. *Suppose that $F_1 \to F_2$ admits a continuous right adjoint. Then*

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] & F_2 \arrow[d] \\ \mathrm{HH}(A, F_1) \arrow[r] & \mathrm{HH}(A, F_2) \end{tikzcd} \tag{5.7}$$

is right adjointable.

Proof. $\mathrm{HH(A, F_1)} = A \otimes_{A \otimes A^{\mathrm{rev}}} F$. Pass the bar resolution to right adjoints to get a co-simplicial object. Taking limits also computes the Hoschild homology. □

Theorem 5.4. *Assume that A is rigid, equipped with a monoidal endomorphism $\phi : A \to A$.*

$$\mathrm{End}_{\mathrm{HH}(A, {}^{\phi}A)}([1_A]_{{}^{\phi}A}) \cong Tr(A, \phi) \tag{5.8}$$

as Λ-algebras. Let M be a left dualizable A-module equipped with

$$\phi : M \to {}^{\phi} \otimes_A M \cong {}^{\phi}M.$$

$$\mathrm{Hom}([1_A]_{{}^{\phi}A}, [M, \phi]_{{}^{\phi}A}) \cong Tr(M, \phi). \tag{5.9}$$

Idea of Proof of Thm. 5.4.

$$\begin{tikzcd} {}^{\phi}A \otimes A \arrow[r] \arrow[d] & {}^{\phi}A \arrow[d, "{[\ \ ]_{{}^{\phi}A}}"] \\ \mathrm{HH}(A, {}^{\phi}A \otimes A) \cong {}^{\phi}A \arrow[r, "{[\ \ ]_{{}^{\phi}A}}"] & \mathrm{HH}(A, {}^{\phi}A) \end{tikzcd} \tag{5.10}$$

□

Theorem 5.5 (S=T)**.** *Let A be rigid and let $a \in A$ be compact ($\varphi = id$ for simplicity). Then define*

$$S_a : [1_A]_{{}^{\phi}A} \to [a \otimes a^{\vee}]_{{}^{\phi}A} \cong [a^{\vee} \otimes a]_{{}^{\phi}A} \to [1_A]_{{}^{\phi}A}. \tag{5.11}$$

Note that $S_a \in \mathrm{Hom}([1_A]_A, [1_A]_A) \cong Tr(A) \ni ch(a) =: T_a$. The theorem is that $S_a = T_a$.

Example 5.6. Larforgue's work. Take $A = \mathrm{Shv}(L^+G \backslash LG / L^+G)$.

Suppose that A is rigid. Let F_1, F_2 be two A-bimodules, both of which admit left duals.

$$\alpha : F_1 \otimes_A F_2 \xrightarrow{\cong} F_2 \otimes_A F_1 \tag{5.12}$$

$$\mathrm{Tr}(\mathrm{HH}(A, F_1), \mathrm{HH}(F_2, \alpha^{-1})) \cong \mathrm{Tr}(\mathrm{HH}(A, F_2), \mathrm{HH}(F_1, \alpha)). \tag{5.13}$$

Now let M be a left smooth and proper A-module equipped with

$$\begin{tikzcd} & M \arrow[dl, "\beta_1"] \arrow[dr, "\beta_2"'] & \\ F_1 \otimes_A M \arrow[d, "\mathrm{id}_{F_1} \otimes \beta_2"] & & F_2 \otimes_A M \arrow[d, "\mathrm{id}_{F_2} \otimes \beta_1"] \\ F_1 \otimes_A F_2 \otimes_A M \arrow[rr, "\alpha \otimes \mathrm{id}_M"] & & F_2 \otimes_A F_1 \otimes_A M \end{tikzcd} \tag{5.14}$$

Theorem 5.7. *(1)*

$$\begin{tikzcd} \mathrm{Mod}_{\Lambda} \arrow[r, "{[M, \beta_1]_{F_1}}"] \arrow[d, "{[M, \beta_1]_{F_1}}"'] & \mathrm{HH}(A, F_1) \arrow[dl, "{\mathrm{HH}(F_2, \alpha^{-1})}"] \\ \mathrm{HH}(A, F_1) \arrow[ur, phantom, "\Rightarrow"] & \end{tikzcd} \tag{5.15}$$

(2) $[M, \beta_i]_{F_i}$ is compact in $\mathrm{HH}(A, F_i)$*.*
(3)$ch([M, \beta_1]_{F_1}, Tr(\mathrm{HH}(F_2, \alpha^{-1}), \eta)) = ch([M, \beta_2]_{F_2}, Tr(\mathrm{HH}(F_1, \alpha, \eta)))$.

Example 5.8. G reductive group over $\mathbb{F}_q$ and B a Borel. Take $A = \mathrm{Ind}(D^b_c(B\backslash G/B, \mathbb{Q}_l))$. Then

Proposition 5.9. *A is rigid.*

There is an algebra structure on A given by the correspondence

$$B\backslash G/B \times B\backslash G/B \leftarrow B\backslash G \times^B G/B \to B\backslash G/B. \tag{5.16}$$

Take $\phi : A \to A$ q-Frobenius pullback. Then

$$\mathrm{Tr}(A, \phi) = C_c(B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)). \tag{5.17}$$

Remark 5.10. It is true that in this case l-adic sheaves on the product is the tensor product of l-adic sheaves. In general this is false. This is a property which generally holds for quasi-coherent sheaves instead of l-adic sheaves.

Theorem 5.11. $\mathrm{HH}(A, {}^{\phi}A) = A \otimes_{A\otimes A^{rev}} {}^{\phi}A$. *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Ind}D^b_c(B\backslash G/B) \arrow[r] \arrow[d, "{[\ \ ]_{\phi A}}"] & \mathrm{Ind}D^b_c(G/\mathrm{Frob}_q B) \arrow[d] \\
\mathrm{HH}(A, {}^{\phi}A) \arrow[r, "\text{fully faithful}"] & \mathrm{Ind}D^b_c(G/\mathrm{Ad}_{\mathrm{Frob}_q}) \cong \mathrm{Rep}(G(\mathbb{F}_q), \mathbb{Q}_l)
\end{tikzcd} \tag{5.18}$$

$$B\backslash G/B \leftarrow G/\mathrm{Ad}_{\mathrm{Frob}_q} B \to G/\mathrm{Ad}_{\mathrm{Frob}_q} G \cong [*\backslash G(\mathbb{F}_q)](\text{by Lang isogeny}), \tag{5.19}$$

where $b \cdot g = b^{-1} g\mathrm{Frob}(b)$. Then note that $B\backslash G/B = \sqcup_{w\in W} S_w$ gives Schubert cells.
$[j_{w,!}]_{\phi A}$ =compactly supported cohomology of Deligne-Lusztig variety associated to w.
$\mathrm{Tr}(A, \phi) = C(B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q))$. Take $w = e$, $[j_{e,!}]_{\phi A} = C(G(\mathbb{F}_q)/B(\mathbb{F}_q))$ and

$$\mathrm{End}_{G(\mathbb{F}_q)}(C(G(\mathbb{F}_q)/B(\mathbb{F}_q))) = C(B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)). \tag{5.20}$$

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