REPRESENTATION SEMINAR AT STANFORD

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Abstract. This is my personal note for the Representation Theory Seminar at Stanford organized by Prof Xinwen Zhu, starting in 2024 Fall. The main goal is to understand the proof of the Deligne-Langlands conjecture by Kazhdan-Lusztig and Ginzburg. Some research talks are recorded as appendices, though maybe incomplete.

Contents

1. Iwahori Hecke Algebras, Xinyu Li

1.1. Notations.

- F: a NA local field;
- $\mathscr{O}$: the ring of integers;
- $\mathfrak{m}$: the maximal ideal of $\mathscr{O}$;
- π: a uniformizer in $\mathfrak{m}$;
- k: the residue field $k = \mathscr{O}/\mathfrak{m}$;
- $q = \#k$;
- G: a split connected reductive group defined over $\mathscr{O}$;
- T: a split maximal torus T;
- B: a Borel subgroup containing T;
- $B = TN$, where N is the unipotent radical of B;
- $\overline{N}$: the unipotent radical of opposite Borel $\overline{B}$;
- $K := G(\mathscr{O}) \subseteq G(F)$ hyperspecial;
- $\mathrm{ev} : G(\mathscr{O}) \to G(k)$;
- Iwahori $\mathcal{I} := \mathrm{ev}^{-1}(B(k))$.

1.2. Some Structure Theory.

1.2.1. *Factorizations.* Iwahori factorization: $\mathcal{I} = N(\mathscr{O})T(\mathscr{O})(\overline{N}(F) \cap \mathcal{I})$. More precisely,

$$N(\mathscr{O}) \times T(\mathscr{O}) \times (\overline{N}(F) \cap \mathcal{I}) \xrightarrow{\text{multiplication}} \mathcal{I} \tag{1.1}$$

is a bijection. Moreover, one can rearrange the order of the three factors.

Define extended affine Weyl group

$$\widetilde{W} = N_{G(F)}(T(F))/T(\mathscr{O}), \tag{1.2}$$

which contains a finite subgroup

$$W = (N_{G(F)}(T(F)) \cap G(\mathscr{O}))/T(\mathscr{O}), \tag{1.3}$$

and contains $T(F)/T(\mathscr{O}) \cong X_*(T)$ coweight lattice, via $\mu \mapsto \pi^\mu$. In fact,

$$\widetilde{W} \cong W \ltimes X_*(T). \tag{1.4}$$

We have Iwahori-Matsumoto decomposition

$$G(F) = \bigsqcup_{w \in \widetilde{W}} \mathcal{I} w \mathcal{I}, \tag{1.5}$$

and Bruhat decomposition

$$K := G(\mathscr{O}) = \bigsqcup_{w \in W} \mathcal{I} w \mathcal{I}. \tag{1.6}$$

We have $B(\mathscr{O}) w \mathcal{I} = I w \mathcal{I}$ by Iwahori factorization.

We have Iwasawa decomposition

$$G(F) = B(F)K = \bigsqcup_{\mu \in X_*(T)} \pi^\mu N(F) K. \tag{1.7}$$

1.2.2. *Hecke Algebra and Universal Unramified Principal Series.*

Definition 1.2.1. Iwahori Hecke algebra $\mathscr{H} := C_c^\infty(\mathcal{I}\backslash G(F)/\mathcal{I}, \mathbb{C})$, whose algebra structure is given by convolution

$$\varphi * \psi(g) := \int_{G(F)} \varphi(gh^{-1})\psi(h)dh, \tag{1.8}$$

where dh is a right Haar measure on $G(F)$, such that $\mathrm{Vol}(\mathcal{I}) = 1$.

By Iwasawa-Mastumuto decomposition, $\mathscr{H}$ has a $\mathbb{C}$-basis given by $\{T_x := 1_{\mathcal{I}x\mathcal{I}}\}_{x\in\widetilde{W}}$.
Auxiliary module $M := C_c^\infty(T(\mathscr{O})N(F)\backslash G(F)/\mathcal{O})$, which is a right $\mathscr{H}$-module by convolution.

$$\begin{aligned} G(F) &= \bigsqcup_{\mu\in X_*(T)} \pi^\mu N(F)K \\ &= \bigsqcup_{\mu\in X_*(T), w\in W} \pi^\mu N(F)B(\mathscr{O})wI \\ &= \bigsqcup_{\mu\in X_*(T), w\in W} T(\mathscr{O})N(F)\pi^\mu w\mathcal{I} \\ &= \bigsqcup_{x\in\widetilde{W}} T(\mathscr{O})N(F)x\mathcal{I}. \end{aligned} \tag{1.9}$$

So M has a basis $v_x = 1_{T(\mathscr{O})N(F)x\mathcal{I}}$, $x \in \widetilde{W}$. Consider

$$R = \mathbb{C}[X_*(T)] = C_c^\infty(T(F)/T(\mathscr{O}), \mathbb{C}) \tag{1.10}$$

the Iwahori Hecke algebra for T. There is a left R-module structure on M given by

$$\pi^\mu \cdot v_x := q^{-<\rho,\mu>} v_{\pi^\mu x}, x \in \widetilde{W}. \tag{1.11}$$

Left translation commutes with right convolution, so M is an $(R, \mathscr{H})$-bimodule.

Proposition 1.2.2. *Via the map $\Xi : \mathscr{H} \to M$ defined by $h \to v_1 \cdot h$, M is a free right $\mathscr{H}$-module of rank* 1.

Corollary 1.2.3. $\mathrm{End}_{\mathscr{H}}(M) \cong \mathscr{H}$.

Proof of Prop.1.2.2. We have to show that in terms of basis $\{T_x\}$ and $\{v_x\}$, Ξ is an upper triangular matrix with invertible diagonal.

$$\begin{aligned} \mathrm{supp}(v_1 \cdot T_x) &= \mathrm{supp}(v_1)\mathrm{supp}(T_x) \\ &= T(\mathscr{O})N(F)\mathcal{I} \cdot \mathcal{I}x\mathcal{I} \\ &= N(F)T(\mathscr{O})\mathcal{I}x\mathcal{I}. \end{aligned} \tag{1.12}$$

Claim: for $y \in \widetilde{W}$, $y \in \mathrm{supp}(v_1 \cdot T_x) = N(F)\mathcal{I}x\mathcal{I}$, then $y \leq x$ is in the Bruhat order. The claim will imply the proposition.

Find an element $n \in N(F)$, such that $ny \in \mathcal{I}x\mathcal{I}$. We can find sufficiently dominant coweight μ, such that $\pi^\mu n \pi^{-\mu} \in N(\mathscr{O}) \subseteq \mathcal{I}$.

$$\begin{aligned}\mathcal{I}\pi^\mu y\mathcal{I} &= \mathcal{I}\pi^\mu n\pi^{-\mu}\pi^\mu y\mathcal{I}\\ &= \mathcal{I}\pi^\mu ny\mathcal{I}\\ &\subseteq (\mathcal{I}\pi^\mu\mathcal{I})(\mathcal{I}x\mathcal{I})\\ &\subseteq \bigsqcup_{x'\leq x} \mathcal{I}\pi^\mu x'\mathcal{I}.\end{aligned}$$

Therefore, $\pi^\mu y \leq \pi^\mu x$ and hence, $y \leq x$. □

Proposition 1.2.4. *(1) For $w \in W$, $v_1 T_w = v_w$.*

(2) For $w \in W$, $\mu \in X_*(T)$, $v_{\pi^\mu} T_w = v_{\pi^\mu w}$.

(3) For $\mu \in X_*(T)$ dominant, $v_1 T_{\pi^\mu} = v_{\pi^\mu}$,

1.3. **Decomposition of $\mathscr{H}$ as Vector Spaces.** Let $\mathscr{H}_0 := C_c^\infty(\mathcal{I}\backslash K/\mathcal{I})$ finite Hecke algebra. Then $\mathscr{H}_0 \subseteq \mathscr{H}$ by extension by zero. Recall that we have a map

$$R \to \mathrm{End}_{\mathscr{H}}(M) \cong \mathscr{H}, \tag{1.13}$$

we denote the image of π^μ by Θ_μ.

Proposition 1.3.1. *We have an isomorphism of vector spaces*

$$R \otimes_{\mathbb{C}} \mathscr{H}_0 \xrightarrow[\cong]{mult} \mathscr{H}, \tag{1.14}$$

Proof. We have

$$\begin{aligned} &R \otimes_{\mathbb{C}} \mathscr{H}_0 \to \mathscr{H} \to M\\ &\pi^\mu \otimes T_w \mapsto \Theta_\mu T_w \mapsto \pi^\mu v_1 T_w = q^{-<\rho,\mu>} v_{\pi^\mu w}.\end{aligned} \tag{1.15}$$

□

1.4. **Intertwiners.** We still fix T, $\mathcal{I}$ and K as before. Set $\mathscr{B}(T)$:=the set of Borels containing T. Now we vary Borels in this family.

Let $B' = TN'$. Define $M_{B'} = C_c^\infty(T(\mathscr{O})N'(F)\backslash G(F)/\mathcal{I}, \mathbb{C})$.

- $J := \{$a set of coroots that is a subset of some system of positive coroots$\}$.
- $\mathbb{C}[J]$ the $\mathbb{C}$-subalgebra of R generated by J;
- $\mathbb{C}[J]^\wedge$ the completion of $\mathbb{C}[J]$ with respect to the ideal generated by J;
- $R_J := \mathbb{C}[J]^\wedge \otimes_{\mathbb{C}[J]} R$ = functions on $X_*(T)$ supported on a finite union of sets of the form $x + C_J$, where $x \in X_*(T)$ and $C_J = \mathbb{Z}_{\geq 0}J$;
- $M_{B',J} = R_J \otimes_R M'_B$ =functions on $G(F)$, $(T(\mathscr{O})N_{B'}(F), \mathcal{I})$-biinvariant, whose support is contained $\cup_{v\in S} T(\mathscr{O})N_{B'}(F)\pi^\nu K$, S is a finite union of $x + C_J$.

Now For $B = TN_B$ and $B' = TN'_B$ containing T, set

$$J := \{\text{coroots that are positive for } B' \text{ and negative for } B\}.$$

Define intertwining operator
$$I_{B',B} : M_{B,J} \to M_{B',J'} \tag{1.16}$$
defined by
$$\varphi \mapsto [\varphi' : g \mapsto \frac{1}{\mathrm{Vol}(N_{B'}(F) \cap \overline{N}_B(F) \cap K)} \int_{N_{B'}(F)\cap\overline{N}_B(F)} \varphi(n'g)dn']. \tag{1.17}$$
Claim: This integral is in fact compactly supported on $N_{B'}(F)\cap\overline{N}_B(F)$. This will imply that $I_{B',B}$ is an $(R_J, \mathscr{H})$-bimodules homomorphism.

Lemma 1.4.1. *For $\nu \in X_*(T)$, set $C_\nu = N_{B'}(F) \cap \overline{N}_B(F) \cap \pi^\nu N_B(F)K$.*

(1) If C_ν is not empty, then $\nu \in C_J = \mathbb{Z}_{\geq 0}J$.

(2) C_ν is compact.

Proof. (1) is by induction on $|J|$ and finally reduce to a brutal compuation for SL_2. (2) is from the fact that $N_B \to \overline{N}_{B'}\backslash G$ is closed immersion (easily checked after base change to G), and hence for any compact $C \subset G(F)$, we will have that $N_B(F)C \cap \overline{N}_B(F)$ is compact. □

Consider 3 Borel subgroups $B_i \in \mathscr{B}(T)$ and define $J_{i,j}$ as above. Assume that
$$J_{31} = J_{21} \bigsqcup J_{32}.$$
Then $I_{B_3,B_1} = I_{B_3,B_2} I_{B_2,B_1}$.

Rephrasing: Fix $B = TN$. Now we let J vary. For any $w \in W$, we have
$$\begin{array}{ccc} M_{B,w^{-1}J} & \xrightarrow{\quad I_w \quad} & M_{B,J} \\ & {\searrow}^{\cong} \qquad {\nearrow}_{I_{B,wB}} & \\ & M_{wB,J} & \end{array} \tag{1.18}$$

Explicitly,
$$I_w(\varphi)(g) = \frac{1}{\mathrm{Vol}(N_w(F) \cap K)} \int_{N_w} \varphi(w^{-1}ng)dn, \tag{1.19}$$
where $N_w := N \cap w\overline{N}w^{-1}$.

Proposition 1.4.2. *(1) $I_w \pi^\mu = \pi^{w(\mu)} I_w$, for $\mu \in X_*(T)$.*

(2) $I_{w_1w_2} = I_{w_1} I_{w_2}$, for $l(w_1) + l(w_2) = l(w_1w_2)$.

(3) $\mathcal{I}_w$ is a right $\mathscr{H}$-module homomorphism.

For SL_2, use α to denote the unique positive coroot and s_α the simple refecltion, represented by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then
$$I_{s_\alpha}(v_1)(g) = \frac{1}{\mathrm{Vol}(\overline{N}(F) \cap K)} \int_{\overline{N}(F)} v_1(s_\alpha g)dn. \tag{1.20}$$
$N(F) \cong F$ parametrized by $x \mapsto \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$. In this case
$$T(\mathscr{O})N(F)\mathcal{I} = \left(\begin{smallmatrix} F & F \\ (\pi) & \mathscr{O}^\times \end{smallmatrix}\right), \tag{1.21}$$

$$\pi^{j\alpha^\vee} = \left(\begin{smallmatrix} \pi^j & 0 \\ 0 & \pi^{-j} \end{smallmatrix}\right). \tag{1.22}$$

Then

$$s_\alpha n \pi^{j\alpha^\vee} = \left(\begin{smallmatrix} 0 & -\pi^j \\ \pi^j & x\pi^{-j} \end{smallmatrix}\right) \tag{1.23}$$

and

$$s_\alpha n \pi^{j\alpha^\vee} s_\alpha = \left(\begin{smallmatrix} -\pi^j & 0 \\ x\pi^{-j} & -\pi^j \end{smallmatrix}\right). \tag{1.24}$$

Therefore,

$$I_{s_\alpha}(v_1)(\pi^{j\alpha^\vee}) = \delta_{j\geq 1} q^{-j}(1-q^{-1}), \tag{1.25}$$

and

$$I_{s_\alpha}(v_1)(\pi^{j\alpha^\vee} s_\alpha) = \delta_{j=0} q^{-1}. \tag{1.26}$$

Finally, we conclude that

$$I_{s_\alpha}(v_1) = q^{-1} v_{s_\alpha} + (1-q^{-1}) \sum_{j\geq 1} \pi^{j\alpha^\vee} v_1. \tag{1.27}$$

Similarly, $I_{S_\alpha}(1_{T(\mathscr{O})N(F)K}) = q^{-1} 1_{T(\mathscr{O})N(F)K} + \sum_{j\geq 0} q^{-j}(1-q^{-1}) 1_{T(\mathscr{O})N(F)\pi^{j\alpha^\vee}K}$.

For general G, using Jacobson-Morozov, we could obtain that

(1) $I_{s_\alpha}(v_1) = q^{-1} v_{s_\alpha} + (1-q^{-1}) \sum_{j\geq 1} \pi^{j\alpha^\vee} v_1$;

(2) $I_{s_\alpha}(v_1 + v_{s_\alpha}) = \frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}}(v_1 + v_{s_\alpha})$.

(3) $I_{s_\alpha}(1_{T(\mathscr{O})N(F)K}) = \frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}} 1_{T(\mathscr{O})N(F)K}$.

Corollary 1.4.3 (Gindikin-Karpelevich formula)**.** *For $w \in W$, R_w :=the set of positive roots α, such that $w^{-1}\alpha$ is negative.Then*

$$I_w(1_{T(\mathcal{O})N(F)K}) = \Big(\prod_{\alpha\in R_w} \frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}}\Big) 1_{T(\mathcal{O})N(F)K}.$$

$J_w := \big(\prod_{\alpha\in R_w}(1-\pi^{\alpha^\vee})\big)\cdot I_w$. Note that J_w preserves the subspace M of $M_{B,w^{-1}J}$ and $M_{B,J}$ and hence can be regarded as an element of H, via our identification of H with $\mathrm{End}_H(M)$.

Corollary 1.4.4. *Hecke relation: $T_{s_\alpha}^2 = q + (q-1)T_{s_\alpha}$.*

Proof. Calculate $I_{s_\alpha}(v_{s_\alpha})$ in two ways: $I_{s_\alpha}(v_{s_\alpha}) = I_{s_\alpha}(v_1)T_{s_\alpha}$ and $I_{s_\alpha}(v_{s_\alpha}) = I_{s_\alpha}(v_{s_\alpha} + v_1) - I_{s_\alpha}(v_1)$. □

Theorem 1.4.5 (Bernstein Presentation)**.** $T_{s_\alpha}\pi^\mu = \pi^{s_\alpha(\mu)}T_{s_\alpha} + (1-q)\frac{\pi^{s_\alpha(\mu)}-\pi^\mu}{1-\pi^{-\alpha^\vee}}$.

1.5. **Center of $\mathscr{H}$.**

Proposition 1.5.1. $R^W \subseteq Z(\mathscr{H})$.

Proof. Using intertwiners J_w to prove that R^W action commutes with $\mathscr{H}_0$. □

Denote $L := \mathrm{Frac}(R)$, then $L^W = \mathrm{Frac}(R^W)$. Then R is a free R^W-module of rank $|W|$. Consider

$$\mathscr{H}_{\mathrm{generic}} := L^W \otimes_{R^W} \mathscr{H}, \tag{1.28}$$

and

$$M_{\mathrm{generic}} := L \otimes_R M = L^W \otimes_{R^W} M. \tag{1.29}$$

Then M_{generic} is an $(L, \mathscr{H}_{\mathrm{generic}})$-bimodule.

Corollary 1.5.2. *$Z(\mathscr{H}_{generic}) = L^W$, and $Z(\mathscr{H}) = R^W \cong H^{sph}$.*

2. Unramified Principal Series, Shurui Liu

The main goal is to prove the following:

- Iwahori block is same as representations of Iwahori-Hecke algebra: $\mathrm{Rep}(G(F))^{[\mathcal{I}]} \cong \mathrm{Rep}(\mathscr{H}_{\mathcal{I}})$
- $\mathrm{Rep}(G(F))^{[cI]}$ is generated by unramified principal series.

2.1. Basics.

A short overview of Bruhat-Tits, smooth representations, admissible representations and so on.

For simplicity, in this talk, I only deal with split case and admissible representations, though most parts work for non-split groups after harder combinatorics.

Briefly talk about non-split case: e.g. replace B with minimal parabolic, G with $\widetilde{G}_{\mathrm{der}}$, etc.

2.2. Unramified Principal Series.

The main reference for this subsection is Casselman's original papers, [Cas80],[CS80] (though he uses B to denote Iwahori and P minimal parabolic; non-split case).

Let G be a split connected reductive group over NA field F as before.

A continuous character $\sigma : T(F) \to \mathbb{C}^\times$ is called unramified, if it is trivial on $T(\mathscr{O})$. The group $\Lambda := T(F)/T(\mathscr{O})$ is a free abelian group of rank r. Therefore,

$$X^{\mathrm{unr}}(T) = \mathrm{Hom}_{\mathrm{gp}}(\Lambda, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^r. \tag{2.1}$$

In this split case, upon a choice of uniformizer $\pi \in \mathscr{O}$, we could identify

$$\Lambda \cong \mathbb{X}_*(T), \tag{2.2}$$

and then we identify

$$X^{\mathrm{unr}}(T) = \mathrm{Hom}_{\mathrm{gp}}(X_*(T), \mathbb{C}^\times). \tag{2.3}$$

Then given $\chi \in X^{\mathrm{unr}}(T)$, we could construct an unramified principal series

$$I(\chi) := i_B^G(\chi) := \mathrm{Ind}_{B(F)}^G(F)\chi\delta_B^{\frac{1}{2}} \tag{2.4}$$

using normalized induction functor, where δ_B is the modulus character of B, i.e. $\delta_B(tn) = \det(\mathrm{Ad}(t)|_{\mathrm{Lie(N)}})$, where $B = TN$, $t \in T$ and $n \in N$.

Remark 2.2.1. We can view $R := \mathbb{C}[X_*(T)]$ defined last time as a $T(F)$-representation, via $(\chi_{\mathrm{univ}})^{-1}$, where

$$\begin{aligned} \chi_{\mathrm{univ}} : T(F)/T(\mathscr{O}) &\to R^\times \\ \pi^\mu &\mapsto \pi^\mu \end{aligned} \tag{2.5}$$

for $\mu \in X_*(T)$ and χ_{univ} does not depend on the choice of π. Then we define the universal unramified principal series

$$M^{\mathrm{unr}}_{\mathrm{univ}} = i_B^G(R). \tag{2.6}$$

Observe that

- $R = \text{c-ind}_{T(\mathscr{O})}^{T(F)}\mathrm{triv}$,
- $i_B^G\text{c-ind}_T(\mathscr{O})^T(F)(\mathrm{triv}) = \text{c-ind}_{T(\mathscr{O})N(F)}^{G(F)}\mathrm{triv} = C_c^\infty(T(\mathscr{O})N(F)\backslash G(F))$

we see that

(1) $(M^{\mathrm{unr}}_{\mathrm{univ}})^{\mathcal{I}} \cong C_c^\infty(T(\mathscr{O})N(F)\backslash G(F)/\mathcal{I}, \mathbb{C}) = M$ defined last time.
(2) $M^{\mathrm{unr}}_{\mathrm{univ}}$ has a natural R-module structure, by $(r\phi)(g) := r(\phi(g))$.
(3) Any character $T(F)/T(\mathscr{O}) \to \mathbb{C}^\times$ gives rises to a $\mathbb{C}$-algebra homomorphism $R := \mathbb{C}[T(F)/T(\mathscr{O})] \to \mathbb{C}$, then

$$M^{\mathrm{unr}}_{\mathrm{univ}} \otimes_{R,\chi} \mathbb{C} \cong I(\chi^{-1}), \tag{2.7}$$

and

$$M \otimes_{R,\chi} \mathbb{C} \cong (I(\chi^{-1}))^{\mathcal{I}}. \tag{2.8}$$

Define G-projection

$$\mathscr{P}_\chi : C_c^\infty(G(F)) \to I(\chi) \tag{2.9}$$

by

$$\mathscr{P}_x(f)(g) := \int_{B(F)} \chi^{-1}\delta^{1/2}(b)f(bg)db, \tag{2.10}$$

where we choose left Haar measure on $B(F)$ such that $B(\mathscr{O})$ has volume 1. For $w \in W_{\mathrm{fin}}$, we define

$$\phi_{w,\chi} := \mathscr{P}_\chi(1_{\mathcal{I}w\mathcal{I}}), \tag{2.11}$$

and

$$\phi_{K,\chi} := \mathscr{P}_\chi(1_{G(\mathscr{O})}). \tag{2.12}$$

Proposition 2.2.2. *The functions* $\{\phi_{w,\chi}\}_{w\in W_{fin}}$ *forms a* $\mathbb{C}$*-basis of* $I(\chi)^{\mathcal{I}}$.

Proof. Notice that

- $\phi_{w,\chi}(x) = 0$ for $x \notin B(F)w\mathcal{I}$,
- $\phi(bwi) = \chi\delta^{1/2}(b)$, for $b \in B(F), w \in W_{\mathrm{fin}}, i \in \mathcal{I}$.

Use the decomposition $G(F) = \bigsqcup_{w\in W_{\mathrm{fin}}} B(F)w\mathcal{I}$. □

Proposition 2.2.3. *The function* $\phi_{K,\chi}$ *is a basis for the one-dimensional space* $I(\chi)^K$.

Proof. This follows from the Iwasawa decomposition. □

Lemma 2.2.4. *The contragredient repn of* $I(\chi)$ *is* $I(\chi^{-1})$.

Proof. This follows the general fact $\widetilde{i_P^G(\sigma)} \cong i_P^G(\widetilde{\sigma})$ for any parabolic subgroup P. See [Cas95, Prop.3.1.2] □

Proposition 2.2.5. *The canonical projection* $I(\chi)^{\mathcal{I}} \to r_B(\chi)^{T(\mathscr{O})}$ *is a linear isomorphism.*

I will provide more general results, which implies this easily.

2.3. **Jacquet's Lemma.** The reference for this subsection is [Cas80] and [Ber].

Let $P = MN$ be a standard parabolic subgroup of $G(F)$. Let $M^\circ \subseteq M$ be the subgroup generated by all compact subgroups. Let Λ denote a lift of $T(F)/T(\mathscr{O})$ in $T(F)$. Let $\Lambda^+ \subset \Lambda$ consisting of those λ, such that $\mathrm{Ad}_\lambda|_U$ is (non-strictly) contracting.

Proposition 2.3.1. *The Jacquet module $r_P(V)$ of an admissible representation V of $G(F)$ is again admissible.*

Take K as an open compact subgroup of $G(F)$.

Definition 2.3.2. We say (K, P) are in good position, if

- Iwahori decomposition holds: $K = K_N K_M K_{\overline{N}}$, where $K_N = K \cap N$, $K_M = K \cap M$ and $K_{\overline{N}} = K \cap \overline{N}$.

We say (K, P) is dominant by Λ^+, if

- K_M is $\mathrm{Ad}(\Lambda)$-invariant;
- K_N is $\mathrm{Ad}(\Lambda^+)$-invariant;
- $K_{\overline{N}}$ is $(\mathrm{Ad}(\Lambda^+))^{-1}$-invariant.

We say $\lambda \in \Lambda^+$ is strictly dominant with respect to (P, K), if P is determined by λ via dynamic approach.

By a nontrivial theorem of Bruhat, given P, there exists an arbitrarily small K, such that (P, K) is in a good position, and there exists λ strictly dominating (P, K). Let (P, K) be in a good position and dominated by Λ^+, V be an admissible representation of $G(F)$. Then there is a natural projection map

$$\mathrm{p} : V^K \to r_P(V)^{K_M}. \tag{2.13}$$

Lemma 2.3.3 (Jacquet's Lemma)**.** *There is a linear operator $\mathscr{A}$ on V^k, and a canonical decomposition $V^K = V_0 \oplus V_*$, such that*

- *$\mathscr{A}$ is nilpotent on V_0;*
- *$\mathscr{A}$ is invertible on V_*;*
- *V_* is isomorphic to $r_P(V)^{K_M}$ under p.*

Proof. Pick λ as a strictly dominant element. We consider the operator $\mathscr{A} := 1_K * 1_\lambda * 1_K$ acting on V^K and $\mathscr{A} := 1_\lambda$. Then p is $\mathscr{A}$-equivariant. Note that $\mathscr{A}$ is invertible on $r_P(V)^{K_M}$.

For any $v' \in r_P(V)^{K_M}$, we can pick a lifting $v \in V^{K_M}$, and then there exists a open compact K' stabilizing v. Consider $v_n := \lambda^n v$, which is stablized by $\lambda^n K' \lambda^{-n}$. Since $\overline{U} = \cup_{n \geq 0} \lambda^n K'_{\overline{U}} \lambda^{-n}$, we could choose a large n, such that v_n is stabilized by $K_{\overline{U}}$. Now set $\widetilde{v} := 1_U * v_n$. Then

$$1_K * \widetilde{v} = \widetilde{v}, \tag{2.14}$$

i.e. $\widetilde{v} \in V^K$. Moreover, $p(\widetilde{v}) = \lambda^n v$. Therefore, for any $v \in r_P(V)^{K_M}$, there exists some n, such that $\lambda^n v \in \mathrm{Im}(p)$. Since $r_P(V)^{K_M}$ is finitely dimensional, we can find some integer n_0, such that $\mathscr{A}^{n_0} v \in \mathrm{Im}(p)$ for any $v \in r_P(V)^{K_M}$. Therefore,

$$\mathrm{Im}(p) = r_P(V)^{K_M}. \tag{2.15}$$

If we decompose $V^K = \mathrm{Ker}(\mathscr{A}^{n_0}) \oplus \mathrm{Im}(\mathscr{A}^{n_0})$, we have $\mathrm{Im}(\mathscr{A}^{n_0}) \cong \mathrm{Im}(p)$. ∎

Remark 2.3.4. As in [Ber], one could refine it to drop admissible condition. The hard input is that any smooth representation V is admissible over $H(G) := C_c^\infty(G)$.

Lemma 2.3.5. *For any smooth representation V, $T_w := 1_{\mathcal{I}w\mathcal{I}} \in \mathscr{H}_{\mathcal{I}}$ is invertible for any $w \in W_{ext}$.*

Proof. We have proved that $T_w^2 = (q-1)T_w + q$ for simple affine reflection w. In particular, each element in $\mathscr{H}_{\text{aff}}$ is invertible. If $w \in \Omega$ has length zero, then w stabilizes the chamber defining $\mathcal{I}$, and hence T_w has inverse given by $T_{w^{-1}}$, and $T_w T_{w'} = T_{ww'}$ for any $w' \in W_{\text{ext}}$. □

Corollary 2.3.6. *Given any smooth (admissible) representation V of $G(F)$, the natural projection*

$$V^{\mathcal{I}} \to r_B(V)^{T(\mathscr{O})} \tag{2.16}$$

is an isomorphism.

Remark 2.3.7. If we take $V = C_c^\infty(G(F)/\mathcal{I})$ acted by $\mathcal{I}$ via left translation, then $V^{\mathcal{I}} \cong C_c^\infty(\mathcal{I}\backslash G(F)/\mathcal{I})$, and $r_B(V)^{T(\mathscr{O})} \cong C_c^\infty(T(\mathscr{O})N(F)\backslash G(F)/\mathcal{I})$. Then we obtain $M \cong \mathscr{H}$ linear isomorphism as in the previous talk. Then for any unramified character χ, $M \otimes_R \chi \cong \mathscr{H} \otimes_R \chi$ gives $I(\chi)^{\mathcal{I}} \cong \mathscr{H}_{fin}$ as linear spaces. This gives another proof of Prop.2.2.2.

Corollary 2.3.8. *If V is an irreducible admissible representation of $G(F)$ with $V^I \neq 0$, if and only if V is a subrepn of an unramified principal series.*

Proof. If V is an (irreducible) admissible representation of $G(F)$, then by adjunction

$$\begin{aligned}
\operatorname{Hom}_{G(F)}(V, I(\chi)) &\cong \operatorname{Hom}_{T(F)}(r_B(V), \chi\delta^{1/2}) \\
&\cong \operatorname{Hom}_{T(F)}(r_B(V), (\chi\delta^{1/2})^{T(\mathscr{O})}) \\
&\cong \operatorname{Hom}_{T(F)}((r_B(V))_{T(\mathscr{O})}, \chi\delta^{1/2}) \\
&\cong \operatorname{Hom}_{T(F)}((r_B(V))^{T(\mathscr{O})}, \chi\delta^{1/2}) \text{ (invariants=coinvariants for cpt grp by [Bor76, sec.1.3])}.
\end{aligned}$$

Then use $r_B(V)^{\mathcal{I}} \cong V^{\mathcal{I}}$ as $\mathbb{C}$-vector spaces. □

Corollary 2.3.9. *The principal unramified series $I(\chi)$ is generated by $I(\chi)^{\mathcal{I}}$.*

Proof. Set V to be the subrepn generated by $I(\chi)^{\mathcal{I}}$. Then $(I(\chi)/V)^{\mathcal{I}} = 0$. Taking contragredient, $\widetilde{I(\chi)/V}$ is a subrepn of $I(\chi^{-1})$. It has no nontrivial $\mathcal{I}$-fixed vectors, and thus by Cor.2.3.8, it is zero, i.e. $I(\chi) = V$. □

Remark 2.3.10. The proof exactly shows that if a G-representation V is a subquotient of some $I(\chi)$, then $V^{\mathcal{I}} = 0$ if and only if $V = 0$.

2.4. **Blocks.** The main reference for this subsection is [Bor76].

Some generalities. Let K be a compact open subgroup of $G(F)$.

Definition 2.4.1. We define $\operatorname{Rep}(G)^{[K]}$ to be the full subcategory of $\operatorname{Rep}(G)$ consisting of smooth representations V of $G(F)$ generated by K-fixed vectors V^K.

We define $\mathscr{H}(G, K) := C_c^\infty(K\backslash G(F)/K, \mathbb{C})$, which is an algebra under convolution and $\mathscr{H}(G) = C_c^\infty(G(F), \mathbb{C})$.

Then there is an adjoint pair

$$F_K := (-) \otimes_{\mathscr{H}(G,K)} \mathscr{H}(G) : \mathrm{Rep}(\mathscr{H}(G,K)) \to \mathrm{Rep}^{\mathrm{sm}}(G)^{[K]} : (-)^K. \tag{2.17}$$

Easy observations:

- $(-)^K$ is exact, since $(-)^K$ is exact.
- F_K is the right quasi-inverse of $(-)^K$.
- F_K is fully faithful, since $\mathrm{Hom}(F_K(x), F_K(y)) \cong \mathrm{Hom}(x, F_K(y)^K) \cong \mathrm{Hom}(x,y)$.

Remark 2.4.2. A fact: $(-)^K$ induces a bijection of (isomorphic classes) of irreducible objects on both sides, although F_K does not preserve irreducible objects in general. But given an irreducible $V \in \mathrm{Rep}(\mathscr{H}(G,K))$, $F_K(V)$ has a irreducible quotient. See [BK91, Prop.4.2.3]

Lemma 2.4.3. *The functor F_K and $(-)^K$ are quasi-inverse (and hence induce equivalence of categories) if and only if $\mathrm{Rep}^{sm}(G)^K \subseteq \mathrm{Rep}^{sm}(G)$ is stable under taking subobjects.*

Proof. For the if part: we only need to show the counit

$$\mathscr{H}(G) \otimes_{\mathscr{H}(G,K)} (V^K) \to V \tag{2.18}$$

is an isomorphism for $V \in \mathrm{Rep}(G)^{[K]}$. Since V is generated by V^K, we know this morphism is surjective. Let W be the kernel and then we have $W^K = 0$. Since $\mathrm{Rep}(G)^{[K]}$ is stable under taking subobjects, W is generated by W^K and thusly $W = 0$. Therefore, the counit is an isomorphism. We omit the proof for "only if" part, since it is easy and it is not needed in our talk. □

Now we know that the unramified principal series $I(\chi)$ lies in Iwahori block $\mathrm{Rep}(G)^{[\mathcal{I}]}$ by Prop.2.3.9.

Proposition 2.4.4. *Let E be an admissible representation of $G(F)$. Assume that*

(1) E is generated by $E^{\mathcal{I}}$;

(2) For any non-zero subrepresentation $W \subseteq E$, $W^{\mathcal{I}} \neq 0$.

Then

(1) If $0 \to E \to V \to Q \to 0$ is a short exact sequence of admissible $G(F)$-representations, and $Q^{\mathcal{I}} = 0$, then the sequence splits.

(2) For any $W \subseteq E$, W is generated by $W^{\mathcal{I}}$.

Proof. (1) implies (2). Let $V \subseteq W$ be the subrepresentation generated by W^I. Consider the short exact sequence $0 \to V \to W \to W/V \to 0$. Then V satisfy the condition in (1). Therefore, the sequence splits $W = V \oplus W/V$. However, $(W/V)^{\mathcal{I}} = W^{\mathcal{I}}/V^{\mathcal{I}} = 0$, and thus $W/V = 0$, i.e. $V = W$.

So we only need to prove(1). Since Jacqute functor is exact, we have

$$0 \to r_B(E) \to r_B(V) \to r_B(Q) \to 0$$

exact, and since $T(\mathscr{O})$ is compact (and we work over $\mathbb{C}$), taking $T(\mathscr{O})$-invaraint is exact. But $r_B(Q)^{T(\mathscr{O})} \cong Q^{\mathcal{I}} = 0$. Therefore, we know that

$$r_B(V)^{T(\mathscr{O})} \cong r_B(E)^{T(\mathscr{O})}. \tag{2.19}$$

Now we have decomposition of $r_B(V)$ under $T(\mathscr{O})$-action (the category of $\mathbb{C}$-representation of compact groups is semisimple):

$$r_B(V) = r_B(E)^{T(\mathscr{O})} \oplus V', \tag{2.20}$$

where V' consists of isotropic spaces with nontrivial $T(\mathscr{O})$-action. Then we have a projection map

$$\beta : r_B(V) \to E^{\mathcal{I}}, \tag{2.21}$$

and by adjunction we get

$$\widetilde{\beta} : V \to i_B(E^{\mathcal{I}}). \tag{2.22}$$

Then we have

$$V \cong \mathrm{Ker}(\widetilde{\beta}) \oplus \mathrm{Im}(\widetilde{\beta}). \tag{2.23}$$

Then we only need to show:

- $E \cap \mathrm{Ker}(\widetilde{\beta}) = 0$.
- $\widetilde{\beta}(V) = \widetilde{\beta}(E)$.

Note that $\widetilde{\beta}|_{\mathcal{I}} : V^{\mathcal{I}} \cong E^{\mathcal{I}} \to (i_B(E^{\mathcal{I}}))^{\mathcal{I}}$ is injective. Then $\ker(\widetilde{\beta}) \cap E = 0$.

Now consider $J := \widetilde{\beta}(V)/\widetilde{\beta}(E)$. Then $J^{\mathcal{I}} = 0$ since $J^{\mathcal{I}}$ is a quotient of $(V/E)^{\mathcal{I}}$. Then by Remark 2.3.10, we see $J = 0$. □

Now apply the proposition to $I(\chi)$ and combine it with our previous subsection, we know that

- any subquotient of $I(\chi)$ is generated by $\mathcal{I}$-fixed points, i.e. any subquotient of $I(\chi) \in \mathrm{Rep}(G)^{[\mathcal{I}]}$;
- any (admissible) V generated by $V^{\mathcal{I}}$ is a sub of $I(\chi)$ for some χ;
- In particular, $\mathrm{Rep}(G)^{[\mathcal{I}]}$ is closed under taking subobjects, and hence $\mathrm{Rep}(G)^{[\mathcal{I}]} \cong \mathrm{Rep}(\mathscr{H}_{\mathcal{I}})$.

Therefore, we proved the main results at the beginning of this talk.

2.5. **K-Type Theory.** I could mention more generality if time permits. May follow [BK98].

3. Algebraic K-theory, Jiahao Niu

3.1. **Introduction.** Let $\mathscr{A}$ be an abelian category. Then $K_0(\mathscr{A})$ is the free abelian group generated by $A \in \mathrm{Ob}(\mathscr{A})$, quotient by the relation $[A] = [A'] + [A'']$, where $0 \to A' \to A \to A'' \to 0$ is a short exact sequence.

Quillen defined an invariant $K(\mathscr{A})$ as a spectra, and $K_i(\mathscr{A}) = \pi_i(K(\mathscr{A}))$.

One can specialize to the case $\mathscr{A} = \mathrm{Coh}(A)$, where A is a Noetherian commutative ring. Then $K_i(A) := K_i(\mathrm{Coh}(A))$. One can extend this construction to X qcqs Noetherian scheme and $K_i(X) = K_i(\mathrm{Coh}(X))$.

If U is open in X with Z closed complement, then we have a long exact sequence

$$\cdots \to K_{i+1}(U) \to K_i(Z) \to K_i(X) \to K_i(U) \to \cdots \tag{3.1}$$

Today our main goal is to review the equivariant version. Consider X a Noetherian scheme acted by G an affine algebraic group and we are interested in $K_i^G(X) = K_i(\mathrm{Coh}_G(X)) = K_i(\mathrm{Coh}([X/G]))$.

Example 3.1.1. $K_0([*/G]) = K_0(\mathrm{Rep}_{f.g.}(G))$. When $G = T$ is a split torus, then $K_0(*/T) = \mathbb{Z}[X^*(T)]$. Higher K-groups are hard to compute.

Example 3.1.2. Let F be a field. Then

- $K_0(F) = \mathbb{Z}$.
- $K_1(F) = \mathbb{F}^{\times}$.
- $K_2(F) = F^{\times} \otimes F^{\times} / < a \otimes (1-a) | a \neq 0, 1 >$.

and

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}/q^j - 1, & \text{if } i = 2j-1; \\ 0, & \text{if } i \text{ even, non-zero.} \end{cases} \tag{3.2}$$

In general, it is hard.

Properties:

(1) projective bundle formula: $K_i(\mathbb{P}^n_X) = K_i(X)^{\oplus n}$.

(2) Thom isomorphism (homotopy invariant): $K_i(\mathbb{A}^n_X) \cong K_i(X)$.

(3) computation tools: cellular fibration, Kunneth formula.

Before doing that, we will give a general construction.

3.2. **Modern Treatment.** Replace abelian category with a small stable (∞-)category, idempotent complete. Denote that category of such categories by $\mathrm{Cat}^{\mathrm{perf}}$, with functors being exact functors. Recall that by $(-)^{\omega}$ and Ind-completion, we have equivalences

$$\mathrm{Cat}^{\mathrm{perf}} \cong \mathrm{Cat}_{st}^{\mathrm{cpt\ gen}}, \tag{3.3}$$

where functors on the right-hand side are continuous functors preserving compact objects.

Fact: $\mathrm{Cat}_{st}^{\mathrm{cpt\ gen}}$ is generated by Sp, the category of spectra. Let S denote the category of spaces (equivalent to the category of CW complexes modulo homotopy equivalences or the category of all topological spaces modulo weak homotopy equivalences). Then Sp is the stablization of S, which is a subcategory of $\mathrm{Fun}(S^{\mathrm{fin}_*}, S_*)$, satisfying $F(*) = *$, and F carries pullback diagrams to pushout diagrams. Morally, Sp classifies all cohomology theories. Actually, Sp is generated under colimits by one object spherical spectra.

Note that $\mathrm{Cat}^{\mathrm{perf}}$ is a presentable category.

If $A \subseteq B$ is a full subcategory, and

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & C \end{array} \tag{3.4}$$

is a pushout diagram, then we call $A \to B \to C$ is a Karoubi sequence, and denote $C = B/A$, and call it Karoubi quotient (Karoubi localization).

Warning: $B \to B/A$ may not be essentially surjective.

For $i : A \to B$, B/A is taking the compact objects of $\mathrm{Ker}(i^R)$.

Example 3.2.1. X qcqs scheme, Then $\mathrm{Perf}(\hat{X|}_Z) \to \mathrm{Perf}(X) \to \mathrm{Perf}(U)$ and simlar for Coh, where $\hat{X}_Z$ is the formal completion of X along Z.

Definition 3.2.2. Let $\mathscr{E}$ be a presentable stable category. A functor $F : \mathrm{Cat}^{\mathrm{pref}} \to \mathscr{E}$ is called localizing invariant, if $F(0) = 0$, and F sends karoubi sequence to cofiber sequence in $\mathscr{E}$. We say F is additive invariant, if $F(B) = F(A) \oplus F(C)$, if (A, C) forms a semi-orthogonal of B.

Localizing invariant implies additive invariant.

Definition 3.2.3. Let A, B be two full subcategories of $C \in \mathrm{Cat}^{\mathrm{perf}}$. We say A, B forms a semi-orthogonal decompostion if $\mathrm{Hom}(a, b) = 0$ for any $a \in A, b \in B$, and for any object $c \in C$, we can find cofiber sequence $a \to c \to b$ where $a \in A, b \in B$.

Example 3.2.4. Consider $A \xrightarrow{i} C \xrightarrow{j} B$ a Karoubi sequence. If j^R is fully faithful, then $i(A), j^R(B)$ forms a semi-orthogonal decomposition.

Fact: if $\mathscr{C}$ admits countable direct sum and F preserves filtered colimit, then $F(C) = 0$.

We can extend F to $\mathrm{Cat}^{\mathrm{cpt\ gen}} \to \mathscr{E}$ by $F(C) : F(C^\omega)$. The problem is that $\mathrm{Ker}(\mathrm{Ind}(A) \to \mathrm{Ind}(B))$ might not be compactly generated, even when the functor is a localization functor. But the Kernel is always dualizable. Effinov constructed continuous K theory

$$F^{\mathrm{cts}} : \mathrm{Cat}^{\mathrm{dualizable}} \to \mathscr{E}. \tag{3.5}$$

Definition 3.2.5. K-theory is the initial localizing invariant $F : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$ commuting with filtered colimit, and together with $\mathscr{C}^{\cong} \to \Omega^\infty K(\mathscr{C})$.

Remark 3.2.6. $\pi_i K(\mathscr{C})$ may not be zero for $i < 0$.

For $A \in \mathrm{CRing}$ (may be derived),

- $\mathrm{Perf}(A) \subset D(A)$ generated by A under finite colimit and retraction, equivalent to $D(A)^\omega$ and $D(A)^{\mathrm{dualizable}}$.
- A Noetherian, $D_{\mathrm{Coh}}(A) \subseteq D(A)$ consisting of bounded complex $X \in D(A)$, and $\pi_i(X)$ are finitely generated $\pi_0(A)$-modules.
- $D(A)$ has a t-structure induces a t-structure on $D_{\mathrm{Coh}}(A)$, such that $D_{\mathrm{Coh}}(A)^\heartsuit \cong \mathrm{Coh}(\pi_0(A))$.

Usually, $K(\mathrm{Perf}(A))$ algebraic K-theory of A and for A Noetherian, $K(D_{\mathrm{Coh}}(A))$ is called algebraic G-theory of A.

Fact: $K(D_{\mathrm{Coh}}A) = K(\pi_0(A))$. Moreover, $K(D_{\mathrm{Coh}}(A))$ is connective.

Extend this construction to algebraic stack $\mathscr{X}$.

$$\mathrm{Perf}(X) = \varprojlim_{\mathrm{Spec}\, A \xrightarrow{\mathrm{smooth}} \mathscr{X}} \mathrm{Perf}(A), \tag{3.6}$$

and simlar for $\mathrm{D_{Coh}}(\mathscr{X})$. Then $K^{\mathrm{perf}}(\mathscr{X}) := K(\mathrm{Perf}(X))$ and $K(\mathscr{X}) := K(\mathrm{D_{Coh}}(X))$ satisfy Zariski descent and Nisevich descent. But $K(\mathrm{D_{Coh}}(X)) \otimes \mathbb{Q}$ satisfy fppf descent when $\mathscr{X}$ is a qcqs scheme.

Passing to étale sheafification, we obtain étale K-theory $K^{\text{ét}}(D_{\mathrm{Coh}}(X))_{\mathbb{Q}}$.

Theorem 3.2.7. *$U \subset \mathscr{X}$ is open with closed complement Z, we have cofiber sequence*

$$K(Z) \to K(\mathscr{X}) \to K(U). \tag{3.7}$$

In particular, $K(Z) = K(Z_{red})$.

Theorem 3.2.8. *$\mathscr{E}$ is a locally free sheaf over $\mathscr{X}$ of rank $n+1$, then*

$$K(\mathbb{P}_{\mathscr{E}}(\mathscr{X})) \cong K(\mathscr{X})^{\oplus n+1}. \tag{3.8}$$

The reason is that $D_{\text{Coh}}(\mathbb{P}_{\mathscr{E}}(X))$ has a semi-orthogonal decomposition $(\mathscr{O}(n), \mathscr{O}(n-1), \dots, \mathscr{O}$. Denote $q : \mathbb{P}_{\mathscr{E}}(\mathscr{X}) \to \mathscr{X}$, then obtain maps

$$p_i : D_{\text{Coh}}(\mathscr{X}) \to D_{\text{Coh}}(\mathbb{P}_{\mathscr{E}}(\mathscr{X})), \tag{3.9}$$

given by $q^*(-) \otimes \mathscr{O}(i)$. Then $D_{\text{Coh}}(\mathbb{P}_{\mathscr{E}}(\mathscr{X}))$ has semi-orthogonal decomposition $(\mathscr{C}_0, \dots, \mathscr{C}_n)$, where $\mathscr{C}_i := \text{Im}(p_i)$.

Corollary 3.2.9. $K(\mathbb{V}_{\mathscr{E}}(\mathscr{X})) \cong K(\mathscr{X})$.

Not true for K^{perf}.

Remark 3.2.10. When X is a smooth scheme, then Perf $= D_{\text{Coh}}$ and two K-theory agrees with each other.

Then $K^G(X) := K_0^G(X) := K_0([X/G])$.

Example 3.2.11. $K^H(X) \cong K^G(G \times^H X)$.

Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of stacks. Then on D_{Coh}, f^* is defined when f has finite Tor dimension, and f_* is defined when f is proper and has finite cohomological dimension.

$K(\mathscr{X})$ is a module over $K^{\text{perf}}(\mathscr{X})$.

Convolution product on $\text{D}_{\text{Coh}}(\mathscr{X} \times_{\mathscr{Y}} \mathscr{X})$ induces algebraic structure on $K(\mathscr{X} \times_{\mathscr{Y}} \mathscr{X})$.

G affine algebraic group and $G = U \rtimes R$, where U is unipotent and R is reductive. Then $K^G(X) \cong K^R(X)$. This is because

$$K^R(X) = K^G(G \times^R X) \cong K^G(G/R \times X) = K^G(U \times X). \tag{3.10}$$

Then filter U by affine spaces and then invoke homotopy invariance.

A G-equivariant morphism $F \xrightarrow{\pi} X$ is called cellular fibration, if $F^\bullet$ a decreasing filtration by closed subschemes, such that

(1) $F^i \to X$ is a locally trivial fibration,
(2) $E_i := F^i \backslash F^{i+1} \to X$ is an affine bundle,

then

- we have short exact sequence $0 \to K^G(F^i) \to K^G(F^{i+1}) \to K^G(E_i) \to 0$.

Proof. The surjective map $K_1^G(F_i) \to K_1^G(E_i) \cong K_1^G(X)$ has a section given by π^*. □

Theorem 3.2.12. *Let G be a reductive group acting on X a smooth projective variety. TFAE*

(1) $\pi : K^G(X) \otimes_{R(G)} K^G(Y) \to K^G(X \times Y)$ is an isomorphism for any Y;
(2) the image of $\pi : K^G(X) \otimes_{R(G)} K^G(X) \to K^G(X \times X)$ contains $\Delta_* \mathscr{O}_X$;
(3) $K^G(X)$ is finitely generated projective $R(G)$-module, and G-variety Y, $K^G(Y \times X) \cong \text{Hom}_{R(G)}(K^G(Y), K^G(X))$;

(4) $K^G(X)$ *is finitely generated projective,* $K^G(X\times X)$ *f.g. projective, and* $\mathrm{rank}(K^G(X\times X)) = \mathrm{rank}(K^G(X))$*, and*

$$<-,->: K^G(X)\times K^G(X)\to R(G) \tag{3.11}$$

given by $<\mathscr{F},\mathscr{G}>= p_*(\mathscr{F}\otimes^{\mathbb{L}}\mathscr{G})$ *is non-degenerate.*

Theorem 3.2.13. *G simply connected,* $K^G(X)\to K^T(X)$.

- $K^G(X)\otimes_{R(G)} R(T)\cong K^T(X)$.
- $K^G(X)\cong K^T(X)^W$.

Theorem 3.2.14. $K_0^{\acute{e}t}(\mathscr{X})_{\mathbb{Q}}\cong A_*(\mathscr{X},\mathbb{Q})$ *chow groups.*

Actually, the usual Chern class map $\mathrm{ch}_* : K(\mathscr{X})\to A_*$ factors through étale K theory.

Let T be a torus over $\mathbb{C}$, acting on X a scheme of finite type over $\mathbb{C}$ and $t\in T$. Denote $i : X^t\hookrightarrow X$ inclusion of t-fixed points. Then $i_* : K^T(X^t)\to K^T(X)$ becomes an isomorphism after localizing $S\subseteq K^T(\mathrm{pt}) = \mathbb{Z}[X_*(T)]$, consisting of those f such that $f(t)\neq 0$.

4. Anti-spherical Modules, Daniel Kim

4.1. **Setup.**

- F: an NA local field, with ring of integers $\mathscr{O}$ and residue field k;
- G: a reductive split group over F;
- $B\supset T$: a Borel containing a split maximal torus;
- $\mathcal{I} = \mathrm{Ker}(G(\mathscr{O})\xrightarrow{\mathrm{ev}} G(k))$ Iwahori;
- Extended affine Weyl group $\widetilde{W} := W_{\mathrm{ext}} := N_{G(F)}(T(F))/T(\mathscr{O})$, $\widetilde{W} = X_*(T)\rtimes W$, where W is the finite Weyl group;
- $\mathscr{H}_{\mathcal{I}} = C_c(\mathcal{I}\backslash G(F)/\mathcal{I},\mathbb{C})$ Iwahori Hecke algebra.

Recall we have Iwasawa decomposition

$$G(F) = \bigsqcup_{w\in\widetilde{W}}\mathcal{I}\dot{w}\mathcal{I}, \tag{4.1}$$

where $\dot{w}$ is any lift of w. Then $T_w := 1_{\mathcal{I}w\mathcal{I}}\in\mathscr{H}_{\mathcal{I}}$. Then we have Iwahori-Matsumoto's presentation

- $(T_s+1)(T_s-q) = 0$, where s is a simple affine reflection in $W_{\mathrm{aff}}\subseteq\widetilde{W}$.
- $T_{vw} = T_vT_w$ if $l(vw) = l(v)+l(w)$.

Remark 4.1.1. We can treat q as a formal variable and work over $\mathbb{Z}[q]$.

If $\lambda,\mu\in\mathbb{X}_*(T)_+$ dominant, then $T_{\lambda+\mu} = T_\lambda T_\mu = T_\mu T_\lambda$. If we work over $\mathbb{Z}[q^{\pm1}]$, then each T_i is invertible.

For $\lambda\in X_*(T)$, we can write $\lambda = \lambda_1-\lambda_2$, for $\lambda_1,\lambda_2\in X_*(T)_+$, we can define

$$e^\lambda := q^{<-\rho,\lambda>}T_{\lambda_1}T_{\lambda_2}^{-1} \tag{4.2}$$

well-defined and $e^{\lambda+\mu} = e^\lambda e^\mu$.

Theorem 4.1.2 (Bernstein)**.** *$e^\lambda T_w$ for $\lambda \in X_*(T)$ and $w \in W = W_{fin}$ forms a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis for $\mathscr{H}_\mathcal{I} \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{Z}[q^{\pm\frac{1}{2}}]$. The relation is given by*

$$T_s e^{s(\lambda)} - e^\lambda T_s = (1-q)\frac{e^\lambda - e^{s(\lambda)}}{1-e^{\alpha^\vee}}, \tag{4.3}$$

where $s = s_\alpha$ is a simple finite reflection.

Remark 4.1.3. If G is simply connected, then we don't need to add $q^{\pm\frac{1}{2}}$.

4.2. **Steinberg Variety.** Let $\hat{G}$, $\hat{B}$, $\hat{T}$ be the dual groups over $\mathbb{C}$.

Definition 4.2.1. An element $x \in \hat{\mathfrak{g}}$ is called nilpotent, when its image under $\hat{\mathfrak{g}} \to \mathfrak{gl}_n$ is.

Fact: $\hat{\mathfrak{n}} := \mathrm{Lie}(\hat{N})$ icovers all nipotent elements under the adjoint action.

Definition 4.2.2. Let $\mathscr{N} \subseteq \hat{\mathfrak{g}}$ be the nilpotent cone, i.e. the set of all nilpotent elements, regarded as a variety.

Definition 4.2.3. Springer resolution

$$\pi : \widetilde{\mathscr{N}} := \{(x, g\hat{B}) \in \mathscr{N} \times \hat{G}/\hat{B} : x \in g\hat{\mathfrak{n}}g^{-1}\} \to \mathscr{N}. \tag{4.4}$$

Note that π is proper and resolution of singularities.

Given $g\hat{B}$, $(\mathrm{Ad}_g\hat{\mathfrak{n}})^\perp = \mathrm{Ad}_g\hat{\mathfrak{b}}$ under the Killing form, and is the fiber of $\mathfrak{g} \to T_{g\hat{B}}(\hat{G}/\hat{B})$. Then $\mathrm{Ad}_g\hat{\mathfrak{n}} \cong T^*_{g\hat{B}}(\hat{G}/\hat{B})$. Therefore, $\widetilde{\mathscr{N}} \cong T^*(\hat{G}/\hat{B})$.

Definition 4.2.4. We define $\mathscr{Z} := \widetilde{\mathscr{N}} \times^{\mathbb{L}}_{\hat{\mathfrak{g}}} \widetilde{\mathscr{N}}$ the Steinberg variety.

Want to understand K-theory of $\mathscr{Z}$ (actually G-theory, i.e. K-groups of Coh). Note that $K(\mathscr{Z})$ has a ring structure given by convolution pattern, and that's why we defined $\mathscr{Z}$ derivedly: to make the pullback functor well-defined (preserving D^b_{Coh}). Namely, we have a base change diagram (both derived and non-derived)

$$\begin{array}{ccc} \widetilde{\mathscr{N}} \times_{\hat{\mathfrak{g}}} \widetilde{\mathscr{N}} \times_{\hat{\mathfrak{g}}} \widetilde{\mathscr{N}} & \xrightarrow{\mathrm{id}\times\Delta\times\mathrm{id}} & \widetilde{\mathscr{N}} \times_{\hat{\mathfrak{g}}} \widetilde{\mathscr{N}} \times \widetilde{\mathscr{N}} \times_{\hat{\mathfrak{g}}} \widetilde{\mathscr{N}} \\ \downarrow & & \downarrow \\ \widetilde{\mathscr{N}} & \xrightarrow[\Delta]{} & \widetilde{\mathscr{N}} \times \widetilde{\mathscr{N}} \end{array} \tag{4.5}$$

We know that Δ is of finite tor dimension, so the map $\mathrm{id} \times \Delta \times \mathrm{id}$ is of fintie tor dimension if the diagram is derived.

Consider that $\hat{G} \times \mathbb{G}_m$ acts on $\widetilde{\mathscr{N}} \to \hat{\mathfrak{g}}$, and $\mathscr{Z}$ has a $\hat{G} \times \mathbb{G}_m$-action. Therefore, $K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z})$ is defined.

Theorem 4.2.5. *We have an isomorphism $K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}) \cong \mathscr{H}_\mathcal{I}$.*

Remark 4.2.6. Another way is to consider Grothendieck alteration $\widetilde{\mathfrak{g}} \to \mathfrak{g}$, and consider $\mathrm{Coh}(\widetilde{\mathfrak{g}} \times_\mathfrak{g} \widetilde{\mathfrak{g}}|_{\mathscr{N}\times_\mathfrak{g}\mathscr{N}})$, the bounded derived category of coherent sheaves on the formal completion along (underived) $\widetilde{\mathscr{N}} \times_\mathfrak{g} \widetilde{\mathscr{N}} \subseteq \widetilde{\mathfrak{g}} \to \mathfrak{g}$. The G-theories obtained are the same (though of course the original stable categories are different).

4.3. **Subalgebras.** Consider $\Delta : \widetilde{\mathscr{N}} \to \widetilde{\mathscr{N}} \times_{\hat{\mathfrak{g}}} \widetilde{\mathscr{N}} = \mathscr{Z}$ is $\hat{G} \times \mathbb{G}_m$-equivariant. Then we have

$$\Delta_* : K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}}) \to K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}). \tag{4.6}$$

Claim: Δ_* is a ring homomorphism.

This is an exercise of base change theorem (another place where we want everything derived, although the G theory only reflects underlying π_0).

Using $\widetilde{\mathscr{N}} = T^*\hat{G}/\hat{B}$

$$\begin{aligned} K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}}) &\cong K^{\hat{G}\times\mathbb{G}_m}(\hat{G}/\hat{B}) \\ &\cong K((\hat{G}/\hat{B})/(\hat{G}\times\mathbb{G}_m)) \\ &\cong K(*/(\mathbb{G}_m\times\hat{B})) \\ &\cong R(\hat{T}\times\mathbb{G}_m) \\ &\cong \mathbb{Z}[X_*(T)\oplus\mathbb{Z}] \\ &\cong \mathbb{Z}[X_*(T)][q^{\pm1}]. \end{aligned} \tag{4.7}$$

Theorem 4.3.1. *There is a diagram of algebras*

$$\begin{array}{ccc} K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}}) & \xrightarrow{\Delta_*} & K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}) \\ \cong\downarrow\gamma & & \downarrow\cong \\ \mathbb{Z}[X_*(T)][q^{\pm1}] & \longrightarrow & \mathscr{H}_{\mathcal{I}} \end{array} \tag{4.8}$$

where the bottom map is $\lambda \mapsto e^\lambda$ and γ matches $[\mathscr{O}_\lambda]$ with $-\lambda$, where $\mathscr{O}_\lambda$ is the line bundle on $\hat{G}/\hat{B}$ corresponding to $\hat{B} \xrightarrow{\lambda} \mathbb{G}_m$, pulled back along $\widetilde{\mathscr{N}} \to \hat{G}/\hat{B}$, with trivial $\mathbb{G}_m$-action.

$\hat{G}$ acts on $\mathbb{Z}$ and diagonally acts on $\hat{G}/\hat{B} \times \hat{G}/\hat{B}$, and the map $\mathscr{Z} \to \hat{G}/\hat{B} \times \hat{G}/\hat{B}$ is equvariant under $\hat{G}$-action, mapping $(x, g_1\hat{B}, g_2\hat{B})$ to $(g_1\hat{B}, g_2\hat{B})$.

We have $\hat{G}/\hat{B} \times \hat{G}/\hat{B} \cong \bigsqcup_{w\in W} Y_w$ decomposition by relative positions (Bruhat decomposition), and pullback to

$$\mathscr{Z} = \bigsqcup_{w\in W} \pi^{-1}(Y_w). \tag{4.9}$$

Each $\pi^{-1}(Y_w) \to Y_w$ is a vector bundle.

Actually

$$\pi^{-1}(Y_w) = T^*_{Y_w}(\hat{G}/\hat{B} \times \hat{G}/\hat{B}) := \mathrm{Ker}(T^*(\hat{G}/\hat{B} \times \hat{G}/\hat{B})|_{Y_w} \to T^*_{Y_w}).$$

We have dimension counting

$$\begin{aligned} \dim(\pi^{-1}(Y_w)) &= \dim Y_w + \dim(\text{fiber}) \\ &= (\dim \hat{G}/\hat{B} + l(w)) + (\dim \hat{G}/\hat{B} - l(w)) \\ &= 2\dim \hat{G}/\hat{B}. \end{aligned} \tag{4.10}$$

and hence $\overline{\pi^{-1}(Y_w)}$ are the irreducible components of $\mathscr{Z}$, where we observed that the fiber is merely $\mathfrak{n} \cap w\mathfrak{n}$.

Fact: for $s \in W$ a simple reflection, $\pi_1, \pi_2 : \overline{Y_s} \to \hat{\mathrm{G}}/\hat{\mathrm{B}}$ are both $\mathbb{P}^1$-bundles.

Consider $\Omega^1_{\overline{Y_s}/(\hat{G}/\hat{B})}$ (relative to π_1), and pullback along $T^*_{\overline{Y_s}}(\hat{G}/\hat{B} \times \hat{G}/\hat{B}) \to \overline{Y_s}$, which we denote by Q_s. Claim: under the isomorphism $\mathscr{H}_{\mathcal{I}} \cong K^{\hat{G} \times \mathbb{G}_m}(\mathscr{Z})$, T_s corresponds to $-[qQ_s] - [\mathscr{O}_0]$.

4.4. **Modules.** Note that whenever $\hat{G} \times \mathbb{G}_m$-equivariant $Y \to \hat{\mathfrak{g}}$, $K^{\hat{G} \times \mathbb{G}_m}(\widetilde{\mathscr{N}} \times_{\hat{\mathfrak{g}}} Y)$ has a convolution action by $K^{G \times \mathbb{G}_m}(\mathscr{Z})$.

Take $Y = \hat{\mathfrak{g}}$, and we have $K^{\hat{G} \times \mathbb{G}_m}(\widetilde{\mathscr{N}}) \cong \mathbb{Z}[X_*(T)][q^{\pm 1}]$ as a left module. Then

$$R(\hat{G} \times \mathbb{G}_m) \xrightarrow{\pi^*} K^{\hat{G} \times \mathbb{G}_m}(\widetilde{\mathscr{N}}) \xrightarrow{\Delta_*} K^{\hat{G} \times \mathbb{G}_m}(\mathscr{Z}) \tag{4.11}$$

acts on $K^{\hat{G} \times \mathbb{G}_m}(\widetilde{\mathscr{N}})$.

Claim:

(1) the composition $\Delta_* \pi^*$ is central.

(2) the action of $K^{\hat{G} \times \mathbb{G}_m}(\widetilde{\mathscr{N}})$ on itself is multiplication.

Proof. For (1), given $V \in \mathrm{Rep}(\hat{G} \times \mathbb{G}_m)$, a $\hat{G} \times \mathbb{G}_m$-equivariant sheaf $\mathscr{F}$ on $\mathscr{Z}$, we have $\mathscr{F} * V \cong F * V$. For (2), use base change isomorphism. □

We have corresponding picture

$$Z(\mathscr{H}_{\mathcal{I}}) = \mathbb{Z}[X_*(T)]^W[q^{\pm 1}] \hookrightarrow \mathbb{Z}[X_*(T)][q^{\pm 1}] \hookrightarrow \mathscr{H}_{\mathcal{I}} \tag{4.12}$$

acts on $\mathscr{H}_{\mathcal{I}} \otimes_{\mathscr{H}_{\mathrm{fin}}, \epsilon} \mathbb{Z}[q^{\pm 1}]$, where ϵ is defined below.

Definition 4.4.1. $\mathscr{H}_{\mathrm{fin}} := \mathbb{Z}[q^{\pm 1}][T_w : w \in W]$ and

$$\begin{aligned} \epsilon : \mathscr{H}_{\mathrm{fin}} &\to \mathbb{Z}[q^{\pm 1}] \\ T_w &\mapsto q^{l(w)} \end{aligned} \tag{4.13}$$

.

The map ϵ is ring homomorphism using $(T_s + 1)(T_s - q) = 0$ for $s \in W$ simple reflection.

Lemma 4.4.2. *Identify $\mathscr{H}_{\mathcal{I}} \otimes_{\mathscr{H}_{fin,\epsilon}} \mathbb{Z}[q^{\pm 1}] \cong K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}})$ as $\mathbb{Z}[q^{\pm}1]$-modules via $e^{\lambda} \otimes 1 \mapsto [\mathscr{O}_{\lambda}]$. Then*

$$\begin{array}{ccccc} Z(\mathscr{H}_{\mathcal{I}}) = \mathbb{Z}[X_*(T)]^W[q^{\pm 1}] & \longrightarrow & Z(\mathscr{H}_{\mathcal{I}}) = \mathbb{Z}[X_*(T)][q^{\pm 1}] & & \mathscr{H}_{\mathcal{I}} \otimes_{\mathscr{H}_{fin,\epsilon}} \mathbb{Z}[q^{\pm 1}] \\ \cong\downarrow & & -\lambda\mapsto[\mathscr{O}_\lambda]\downarrow\cong & & \downarrow\cong \\ R(\hat{G}\times\mathbb{G}_m) & \longrightarrow & K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}}) & & K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}}) \end{array} \tag{4.14}$$

commutes.

Then

$$K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}) \xrightarrow{\alpha} \mathrm{End}_{R(\hat{G}\times\mathbb{G}_m)}(K^{\hat{G}\times\mathbb{G}_m}(\mathscr{N})) \cong \mathrm{End}_{\mathbb{Z}[X_*(T)]^W[q^{\pm 1}]}(\mathscr{H}_{\mathcal{I}} \otimes_{\mathscr{H}_{\mathrm{fin}},\epsilon} \mathbb{Z}[q^{\pm 1}]) \hookleftarrow \mathscr{H}_{\mathcal{I}}.$$

(1) Will prove that α is injective.
(2) The action of T_s and $-[qQ_s] - [\mathscr{O}_0]$ agrees, which will imply that $i : \mathscr{H}_{\mathcal{I}} \hookrightarrow K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z})$.
(3) Check that i is moreover surjective.

Recall that $\hat{G}/\hat{B} \times \hat{G}/\hat{B} = \bigsqcup_{w\in W} Y_w$ and $\mathscr{Z} = \bigsqcup_{w\in W} T^*_{Y_w}(\hat{G}/\hat{B} \times \hat{G})$. We write

$$\mathscr{Z}_{\leq w} := \bigcap_{v\leq w} T^*_{Y_v}(\hat{G}/\hat{B} \times \hat{G}). \tag{4.15}$$

and

$$\mathscr{Z}_{< w} := \bigcap_{v< w} T^*_{Y_v}(\hat{G}/\hat{B} \times \hat{G}). \tag{4.16}$$

Proposition 4.4.3. *There is a split short exact sequence*

$$0 \to K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}_{<w}) \to K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}_{\leq w}) \to K^{\hat{G}\times\mathbb{G}_m}(Y_w) \to 0. \tag{4.17}$$

Proof. Cell decomposition for $\pi_1 : \mathscr{Z} \to \hat{G}/\hat{B}$. Claim:

(1) $K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}_{\leq w})$ are $K^{\hat{G}\times\mathbb{G}_m}(\mathscr{N})$-submodules.
(2) $K^{\hat{G}\times\mathbb{G}_m}(Y_w) \cong \mathbb{Z}[X_*(T)][q^{\pm 1}]$. This is because $Y_w = \hat{G}/(\hat{B}\cap w\hat{B}w^{-1})$, and $R(\hat{B}\cap w\hat{B}w^{-1}) = R(T)$ by solvability.

□

Question: do we need the group G to be simply connected?

5. Coherent Realization of Affine Hecke Algebras, Vaughan Macdonald

5.1. Goal and Strategy. The goal of this talk is to finish the establishment of isomorphism

$$\theta : \mathscr{H}_{\mathcal{I}} \to K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}), \tag{5.1}$$

where $\mathscr{Z} := \widetilde{\mathscr{N}} \times^{\mathbb{L}}_{\hat{\mathfrak{g}}} \widetilde{\mathscr{N}}$ is the Steinberg variety.

Strategy: Fill in a diagram

$$\begin{array}{ccc} \mathscr{H}_{\mathcal{I}} & \hookrightarrow & \mathrm{End}_{R(\hat{T})^W[q^{\pm 1}]}(\mathscr{H}_{\mathcal{I}} \otimes_{\mathscr{H}_{\mathrm{fin},\epsilon}} \mathbb{Z}[q^{\pm 1}]) \\ & & \downarrow \cong \\ K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}) & \xrightarrow{\alpha} & \mathrm{End}_{K^{\hat{G}\times\mathbb{G}_m}(\mathrm{pt})}(K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}})) \end{array} \tag{5.2}$$

We need to show that

1. Show that α is injective;
2. Show that $\mathrm{im}(\mathscr{H}_{\mathcal{I}}) \subseteq (\alpha)$, therefore we obtain an algebra homomorphism θ.
3. Show surjectivity of θ.

Ideas for each step:

1. Use flatness of $K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z})$ over $R(\hat{T})^W[q^{\pm 1}]$, and use Kunneth formula and localization theorem.
2. For each simple reflection s, reduce to computing on $\mathscr{Z}_s := \widetilde{\mathscr{N}} \times_{\widetilde{\mathscr{N}_s}} \widetilde{\mathscr{N}} \subseteq \mathscr{Z}$ subvariety which is $\mathbb{P}^1$-bundle over $\widetilde{\mathscr{N}}$. This is done via explicit SL_2-calculation.
3. Show that θ is filtered and associated graded are isomorphisms under θ.

5.2. Injectivity of α. Consider the diagram

$$\begin{array}{ccc} K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}) & \longrightarrow & \mathrm{End}_{K^{\hat{G}\times\mathbb{G}_m}(\mathrm{pt})}(K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}})) \\ \downarrow \gamma_1 & & \downarrow \gamma_2 \\ K^{\hat{G}\times\mathbb{G}_m}(\mathrm{Fl}\times\mathrm{Fl}) & \xrightarrow{\beta} & \mathrm{End}_{R(\hat{G}\times\mathbb{G}_m)}(\mathrm{Fl}) \end{array} \tag{5.3}$$

where $\mathrm{Fl} := \hat{G}/\hat{B}$ is finite flag variety.

Then $\pi : \widetilde{\mathscr{N}} = T^*\mathrm{Fl} \to \mathrm{Fl}$ is an affine bundle, and pullback π^* induces isomorphism on equivariant K, i.e.

$$\pi^* : K^{\hat{G}\times\mathbb{G}_m}(\mathrm{Fl}) \xrightarrow{\cong} K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}}). \tag{5.4}$$

We have a projection map $\mathrm{id}\times\pi : T^*(\mathrm{Fl}\times\mathrm{Fl}) \to \widetilde{\mathscr{N}} \times \mathrm{Fl}$.

Then

- Fact: $(\mathrm{id}\times\pi)|_{\mathscr{Z}}$ is a closed embedding. Consider $j : \mathscr{Z} \hookrightarrow \widetilde{\mathscr{N}} \times \mathrm{Fl}$ and zero section $i : \mathrm{Fl}\times\mathrm{Fl} \hookrightarrow \widetilde{\mathscr{N}} \times \mathrm{Fl}$. We define $\gamma_1 := i^* j_*$.

- Fact: this diagram commutes (follows from some base change properties). Also β is an isomorphism precisely induced by Kunneth formula

$$\begin{aligned} K^{\hat{G}\times\mathbb{G}_m}(\mathrm{Fl}\times\mathrm{Fl}) &\cong K^{\hat{G}\times\mathbb{G}_m}(\mathrm{Fl})\otimes_{R(\hat{G}\times\mathbb{G}_m)} K^{\hat{G}\times\mathbb{G}_m}(\mathrm{Fl}) \\ &\cong K^{\hat{G}\times\mathbb{G}_m}(\mathrm{Fl})^\vee \otimes_{R(\hat{G}\times\mathbb{G}_m)} \quad \text{(using nondegenerate pairing)} K^{\hat{G}\times\mathbb{G}_m}(\mathrm{Fl}) \\ &\cong \mathrm{End}_{R(\hat{G}\times\mathbb{G}_m)}(K^{\hat{G}\times\mathbb{G}_m}(\mathrm{Fl})). \end{aligned}$$

Therefore, it suffices to show that $\gamma_1 := i^* j_*$ is injective.

From Daniel's talk, $\mathscr{Z}$ has a filtration given by $\mathscr{Z}_{\le w} := \bigsqcup_{w'\le w} T^*_{Y_w}(\mathrm{Fl}\times\mathrm{Fl})$, and we have short exact sequence

$$0 \to K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}_{<w}) \to K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}_{\le w}) \to K^{\hat{G}\times\mathbb{G}_m}(T^*_{Y_w}(\mathrm{Fl}\times\mathrm{Fl})) \to 0. \tag{5.5}$$

Consider

$$\mathscr{Z} \xhookrightarrow{j} \widetilde{\mathscr{N}}\times\mathrm{Fl} \xhookleftarrow{i} \mathrm{Fl}\times\mathrm{Fl} \xrightarrow{\pi_2} \mathrm{Fl} \tag{5.6}$$

which induces maps

$$\begin{array}{ccccc} K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}) & \longrightarrow & K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}}\times\mathrm{Fl}) & \longrightarrow & \mathrm{Fl}\times\mathrm{Fl} \\ \downarrow{\scriptstyle\mathrm{Res}} & & \downarrow{\scriptstyle\mathrm{Res}} & & \downarrow \\ K^{\hat{T}\times\mathbb{G}_m}(\mathscr{Z}_0) & \longrightarrow & K^{\hat{T}\times\mathbb{G}_m}(\widetilde{\mathscr{N}}) & \longrightarrow & K^{\hat{T}}(\mathrm{Fl})[q^{\pm1}] \end{array} \tag{5.7}$$

where Z_0 is the preimage of B under $\pi_2 : \mathscr{Z} \hookrightarrow \mathrm{Fl}\times\mathrm{Fl} \xrightarrow{\mathrm{pr}_2} \mathrm{Fl}$ Fact: Res are canonical isomorphisms. So it suffices to show the results for $\hat{T}$-equivariant K-theory.

In Daniel's talk, $K^{\hat{G}\times\mathbb{G}_m}(T^*_{Y_w}(\mathrm{Fl}\times\mathrm{Fl}))$ are free modules over $R(\hat{T})[q^{\pm1}]$.

Fix $a = (1, z) \in \hat{T}\times\mathbb{C}^\times$, and $z \neq 1$. Localize at a,

$$K^{\hat{T}\times\mathbb{G}_m}(\mathscr{Z}_0)_a \to K^{\hat{T}\times\mathbb{G}_m}(\widetilde{\mathscr{N}})_a \to K^{\hat{T}\times\mathbb{G}_m}(\mathrm{Fl})_a. \tag{5.8}$$

By localization theorem, just need to look at fixed points

$$\begin{array}{ccc} \mathscr{Z}_0^a & \hookrightarrow & \widetilde{\mathscr{N}}^a \\ \uparrow & & \uparrow \\ (Z_0\cap\mathrm{Fl})^a & \hookrightarrow & (\mathrm{Fl})^a \end{array} \tag{5.9}$$

All inclusions above become equalities on a-fixed points.

Remark 5.2.1. Recall that we have short exact sequence

$$0 \to K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}_{<w}) \to K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}_{\le w}) \to K^{\hat{G}\times\mathbb{G}_m}(T^*_{Y_w}(\mathrm{Fl}\times\mathrm{Fl})) \to 0.$$

By cell decomposition of $\mathrm{Fl}\times\mathrm{Fl}$, we also have short exact sequence

$$0 \to K^{\hat{G}\times\mathbb{G}_m}(\mathrm{Fl}\times\mathrm{Fl})_{<w}) \to K^{\hat{G}\times\mathbb{G}_m}((\mathrm{Fl}\times\mathrm{Fl})_{\le w}) \to K^{\hat{G}\times\mathbb{G}_m}((\mathrm{Fl}\times\mathrm{Fl})_w \to 0. \tag{5.10}$$

Since $\pi : T^*_{Y_w}(\mathrm{Fl}\times\mathrm{Fl}) \to (\mathrm{Fl}\times\mathrm{Fl})_w$ is an affine bundle, π^* induces an isomorphism on K-theory. However, pushforward along zero section is not isomorphism in general. Think of the simplest case: $0/\mathbb{G}_m \xrightarrow{i} \mathbb{A}^1/\mathbb{G}_m \xrightarrow{j} 0/\mathbb{G}_m$. Then j^* is an ismorphism $\mathbb{Z}[q,q^{-1}][\mathscr{O}] \cong \mathbb{Z}[q,q^{-1}]$ mapping 1 to $[\mathscr{O}]$. However i_* is not. Suppose this $\mathbb{G}_m$-action is weight n. Then δ_0 is resolved by

$$0 \to q^n\mathscr{O} \to \mathscr{O} \to \delta_0 \to 0. \tag{5.11}$$

Therefore, $[\delta_0] = (1-q^n)[\mathscr{O}]$. But away from roots of unities, it is an isomorphism.

5.3. **Matching the Actions.** Consider $G = \mathrm{SL}_2(\mathbb{C})$. Then $\mathscr{H}_{\mathcal{I}}$ is a $\mathbb{C}[q^{\pm 1}]$-algebra with 3 generators T, x, x^{-1}, such that

- $(T+1)(T-q) = 0$,
- $xx^{-1} = x^{-1}x = 1$,
- $Tx^{-1} - xT = (1-q)X$.

Write $C = -(T+1)$, and then

- $C^2 = -(q+1)C$,
- $Cx^{-1} - xC = qx - x^{-1}$.

On Steinberg side, $\mathrm{Fl} = \mathbb{P}^1$ and Bruhat decomposition reads as

$$\mathrm{Fl}\times\mathrm{Fl} = \mathbb{P}^1\times\mathbb{P}^1 = \mathbb{P}^1_\Delta \sqcup (\mathbb{P}^1\times\mathbb{P}^1 - \mathbb{P}^1_\Delta). \tag{5.12}$$

Denote $Y := \mathbb{P}^1\times\mathbb{P}^1 - \mathbb{P}^1_\Delta$. Then $\mathscr{Z} = T^*_{\mathbb{P}^1_\Delta}(\mathbb{P}^1\times\mathbb{P}^1) \sqcup T^*_Y(\mathbb{P}^1\times\mathbb{P}^1)$. Observe that we have an isomorphism $\pi_Y : T^*_Y(\mathbb{P}^1\times\mathbb{P}^1) \xrightarrow{\cong} \mathbb{P}^1\times\mathbb{P}^1$.

For sheaves, $\mathscr{O}_n := \Delta_*(\mathscr{O}(n))$ and $Q := \pi_Y^*\Omega_{\mathbb{P}^1\times\mathbb{P}^1/\mathbb{P}^1}$.

Theorem 5.3.1. *The assignment*

- $C \mapsto [qQ]$,
- $x \mapsto [\mathscr{O}_{-1}]$,
- $x^{-1} \mapsto [\mathscr{O}_1]$,

naturally extends to an algebra homomorphism $\theta : \mathscr{H}_I \xrightarrow{\cong} K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z})$, *i.e.*

- $qQ * qQ = -q(q+1)Q$;
- $qQ * \mathscr{O}_1 = \mathscr{O}_{-1} * qQ = q\mathscr{O}_{-1} - \mathscr{O}_1$.

Proof. Let $\pi : T^*\mathbb{P}^1 \to \mathbb{P}^1$ natural projection and $i : \mathbb{P}^1 \to T^*\mathbb{P}^1$ inclusion of zero section. Use two short exact sequence

(1) $0 \to \mathscr{O}_{T^*\mathbb{P}^1} \to \pi^*\Omega^1_{\mathbb{P}^1} \to i_*\Omega^1_{\mathbb{P}^1} \to 0$ Koszul complex of sheaves on $T^*\mathbb{P}^1$.

(2) $0 \to \mathscr{O}(-1)\boxtimes\Omega^1(1) \to \mathscr{O}\boxtimes\mathscr{O} \to \mathscr{O}_0 \to 0$ Beilinson resolution for sheaves on $\mathbb{P}^1\times\mathbb{P}^1$.

First note that $Q = \pi_Y^*\pi_2^*\Omega^1_{\mathbb{P}^1} = \mathscr{O}_{\mathbb{P}^1}\boxtimes\Omega^1_{\mathbb{P}^1}$. By short exact sequence (1), one has a short exact sequence

$$0 \to q^{-1}\mathscr{O}_{\mathbb{P}^1}\boxtimes\mathscr{O}_{T^*\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}\boxtimes\pi^*\Omega^1_{\mathbb{P}^1} \to Q \to 0. \tag{5.13}$$

We add q^-1 to make it an exact sequence of $\mathbb{G}_m$-equivariant sheaves. Then pass to K-theory, one gets

$$[qQ] = q\mathscr{O}_{\mathbb{P}^1}\boxtimes\pi^*\Omega^1_{\mathbb{P}^1} - \mathscr{O}_{\mathbb{P}^1}\boxtimes\mathscr{O}_{T^*\mathbb{P}^1}. \tag{5.14}$$

This happens in $\mathscr{Z} \subseteq \mathrm{Fl} \times T^*(\mathrm{Fl})$, and we implicitly used injectivity of i_*.

Then compute

$$[qQ * qQ] = (q\mathscr{O}_{\mathbb{P}^1} \boxtimes \pi^*\Omega^1_{\mathbb{P}^1} - \mathscr{O}_{\mathbb{P}^1} \boxtimes \mathscr{O}_{T^*\mathbb{P}^1}) * q\mathscr{O}_{\mathbb{P}^1} \boxtimes \Omega^1_{\mathbb{P}^1}. \tag{5.15}$$

By defintion, $(\mathscr{F} \boxtimes \mathscr{G}) * (\mathscr{G}' \boxtimes \mathscr{F}') =< \mathscr{G}, \mathscr{G}' > (\mathscr{F} \boxtimes \mathscr{F}')$. Then

$$\begin{aligned}[qQ * qQ] &= q^2 < \pi^*\Omega^1_{\mathbb{P}^1}, \mathscr{O}_{\mathbb{P}^1} > \mathscr{O}_{\mathbb{P}^1} \boxtimes \Omega^1_{\mathbb{P}^1} - q < \mathscr{O}_{T^*\mathbb{P}^1}, \mathscr{O}_{\mathbb{P}^1} > \mathscr{O}_{\mathbb{P}^1} \boxtimes \Omega^1_{\mathbb{P}^1} \\ &= (q^2 - q)Q,\end{aligned}$$

using that $\Omega^1_{\mathbb{P}^1} = \mathscr{O}(-2)$. If we identify $K^{\hat{G}\times\mathbb{G}_m}(\mathrm{pt}) = R[x, x^{-1}]^W[q^{\pm 1}]$, then $p_*[\mathscr{O}(n)] = \frac{x^{n+1} - x^{-(n+1)}}{x - x^{-1}}$ Weyl character formula, where $p : \mathbb{P}^1 \to \mathrm{pt}$.

For the second relation, use injectivity of i^*j_* and compute in $K^{\hat{G}\times\mathbb{G}_m}(\mathrm{Fl} \times \mathrm{Fl})$

- $i^*j_*(qQ) = qQ \boxtimes \mathscr{O}(-1)$,
- $i^*j_*(\mathscr{O}_n) = \Delta_*\mathscr{O}(n)$,

and then use the first relation we proved just now combined with Beilinson resolution. □

Theorem 5.3.2.
- *Let $\mathscr{H}_{\leq w} :=< e^\lambda T'_w : w' \leq w >$, then the map $\theta : \mathscr{H}_{\mathcal{I}} \to K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z})$ satisfies $\theta(\mathscr{H}_{\leq w}) \subseteq K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}_{\leq w})$.*
- *the induced associated graded*
$$\mathscr{H}_w \cong R(T)[q^{\pm 1}][T_w] \xrightarrow{\theta} K^{\hat{G}\times\mathbb{G}_m}(T^*_{Y_w}(\mathrm{Fl} \times \mathrm{Fl})) \cong R(T)[q^{\pm 1}][\mathscr{O}_{T^*_{Y_w}}(\mathrm{Fl} \times \mathrm{Fl})] \tag{5.16}$$
is an isomorphism.

Proof. (1) By construction, only need to verify that

- $e^\lambda \mapsto [\Delta_*\mathscr{O}(-\lambda)] \in K^{\hat{G}\times\mathbb{G}_m}(\Delta(\widetilde{\mathscr{N}}))$.
- $T_s \mapsto [qQ_s] - [\mathscr{O}_0]$ for simple reflection s, where $Q_s \in K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}} \times_{\widetilde{\mathscr{N}_s}} \widetilde{\mathscr{N}})$, where $\widetilde{\mathscr{N}_s}$ is a relative $\mathbb{P}^1$-bundle over $\widetilde{\mathscr{N}}$.
- In general, $w = s_1 \cdots s_r$ reduced word, then $\theta(T_w) = \theta(s_1) * \cdots * \cdots \theta(s_r)$ is supported on the image of
$$\widetilde{\mathscr{N}} \times_{\widetilde{\mathscr{N}_{s_1}}} \widetilde{\mathscr{N}} \times \cdots \times_{\widetilde{\mathscr{N}_{s_r}}} \widetilde{\mathscr{N}} \to \widetilde{\mathscr{N}} \times_{\hat{\mathfrak{g}}} \widetilde{\mathscr{N}} = \mathscr{Z}. \tag{5.17}$$

Claim: image is $\mathscr{Z}_{\leq w}$.

(2)Note that $\theta(T_{s_i})|_{\mathscr{Z}_{s_i}} = qQ_{s_i}$ is a line bundle. Roughly want $[\theta(T_{s_1}) * \cdots * \theta(T_{s_r})]|_{\mathscr{Z}_w} = [Q_{s_1}] * \cdots * [Q_{s_r}]$ in $K(\mathscr{Z}_w)$. Q_i's being Line bundles implies the right-hand side is invertible. Then obtain an isomorphism. □

6. Convolution Pattern of Constructible Sheaves, Shurui Liu

6.1. Motivation.

Recall last time, we proved that

$$K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}} \times_{\hat{\mathfrak{g}}} \widetilde{\mathscr{N}}) \cong \mathscr{H}_{\mathcal{I}}. \tag{6.1}$$

One formal feature of Steinberg variety is that $\mathscr{Z} := \mathrm{St} := \widetilde{\mathscr{N}} \times_{\hat{\mathfrak{g}}} \widetilde{\mathscr{N}}$ is of the form $X \times_Y X$. In particular, for any (reasonable) $Z \to Y$, we will have $K(X \times_Y X)$ acts on $K(X \times_Y Z)$ by convolution.

However, it is hard to directly classify representations of $K(X \times_Y X)$. Instead, we look at Borel-Moore Homology. The upshot is that we will reduce the classification of $\mathscr{H}_{\mathcal{I}}$ representations to simple modules of BM homology of a fiber product space (convolution algebra). Before elaborating on this point, let me briefly review the basics of Borel-Moore homology.

6.2. A Brief Introduction to Borel-Moore Homology.

Definition 6.2.1. Let S be a scheme of finite presentation over $\mathbb{C}$ and ω_S the dualizing sheaf of S. Then we define the Borel-Moore homology chain to be

$$C_{-*}^{\mathrm{BM}}(S) := R\Gamma(S, \omega_S). \tag{6.2}$$

Remark 6.2.2. The right-hand side is a cochain complex, and we transfer it to a chain complex via the standard procedure: given $A^\bullet$ a cochain complex, we take $A_n := A^{-n}$, $d_n = d^{-n}$. In particular, $H_k^{\mathrm{BM}}(X) := H^{-k}(X, \omega_X)$

Remark 6.2.3. For BM homology on a smooth manifold, one may consult [CG97, Chapter 2].

Remark 6.2.4. Let $f : S \to \operatorname{Spec} \mathbb{C}$ be the structure morphism. In this case $\omega_S = f^!\mathbb{C}$ and $C_{-*}^{\mathrm{BM}}(S) = f_* f^! \mathbb{C}$. Then taking Verdier dual

$$\mathbb{D} f_* f^! \mathbb{C} = f_! f^* \mathbb{C} = R\Gamma_c(S, \mathbb{C}), \tag{6.3}$$

we see that Borel-Moore homology is dual to compact support cohomology.

Note that

$$\operatorname{Hom}(\mathbb{C}, \mathbb{C}[k]) \times \operatorname{Hom}(\mathbb{C}[k], \mathbb{C}[k+l]) \to \operatorname{Hom}(\mathbb{C}, \mathbb{C}[k+l]), \tag{6.4}$$

gives cup product on cohomology $H^*(X)$. If X is smooth of pure dimension $\dim X$, then passing to Poincare duality, we obtain intersection pairing

$$\cap : H_k^{\mathrm{BM}}(X) \times H_l^{\mathrm{BM}}(X) \to H_{k+l-2\dim X}^{\mathrm{BM}}(X). \tag{6.5}$$

In general, take $i : X \to M$ closed embedding, where M is smooth of pure dimension d. Then we a natural map by adjunctions (and identifying $\omega_M \cong \mathbb{C}_M[2d]$)

$$i_*\omega_X[-d] = i_! i^! \omega_M[-d] \mapsto \omega_M[-d] \cong \mathbb{C}_M[d] \mapsto i_* i^* \mathbb{C}_M[d] = i_* \mathbb{C}_X[d]. \tag{6.6}$$

Taking global sections, one gets

$$H_{-*+d}^{\mathrm{BM}}(X) \times H_{*-d}(X) \to \mathbb{C}. \tag{6.7}$$

Remark 6.2.5. When X is smooth, this is a non-degenerate pairing and gives the Poincare duality mentioned above.

If X is proper, get $H_{-*+d}^{\mathrm{BM}}(X) \times H_{*-d}^{\mathrm{BM}}(X) \to \mathbb{C}$. In this case, by Kashiwara-Schapira, this construction could be enhanced to

$$\cap : H_k^{\mathrm{BM}}(X) \times H_l^{\mathrm{BM}}(X) \to H_{k+l-2\dim M}^{\mathrm{BM}}(X). \tag{6.8}$$

Remark 6.2.6. Assume $f : X \to Y$ is proper and X is smooth algebraic variety. Then for cap product on fiber product spaces $X \times_Y X$, or $X \times_Y X \times_Y X$, one could take the natural closed embedding

$$X \times_Y X \hookrightarrow X \times X \tag{6.9}$$

and

$$X \times_Y X \times_Y X \hookrightarrow X \times X \times X, \tag{6.10}$$

which will be used later on.

6.3. **Continuum on Motivation.** So our goal is to classify all simple modules over

$$\mathscr{H}_{\mathcal{I}} \cong K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}} \times_{\hat{\mathfrak{g}}} \widetilde{\mathscr{N}}).$$

Recall that we already identify its center

$$Z(\mathscr{H}_{\mathcal{I}}) \cong R(\hat{G} \times \mathbb{G}_m).$$

Let M be a simple $\mathscr{H}_{\mathcal{I}}$-module. By Schur's lemma, $Z(\mathscr{H}_{\mathcal{I}})$ will act by a character. But a character

$$R(\hat{G} \times \mathbb{G}_m) \to \mathbb{C} \tag{6.11}$$

is same as

$$\operatorname{Spec} \mathbb{C} \to (\hat{G} \times \mathbb{G}_m)//(\hat{G} \times \mathbb{G}_m), \tag{6.12}$$

i.e. a semi-simple conjugacy class $a := (s, t) \in \hat{G} \times \mathbb{G}_m$. Define

$$\mathscr{H}_a := \mathscr{H}_{\mathcal{I}} \otimes_{Z(\mathscr{H}_{\mathcal{I}})} \mathbb{C}_a. \tag{6.13}$$

Then the action of $\mathscr{H}_{\mathcal{I}}$ on M_a factors through $\mathscr{H}_a$. In other words, we only need to classify all simple $\mathscr{H}_a$-modules, where a runs through all semi-simple conjugacy classes of $\hat{G} \times \mathbb{G}_m$.

Lemma 6.3.1. *Notation as above, $\mathscr{H}_a$ is of dimension $(\#W)^2$ over $\mathbb{C}$. In particular, any simple module $\mathscr{H}_{\mathcal{I}}$ is finite-dimensional.*

This algebraic construction has a geometric explanation. Recall that $a = (s, t) \in \hat{G} \times \mathbb{G}_m$ acts on $\mathscr{N}$ by $(s, t)x = t^{-1}sxs^{-1}$, and similar for $\widetilde{\mathscr{N}}$. Define $\widetilde{\mathscr{N}_a}, \mathscr{N}_a$ to be the a-fixed points, and $\mathscr{Z}_a := \widetilde{\mathscr{N}_a} \times_{\mathscr{N}_a} \widetilde{\mathscr{N}_a}$. Then there is an algebra structure on $C_{-*}^{\mathrm{BM}}(\mathscr{Z}_a)$ by convolution product, to be reviewed later.

Theorem 6.3.2. *Let $a = (s, t) \in \hat{G} \times \mathbb{G}_m$ be a semi-simple element. Then there is an algebra isomorphism*

$$\mathscr{H}_a \cong C_{-*}^{\mathrm{BM}}(\mathscr{Z}_a). \tag{6.14}$$

Sketch of Proof. Let $\mathscr{A}$ be the closed subgroup of $\hat{G} \times \mathbb{G}_m$ generated by a. In particular, $\mathscr{A}$ is an abelian reductive subgroup.

$$\begin{aligned}
\mathscr{H}_a &\cong \mathbb{C}_a \times_{R(\hat{G}\times\mathbb{G}_m)} K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}) \quad \text{using Daniel, Vaughan's talk} \\
&\cong \mathbb{C}_a \otimes_{R(\mathscr{A})} K^{\mathscr{A}}(\mathscr{Z}) \quad \text{covered in Jiahao's talk??} \\
&\cong \mathbb{C}_a \otimes R(\mathscr{A}) \otimes K(\mathscr{Z}_a) \\
&\cong K_{\mathbb{C}}(\mathscr{Z}_a) \\
&\cong C^{\mathrm{BM}}_{-*}(\mathscr{Z}_a) \quad \text{Riemann-Roch.}
\end{aligned}$$

Relevant facts about K-theory could be found in [CG97, Ch.5]. □

The upshot is that, our problem is reduced to classifying simple modules over Borel Homology with convolution product.

6.4. Convolution Algebra as Endomorphism Algebra of "Springer Sheaf".

We now switch to a general setting and later apply it to fixed points of the Steinberg variety.

Let $f : X \to Y$ be a proper morphism of schemes of finite presentation over $\mathbb{C}$. Define $\mathscr{Z} := X \times_Y X$. Assume that X is smooth. Then

$$\begin{array}{ccccc}
 & & X \times_Y X \times_Y X & & \\
 & \swarrow^{p_{12}} & \downarrow^{p_{13}} & \searrow^{p_{23}} & \\
X \times_Y X & & X \times_Y X & & X \times_Y X
\end{array} \tag{6.15}$$

Then $(p_{13})_*(p_{12}^*(-) \cap p_{23}^*(-))$ defines an algebra structure on $C^{\mathrm{BM}}_{-*}(X \times_Y X)$. Moreover, the unit is given by the fundamental class of $[\Delta(X)]$.

We work in a slightly more general setting. Let $f_i : X_i \to Y$, $i = 1, 2$.

Lemma 6.4.1. *Assume that f_i are proper, and X_i are smooth of pure dimension d_i, for $i = 1, 2$. Then as graded $\mathbb{C}$-vector spaces, we have*

$$C^{\mathrm{BM}}_{-*+d_2+d_1}(X_1 \times_Y X_2) \cong R\mathrm{Hom}((f_1)_*\mathbb{C}[d_1], (f_2)_*\mathbb{C}[d_2]). \tag{6.16}$$

Proof. We have a Cartesian square

$$\begin{array}{ccc}
X_1 \times_Y X_2 & \xrightarrow{\alpha} & X_1 \\
\beta \downarrow & & \downarrow f_1 \\
X_2 & \xrightarrow{f_2} & Y
\end{array} \tag{6.17}$$

Then

$$\begin{aligned} R\mathrm{Hom}((f_1)_*\mathbb{C}[d_1],(f_2)_*\mathbb{C}[d_2]) &\cong R\mathrm{Hom}(\mathbb{C}, f_1^!(f_2)_*\mathbb{C})[d_2-d_1] \\ &\cong R\mathrm{Hom}(\mathbb{C}, \alpha_*\beta^!\mathbb{C})[d_2-d_1] \\ &\cong R\mathrm{Hom}(\alpha^*\mathbb{C}, \beta^!\omega_{X_2}[-d_2-d_1]) \\ &\cong R\Gamma(X_1\times_Y X_2, \omega_{X_1\times_Y X_2})[-d_2-d_1] \\ &\cong C^{\mathrm{BM}}_{-*+d_2+d_1}(X_1\times_Y X_2). \end{aligned}$$

□

Corollary 6.4.2. *Let $f : X \to Y$ be a proper morphism of schemes of finite presentation over $\mathbb{C}$ and connected components of X are smooth. Then we have an isomorphism of vector spaces*

$$C^{\mathrm{BM}}_{-*}(X\times_Y X) \cong R\,\mathrm{End}(f_*\mathrm{IC}(X)). \tag{6.18}$$

Theorem 6.4.3. *The isomorphism in (6.18) is an isomorphism of algebras. Namely, the following diagram commutes*

$$\begin{array}{ccc} C^{\mathrm{BM}}_{-*}(X\times_Y X)\times C^{\mathrm{BM}}_{-*}(X\times_Y X) & \xrightarrow{\text{convolution}} & C^{\mathrm{BM}}_{-*}(X\times_Y X) \\ \downarrow\cong & & \downarrow\cong \\ R\,\mathrm{End}(f_*\mathrm{IC}(X))\times R\,\mathrm{End}(f_*\mathrm{IC}(X)) & \xrightarrow{\text{composition}} & R\,\mathrm{End}(f_*\mathrm{IC}(X)) \end{array} \tag{6.19}$$

Remark 6.4.4. Here I abuse notation, and treat $C^{\mathrm{BM}}_{-*} = H^{\mathrm{BM}}_{*}$, and $R\mathrm{Hom} = \mathrm{Ext}^*$ (ignoring the difference between chain/cochain complex and their homology/cohomology ring).

Sketch of Proof. A similar base change theorem argument shows that the convolution product of Borel-Moore homology is realized as a sheaf theoretic convolution $*$. Then it is an exercise of six functor formalism calculations (called yoga). □

6.5. **Classifying Simple Modules by Decomposition Theorem.** Since f is proper, $f_*\mathrm{IC}(X)$ is constructible with respect to some stratification $\{Y_\alpha\}_{\alpha\in\Lambda}$ of Y. By decomposition theorem,

$$f_*\mathrm{IC}(X) \cong \bigoplus_{\alpha\in\Lambda, k\in\mathbb{Z}} M_{\mathscr{L}_{\alpha,k}}\otimes \mathrm{IC}(\mathscr{L}_\alpha)[k], \tag{6.20}$$

where $\mathscr{L}_\alpha$ is an irreducible local system on Y_α, $M_{\mathscr{L}_{\alpha,k}}$ is a $\mathbb{C}$-vector space served as multiplicity space.

Therefore,

$$R\,\mathrm{End}(f_*\mathrm{IC}(X)) \cong \bigoplus_{\alpha,\beta\in\Lambda, i,j,k\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}}(M_{\mathscr{L}_{\alpha,i}}, M_{\mathscr{L}_{\beta,j}})\otimes \mathrm{Ext}^{k+j-i}(\mathrm{IC}(\mathscr{L}_\alpha),\mathrm{IC}(\mathscr{L}_\beta)) \tag{6.21}$$

$$\cong \bigoplus_{\alpha,\beta\in\Lambda, i,j,k\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}}(M_{\mathscr{L}_{\alpha,i}}, M_{\mathscr{L}_{\beta,j}})\otimes \mathrm{Ext}^{k}(\mathrm{IC}(\mathscr{L}_\alpha),\mathrm{IC}(\mathscr{L}_\beta)) \tag{6.22}$$

Fact: (see [Bei+18])

(1) $\mathrm{Perv}(Y)$ is an abelian category, whose derived category is $\mathrm{Shv}_c(Y)$.

(2) Irreducible objects of $\mathrm{Perv}(Y)$ are of the form $\mathrm{IC}(\mathscr{L}_\alpha)$, where $\mathscr{L}_\alpha$ is an irreducible local system on some smooth locally closed subvariety Y_α of Y.

In particular, we have

(1) $\mathrm{Ext}^k(\mathscr{F},\mathscr{G})=0$, for $k<0$, $\mathscr{F},\mathscr{G}$ perverse.

(2) $\mathrm{Hom}(\mathrm{IC}(\mathscr{L}_\alpha),\mathrm{IC}(\mathscr{L}_\beta))=0$, if $\mathscr{L}_\alpha\neq\mathscr{L}_\beta$ irreducible local systems.

(3) (non-derived) $\mathrm{End}(\mathrm{IC}(\mathscr{L}_\alpha))=\mathbb{C}$, for $\mathscr{L}_\alpha$ irreducible.

Set

$$M_\alpha := \oplus_i M_{\alpha,i} \tag{6.23}$$

Therefore,

$$R\,\mathrm{End}(f_*\mathrm{IC}(X)) \cong \oplus_\alpha \mathrm{End}(M_\alpha)\bigoplus \mathrm{Nil}, \tag{6.24}$$

where

$$\mathrm{Nil} := \bigoplus_{\alpha,\beta,k\geq 0} \mathrm{Hom}(M_\alpha, M_\beta)\otimes \mathrm{Ext}^k(\mathrm{IC}(\mathscr{L}_\alpha),\mathrm{IC}(\mathscr{L}_\beta))$$

is concentrated in degree >0 and hence nilpotent in $R\,\mathrm{End}(f_*\mathrm{IC}(X))$.

Moreover, $\oplus_\alpha\mathrm{End}(M_\alpha)$ is a direct sum of matrix algebras and hence semisimple. Therefore, we see that Nil is the radical and the semi-simple quotient is identified with $\oplus_\alpha\mathrm{End}(M_\alpha)$. Therefore, we obtain the following classification.

Proposition 6.5.1. $\{M_\alpha\}_{\alpha\in\Lambda}$ *forms a complete set of isomorphic classes of simple* $R\,\mathrm{End}(f_*\mathrm{IC}(X))$*-modules.*

I implicitly used the following algebra fact.

Lemma 6.5.2. *If M is a simple module over a matrix algebra $R=\mathrm{End}(V)$ with V dimension d $\mathbb{C}$-vector space, then M is isomorphic to V as R-modules.*

Proof. Let v be a non-zero vector in M. Since M is simple, $R\cdot v = M$. Consider the map

$$f : M \to \mathbb{C}^n, \tag{6.25}$$

defined by $r\cdot v\mapsto r\cdot e_1$, where $e_1=(1,0,\dots,0)\in\mathbb{C}^n$. Then f is injective since M is simple and f is surjective since $\mathbb{C}^n$ is simple. □

6.6. **Describing Multiplicity spaces M_α.** Our goal now is to give M_α geometric meanings.

6.6.1. *Non-equivariant Setting.* Consider $i : Z\hookrightarrow Y$ a locally closed subvariety. Then i^* induces

$$R\mathrm{End}(\mathscr{F}) \to R\mathrm{End}(i^*\mathscr{F}) \tag{6.26}$$

acts on $i^*\mathscr{F}$, for any complex $\mathscr{F}$ on Y. One obtain a $R\,\mathrm{End}(\mathscr{F})$-module structure on $R\Gamma(Z,i^*\mathscr{F})$. Similar results hold for $i^!$. Now take $i:\{y\}\hookrightarrow Y$ a closed point. By base change, we see that

$$C^{*+d}(X_y) \cong R\Gamma(\{y\}, i^*f_*\mathrm{IC}(X)), \tag{6.27}$$

$$C^{\mathrm{BM}}_{-*+d}(X_y) \cong R\Gamma(\{y\}, i^!f_*\mathrm{IC}(X)). \tag{6.28}$$

Therefore, $C^*(X_y)$ and $C^{\mathrm{BM}}_{-*}(X_y)$ are $R\mathrm{End}(f_*\mathrm{IC}(X))$-modules. They are also $C^{\mathrm{BM}}_{-*}(X\times_Y X)$-modules by convolution product.

Theorem 6.6.1. *These two module structures are compatible with the algebra isomorphism* (6.18).

Note that there is a canonical map $i^! \to i^*$, obtained by applying i^* to the adjunction counit $i_! i^! \to \mathrm{id}$. Therefore, there is a canonical map

$$C_{-*}^{\mathrm{BM}}(X_y) \to C^*(X_y) \tag{6.29}$$

of $C_{-*}^{\mathrm{BM}}(\mathscr{Z})$-modules.

6.6.2. *Equivariant Setting.* Now if $f : X \to Y$ is G-equivariant for a reductive group G, and N has a (finite) stratification given by G-orbit $N = \sqcup_{\alpha\in\Lambda}\mathcal{O}_\alpha$.

Let $i_\alpha : \mathcal{O}_\alpha \to Y$ be a locally closed immersion of a stratum. Then in the decomposition theorem, Y_α are exactly $\mathcal{O}_\alpha$, and $\mathscr{L}_\alpha$ are irreducible local systems on $\mathcal{O}_\alpha$. Let $y \in \mathscr{O}_\alpha$ be a closed point, with stabilizer $G(y)$. Then the irreducible local system $\mathscr{L}_\alpha$ corresponds to an irreducible representation χ of $G(y)/G(y)^\circ$. ($[\mathscr{O}_\alpha/G] \cong [*/G(y)]$.).

Given an irreducible representation χ of $G(y)/G(y)^\circ$, one define

$$L_{y,\chi} := \mathrm{Im}(C_*^{\mathrm{BM}}(X_y)_\chi \to C^*(X_y)_\chi), \tag{6.30}$$

which inherits an $R\mathrm{End}(f_*\mathrm{IC}(X))$-module structure.

Remark 6.6.2. The source is identified with standard module and the target is identified with co-standard module. These will be discussed in the next talk.

Theorem 6.6.3. *Notation as before. Let $\mathscr{L}_\alpha$ be an irreducible local system on $\mathscr{O}_\alpha$, which appears in the decomposition. Let $y \in \mathscr{O}_\alpha$ and χ be the irreducible representation of $G(y)/G(y)^\circ$ corresponding to $\mathscr{L}_\alpha$. Then there is an isomorphism of $R\mathrm{End}(f_*\mathrm{IC}(X))$-modules*

$$M_\alpha \cong L_{y,\chi}. \tag{6.31}$$

It is easy to see that the isomorphic class $L_{y,\chi}$ is independent of the choice of $y \in \mathscr{O}_\alpha$. Hence we denote it by $L_{\mathscr{O},\chi}$ or $L_{\mathscr{O},\mathscr{L}_\alpha}$.

6.7. **Deligne-Langlands Correspondence.** Apply these general results to Steinberg variety, one obtains the classification result. Namely, given $a = (s, q) \in \hat{\mathrm{G}} \times \mathbb{G}_m$ semi-simple, a $\hat{G}$-orbit $\mathscr{O}$ in $\mathscr{N}_a$, an $\hat{G}$-equivariant local system $\mathscr{L}$ on $\mathscr{O}$, then the $\mathscr{H}_{\mathcal{I}}$-module $L_{\mathscr{O},\mathscr{L}}$ is simple and every simple module with q action by multiplication by q is isomorphic to such form.

Note that $\mathscr{N}_a = \{x \in \mathscr{N} : sxs^{-1} = qx\}$. Fix q. Then we see

$$\begin{array}{c}\{\text{Simple } \mathscr{H}_{\mathcal{I}}\text{-module } M \text{ with } q \text{ acting by mult } q\}/\cong \\ \Big\downarrow {\scriptstyle \text{finite:1}} \\ \{(s,x) \in \hat{G} \times \hat{\mathfrak{g}} : s \text{ s.s.}, x \text{ nilpotent}, sxs^{-1} = qx\}/\cong \end{array} \tag{6.32}$$

The L-package is also precisely described, i.e.

$$\begin{array}{c}\{\text{Simple } \mathscr{H}_{\mathcal{I},q}\text{-module } M\}/\cong \\ \big\downarrow {\scriptstyle 1:1} \\ \{(s,x)\in \hat{G}\times\hat{\mathfrak{g}} : s \text{ s.s.}, x \text{ nilpotent}, sxs^{-1}=qx, \mathscr{L} \text{ an irr equivariant local system on } \hat{G}\text{-orbit}\}/\cong\end{array} \tag{6.33}$$

One could also do group version (replacing nilpotent cone with unipotent cone) as in the original paper [KL87]. Combining my previous talk, Iwahori-Matsumoto relation, one could understand this result as a description of Iwahori principal block, aka a local Langlands correspondence in tamely ramified unipotent monodromy case.

Remark 6.7.1. Note that $\mathscr{N}_q := \{(s,u)\in\hat{G}\times\hat{G} : s \text{ s.s.}, u \text{ nilpotent}, sus^{-1}=u^q\}$, also appears in the geometry of (discretized) moduli of (unipotent) Langlands parameters.

Things not done in this talk (left to later speakers):

(1) Proof of Theorem 6.6.3: identifying multiplicity space as image from standard to costandard. More details on standard and costandard.
(2) Kazhdan-Lusztig multiplicity: [CG97, Thm.8.6.23].
(3) Group version: [KL87] (it seems no assumption on simply-connectedness in the paper, while in Lie algebra case in [CG97] we assumed so).

7. Standard and Costandard Modules, Xinwen Zhu

Recall that

$$\mathbb{H} = \mathbb{C}[v,v^{-1}]\langle e^{\lambda}T_w\rangle_{\lambda\in\mathbb{X}_*(T), w\in W_{\mathrm{fin}}} \supseteq R := \mathbb{C}[v,v^{-1}]\langle e^{\lambda}\rangle \supseteq Z(\mathscr{H}).$$

This is identified as

$$K^{\hat{G}\times\mathbb{G}_m}(\mathscr{Z}) \supseteq K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}}) \supseteq K^{\hat{G}\times\mathbb{G}_m}(\mathrm{pt}),$$

where $\widetilde{\mathscr{N}}$ is the Springer resolution for dual group and $\mathscr{Z} := \widetilde{\mathscr{N}}\times_{\hat{\mathfrak{g}}}\widetilde{\mathscr{N}}$. Moreover, $\mathbb{H}$ action on $\mathrm{M}_{\mathrm{asp}} = \mathbb{H}\otimes_{H_f}\mathbb{C}[v,v^{-1}]$ defined by $T_w\mapsto(-1)^{l(w)}$ is identified with $K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}})$.

Then for any $W\to\hat{\mathfrak{g}}$, we will have a module $K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathscr{N}}\times_{\hat{\mathfrak{g}}} W)$. Now take W to be a nilpotent orbit $\mathscr{O}$.

Take $a=(s,t)\in\hat{G}//\hat{G}\times\mathbb{G}_m = \mathrm{Spec}\, K^{\hat{G}\times\mathbb{G}_m}(\mathrm{pt})$. Choose $X\in\mathscr{O}^a$, i.e. $\mathrm{Ad}_s(X)=tX$.

$$\begin{array}{ccc} \widetilde{\mathscr{N}}^a & \xrightarrow{\pi^a} & \mathscr{N}^a \\ & & \cup\big\uparrow \\ & & \mathscr{O}^a \\ & & \cup\big\uparrow \\ \mathscr{B}_X^s = \{\mathfrak{b}'\in\hat{G}/\hat{B} \mid X, s\in\mathfrak{b}'\} & \longrightarrow & X \end{array} \tag{7.1}$$

Proposition 7.0.1. *We consider the localization*

$$K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathcal{N}_{\mathcal{O}}})\otimes_{K^{\hat{G}\times\mathbb{G}_m}}\mathbb{C}_a \xrightarrow[\text{localization}]{\cong} \bigoplus_{X\in\mathcal{O}^a/C_{\hat{G}}(s)} K(\mathscr{B}_X^s). \tag{7.2}$$

Let $u=\exp(X)$. Let $C(s,u):=\pi_0(C_{\hat{G}})(s,u)$, which acts on $\mathscr{B}_X^s$.

Definition 7.0.2. Let $a=(s,t)$ and X be as above, and $\rho\in\operatorname{Irr}(C(s,u))$. We define the standard module of $\mathbb{H}$ by

$$\mathscr{M}_{(a,u,\rho)}:=\operatorname{Hom}_{C(s,u)}(\rho,K(\mathscr{B}_X^s)). \tag{7.3}$$

Fix $a=(s,t)\in\hat{G}//\hat{G}\times\mathbb{G}_m$, $\mathbb{H}_a:=\mathbb{H}\otimes_{Z(H)}\mathbb{C}_a$. Last time we showed

$$\mathbb{H}_a\cong H^{\mathrm{BM}}(\mathscr{Z}_a), \tag{7.4}$$

which acts on $H^{\mathrm{BM}}(\widetilde{\mathcal{N}^a})$. Also last time this is identified with

$$H^{\mathrm{BM}}(\mathscr{Z}_a)\cong R\operatorname{End}(\pi_*^a\omega_{\widetilde{\mathcal{N}^a}}) \tag{7.5}$$

acting on $R\mathrm{Hom}(\underline{\mathbb{C}},\pi_*^a\omega_{\widetilde{\mathcal{N}^a}})$.

Lemma 7.0.3. $\widetilde{\mathcal{N}^a}$ *is smooth.*

Proof. It is a general fact that the fixed point of a torus action on a smooth variety is still smooth, which is proved affine locally. ☐

Recall the general machinery: given $f:X\to Y$, with X smooth and f proper. Then $R\operatorname{End}(Rf_*\omega_X[-\dim X])$ acts on $\mathscr{A}:=R\mathrm{Hom}(\mathscr{F},R_f*\omega_X[-\dim X])$. By the decomposition theorem, there is a non-canonical isomorphism

$$Rf_*\omega_X[-\dim X]\cong\bigoplus V_{(Z,\mathscr{L})}\otimes\mathrm{IC}(Z,\mathscr{L}), \tag{7.6}$$

where $Z\hookrightarrow Y$ closed, $Z^0\subseteq Z$ open smooth, $\mathscr{L}$ a local system on Z^0. Then $V_{(Z,\mathscr{L})}$ gives a complete list of irreducible representations of $\operatorname{Ext}^\bullet(\mathscr{A},\mathscr{A})$.

Remark 7.0.4. In general, describing $V_{(Z,\mathscr{L})}$ and IC's is really hard, e.g. the case of the Hitchin system is related to Ngo's proof of the fundamental lemma. Our situation is $\pi^a:\widetilde{\mathcal{N}^a}\to\mathcal{N}^a$. When $a=1$, this is classical Springer theory.

Proposition 7.0.5. *(1) π^a is subjective.*

(2) $\widetilde{\mathcal{N}^a}$ has only finitely many $C_{\hat{G}}(s)$-orbits.

Corollary 7.0.6. *In this case of $\pi^a:\widetilde{\mathcal{N}^a}\to\mathcal{N}^a$, Z appearing in the decomposition must be of the form $Z=\overline{\mathcal{O}}$, for some orbit $\mathcal{O}\subseteq\widetilde{\mathcal{N}^a}$. Moreover, $\mathscr{L}$ is a $C_{\hat{G}}(s)$-equivariant local system on $C_{\hat{G}}(s)/C_{\hat{G}}(s,u)=\mathcal{O}$, which corresponds to an irreducible representation of $\pi_0(C_{\hat{G}}(s,u))=C(s,u)$.*

Proof of Prop. 7.0.5. For (1), π^a is subjective, if and only if $\mathscr{B}_u^s \neq \emptyset$, if and only if there exists Borel of $\hat{G}$ containing (s, u), such that $sus^{-1} = u^t$. However, s, u generate a solvable subgroup of $\hat{G}$, and hence they are contained in some Borel. For (2), consider

$$\begin{array}{ccc} & \{(s, X) \in \hat{G} \times \hat{\mathfrak{g}} : \mathrm{Ad}_s(X) = tX, s \mathrm{ss}, X \text{ nilp}\}/\hat{G} & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ \hat{G}/\hat{G} & & \hat{\mathscr{N}}/\hat{G} \end{array} \tag{7.7}$$

Observe that $p_1^{-1}(s) = \mathscr{N}^a$. We need to show there are only finitely many $C_{\hat{G}}(s)$-orbits. Let C be the conjugacy class of s. Then

$$\widetilde{\mathscr{N}^a}/C_{\hat{G}}(s) \cong p^{-1}(C)/\hat{G}. \tag{7.8}$$

Since $\mathscr{N}$ has only finitely many $\hat{G}$-orbits. Therefore, it is enough to show that

$$p_1^{-1}(C)/\hat{G} \to \mathscr{N}/\hat{G} \tag{7.9}$$

is quasi-finite (i.e. has finite fibers). For $X \in \mathscr{N}$, the fiber is $C_{\hat{G}}(X)^{\mathrm{ss}}/\mathrm{Ad}_s(C_{\hat{G}}(X))$. Then semisimple $C_{\hat{G}}(X)^{\mathrm{ss}}$ is conjugate into some maximal torus T_X. Then $T_X \to T_X//W$ is quasi-finite. □

Still fix $a = (s, t)$.

$$\hat{\mathfrak{g}} = \oplus_\lambda \hat{\mathfrak{g}}_\lambda, \tag{7.10}$$

where λ is an eigenvalue of Ad_s. Then $\mathscr{N}^a = \hat{\mathfrak{g}}_t \cap \mathscr{N}$. Note that $[\hat{\mathfrak{g}}_\lambda, \hat{\mathfrak{g}}_\mu] \subseteq \hat{\mathfrak{g}}_{\lambda\mu}$. Then observe that if t is not a root of 1, then $\mathrm{ad}_{X'} : \hat{\mathfrak{g}}_\lambda \to \hat{\mathfrak{g}}_{t\lambda} \to \cdots$ eventually zero. So we have $\mathscr{N}^a = \hat{\mathfrak{g}}_t$.

Theorem 7.0.7. *We have a parity result*

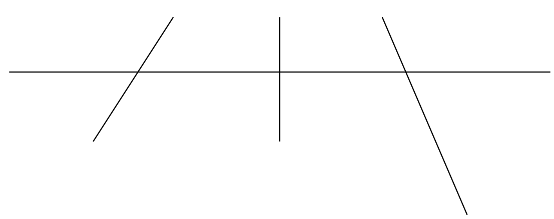
$$H^{\mathrm{BM}}_{odd}(\mathscr{B}_X^s, \mathbb{C}) = 0, \tag{7.11}$$

and

$$H^{\mathrm{BM}}_{odd}(\mathscr{B}_X, \mathbb{C}) = 0. \tag{7.12}$$

Fact: $\mathscr{B}_X$ is connected and equidimensional while $\mathscr{B}_X^s$ may not be connected nor equidimensional.

Example 7.0.8. $X \in \hat{N}$ is called subregular if $\hat{G}X \subset \hat{\mathscr{N}}$ is of codimension 2. Then $\mathscr{B}_X$ is always a chain of $\mathbb{P}^1$'s. For example, $\hat{G} = G_2$, then it looks like



the dual Dykin diagram is D_4. For a particular s, it will preserve the central $\mathbb{P}^1$ and permute the other 3 $\mathbb{P}^1$'s. Then $C(s, u) = S_3$.

Consider $\pi^a : \widetilde{\mathscr{N}^a} \to \mathscr{N}^a$. Denote $\mathscr{A} := R\pi^a_* \omega_{\widetilde{\mathscr{N}^a}}[-\dim \mathscr{N}_a]$. Then

$$\mathscr{A} \cong \oplus_{(\mathscr{O},\mathscr{L})} V_{(\mathscr{O},\mathscr{L})} \otimes \mathrm{IC}(\mathscr{O},\mathscr{L}), \tag{7.13}$$

where $\mathscr{O}$ is a $C_{\hat{G}(s)}$-orbit in $\mathscr{N}^a$, and $\mathscr{L}$ corresponds to $\chi \in \mathrm{Irr}(C(s,u))$. Let $i_{\mathscr{O}} : \mathscr{O} \hookrightarrow \mathscr{N}^a$ be the locally closed embedding. Then

$$\epsilon_{\mathscr{O}} : i^!_{\mathscr{O}'} \mathrm{IC}(\mathscr{O},\mathscr{L}) \to i^*_{\mathscr{O}'} \mathrm{IC}(\mathscr{O},\mathscr{L}). \tag{7.14}$$

Lemma 7.0.9. *$\epsilon_{\mathscr{O}'} = 0$ if $\mathscr{O}' \neq \mathscr{O}$; $\epsilon_{\mathscr{O}'}$ is an isomorphism if $\mathscr{O}' = \mathscr{O}$.*

Proof. Clearly when $\mathscr{O}' \nsubseteq \overline{\mathscr{O}}$, $\epsilon_{\mathscr{O}'} = 0$. If $\mathscr{O}' \subseteq \overline{\mathscr{O}}$, and $\mathscr{O}' \neq \mathscr{O}$, then $\mathscr{F} \in \mathrm{Perv}$ implies that

$$i^!_{\mathscr{O}'} \mathscr{F} \in D(\mathrm{Loc}(\mathscr{O}'))^{\geq -\dim \mathscr{O}'} \tag{7.15}$$

$$i^*_{\mathscr{O}'} \mathscr{F} \in D(\mathrm{Loc}(\mathscr{O}'))^{\leq -\dim \mathscr{O}'} \tag{7.16}$$

In addition, if $\mathscr{F}$ is IC sheaf, then the inequalities above are strict unless $\mathscr{O}'$ is the largest stratum in the support of $\mathscr{F}$. □

If we pick up some $X \in \mathscr{O}'$, $i_X : \{X\} \hookrightarrow \mathscr{O}$,

$$i^!_X i^!_{\mathscr{O}} \mathscr{A} \to i^!_X i^*_{\mathscr{O}} \mathscr{A}. \tag{7.17}$$

Corollary 7.0.10. $V_{(\mathscr{L},\mathscr{O})} = \mathrm{Im}(i^!_X i^!_{\mathscr{O}} \mathscr{A} \to i^!_X i^*_{\mathscr{O}} \mathscr{A})$.

$$\{X\} \xhookrightarrow{i_X} \mathscr{O} \xhookrightarrow{i_{\mathscr{O}}} \widetilde{\mathscr{N}^a}, \qquad \{X\} \xrightarrow{i_X} \widetilde{\mathscr{N}^a} \tag{7.18}$$

Definition 7.0.11. $M_{(a,X,\mathscr{L})} := (i^!_X \mathscr{A})_{\mathscr{L}} \cong R\mathrm{Hom}(i^!_X \mathbb{C}, \mathscr{A})_{\mathscr{L}} \cong (H^{\mathrm{BM}}(B^u_X))_{\mathscr{L}} \xrightarrow[RR]{\cong} K(B^s_X)_{\mathscr{L}}$ is called standard module of $\mathbb{H}$.

Corollary 7.0.12. *Every irreducible representation of H appears as a quotient of a standard module.*

Corollary 7.0.13. *Suppose that t is not a root of 1. Let $u \in \mathscr{O} \subseteq \mathscr{N}^a = \hat{\mathfrak{g}}_t$. Then the standard module $M_{(a,X,\mathscr{L})}$ is irreducible.*

Theorem 7.0.14. *Suppose that t is not a root of 1. Then every irreducible $V_{(\mathscr{O},\mathscr{L})} \neq 0$ as soon as $M_{(a,u,\mathscr{L})} \neq 0$, i.e. irreducible representation χ of $C(s,u)$ corresponding to $\mathscr{L}$ appears in $H^{\mathrm{BM}}(\mathscr{B}^s_X)$.*

Example 7.0.15. $\hat{G} = G_2$, $a = (s,q)$, u subregular, then $C(s,u) = S_3$ acts on $H^{\mathrm{BM}}(\mathscr{B}^s_u)$. $\mathrm{Irr}(S_3) = \{\mathrm{triv}, \mathrm{sign}, \text{2-dim perm}\}$. Only trivial representation and the 2-dim representation appears in $H^{\mathrm{BM}}(\mathscr{B}^s_u)$. In fact (s,u,sign) will correspond to a supercuspidal representation in the local Langlands correspondence. Moreover, only triv have Whittaker models. Besides, in this case, standard modules are irreducible.

Definition 7.0.16. Similarly, $(i^!_X i^*_{\mathscr{O}} \mathscr{A})_{\mathscr{L}} = \check{M}_{(a,u,\mathscr{L})}$ is called costandard represetnation.

Then

$$M_{(a,u,\mathscr{L})} \twoheadrightarrow L_{(a,u,\mathscr{L})} \hookrightarrow \check{M}_{(a,u,\mathscr{L})}. \tag{7.19}$$

Take S a transversal slice, acted by reductive quotient $C_{\hat{G}}(s,u)^{\mathrm{red}}$, in particular by $\pi_0 = C(s,u)$.

Lemma 7.0.17.

$$H^{\mathrm{BM}}(\widetilde{S})_{\mathscr{L}} \cong R\mathrm{Hom}(i_S^!\underline{\mathbb{C}}, \mathscr{A})_{\mathscr{L}} \cong \check{M}_{(a,u,\mathscr{L})}. \tag{7.20}$$

$$\widetilde{S} := \pi_a^{-1}(S) \tag{7.21}$$

is smooth, and homotopy retracts to $\mathscr{B}_u^s$. In particular, the map from standard to costandard is induced by

$$(i_S)_!\mathbb{C}_S \to (i_S)_!(i_X)_*i_X^*\mathbb{C}_S = (i_X)_!\mathbb{C}, \tag{7.22}$$

since standard is $R\mathrm{Hom}((i_X)_!\mathbb{C}, \mathscr{A})$ and costandard is $R\mathrm{Hom}((i_S)_!\mathbb{C}, \mathscr{A})$. In particular, this implies the map from standard to costandard is actually a homomorphism of $\mathbb{H}$-modules.

8. Proof of Classification, Calder Morton-Ferguson

Recall that we pass from $\mathscr{H}_{\mathcal{I}}$ to $\mathscr{H}_a$, where $a = (s,t) \in G \times \mathbb{C}^\times$, where we write G for the dual group notation, since we only care about the spectral side in this talk.

Recall that $\mathscr{H}_{\mathcal{I}} = K^{G\times\mathbb{C}}(\mathscr{Z})$, and $\mathscr{H}_a \cong \mathrm{Ext}^\bullet(\mathbb{L},\mathbb{L})$. We have that

$$\mathbb{L} \cong \mu_*\mathrm{IC}(\widetilde{\mathscr{N}^a}) \cong \oplus_\phi L_\phi \otimes \mathrm{IC}_\phi, \tag{8.1}$$

where ϕ are parameterized by a nilpotent orbit $\mathbb{O} \subseteq \mathscr{N}^a$ and an irreducible representation χ of $C(y)$, where $y \in \mathbb{O}$, and $C(y) := \mathrm{Stab}_G(y)/\mathrm{Stab}(y)^\circ$, i.e. a local system on $\mathbb{O}$.

To understand L_ϕ concretely, we exhibited them as image of a map between standard and costandard modules

$$H^\bullet(i_{\mathbb{O}}^!\mathbb{L}) \to H^\bullet(i_{\mathbb{O}}^*\mathbb{L}). \tag{8.2}$$

The standard modules can be described as $H_\bullet^{\mathrm{BM}}(\mathscr{B}_\chi^s)$ while the standard modules have similar interpretations in terms of transversal slices.

The image is $L_\phi \cong L_{a,x,\chi}$.

Theorem 8.0.1. *For any semisimple element $a = (s,t) \in G \times \mathbb{C}^\times$, and any $x \in \mathscr{N}^a$, $\chi \in \mathrm{Irr}(C(s,x))$, the $\mathscr{H}_a$-module $L_{a,x,\chi}$ is simple provided it is non-zero. Thus such modules $L_{a,x,\chi}$ and $L_{a,x',\chi'}$ are isomorphic if and only if (x,χ) is conjugate to (x',χ').*

Theorem 8.0.2 (Deligne-Langlands Conjecture)**.** *Irreducible $\mathscr{H}$-modules are parameterized by G-conjugacy classes of triples (s,x,χ) where $sxs^{-1} = qx$, and $\chi \in \mathrm{Irr}(C(s,x))$, i.e. χ shows up in $H_\bullet^{\mathrm{BM}}(\mathscr{B}_x^s)$.*

Theorem 8.0.3 (Non-vanishing theorem, Kazhdan-Lusztig-Grojnawski-Ginzburg)**.** *If χ shows up in $H_\bullet(\mathscr{B}_x^s)$, then $L_{a,x,\chi}$ is nonzero.*

Proposition 8.0.4. *Assume that $t \in \mathbb{C}^\times$ is not a root of unity. There exists a $G(s)$-stable union of connected components $\hat{\mathbb{O}}$ of $\widetilde{\mathscr{N}^a}$ such that $\mu(\hat{\mathbb{O}}) = \overline{\mathbb{O}}$, for any $G(s)$-orbit $\mathbb{O}$.*

Let $\hat{\mathscr{B}}_x^s = \mathscr{B}_x^s \cap \mathbb{O}$.

Proposition 8.0.5. *Assume $a = (s, t)$, t is not a root of unity. Then any simple $C(s, x)$-module occurring in $H^\bullet(\mathscr{B}^s_x)$ with non-zero multiplicity, also occurs in $H^\bullet(\hat{\mathscr{B}}^s_x)$ with non-zero multiplicity.*

Proof of Non-vanishing Theorem assuming Prop.8.0.4 and Prop.8.0.5. Recall that we have that

$$\mu_* \mathrm{IC}(\widetilde{\mathscr{N}^a}) = \oplus_{\chi,\mathscr{O}} L_{a,x,\chi} \otimes \mathrm{IC}(\mathbb{O}, \chi). \tag{8.3}$$

The complex $\mu_*\mathrm{IC}(\widetilde{\mathscr{N}^a})$ contains $\mu_*\mathrm{IC}(\hat{\mathbb{O}})$ as a direct summand.

$$\mu_*\mathrm{IC}(\hat{\mathbb{O}}) = (\oplus_\chi \hat{L}_\chi \otimes \mathrm{IC}(\mathbb{O}, \chi)) \oplus B, \tag{8.4}$$

where B is supported on the boundary $\overline{\mathbb{O}} - \mathbb{O}$ and L_χ are exactly the multiplicities of χ in $H^\bullet(\hat{\mathscr{B}}^s_x)$. □

Sketch of Proof of Prop.8.0.4.

Lemma 8.0.6. *(1) The group $G(s)$ is a connected reductive group, and each connected component of $\mathscr{B}^s$ is a submanifold of $\mathscr{B}$, which is $G(s)$-equivariantly isomorphic to the flag variety for $G(s)$. (need to assume the dual group G to have simply connected derived group, i.e. the original group has connected center).*

(2) If $t \in \mathbb{C}^\times$ is not a root of unity, then $\mathfrak{g}^a = \mathscr{N}^a$. (This is proved last time.)

Lemma 8.0.7. *(1) There is an embedding $\mathfrak{sl}_2(\mathbb{C}) \xrightarrow{\gamma} \mathfrak{g}$ associated to x, such that $s = s_\gamma s_0$, where s_0 is a semi-simple element commuting with the image of γ and $s_\gamma = \gamma \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}$, $\tau^2 = t$.*

(2) There is a group homomorphism $v : \mathbb{C}^\times \mathbb{R}$ with $v(\tau) > 0$.

Let γ, τ, s_0 be as in the lemma above. We have a weight decomposition of $\mathfrak{g}$ by conjugation of s_0, $\mathfrak{g} = \oplus_{\alpha \in \mathbb{C}^\times} \mathfrak{g}_\alpha$, and we define

$$\mathfrak{p} := \oplus_{v(\alpha) \le 0}, \tag{8.5}$$

$$\mathfrak{l} := \oplus_{v(\alpha) = 0}, \tag{8.6}$$

$$\mathfrak{u} := \oplus_{v(\alpha) < 0}. \tag{8.7}$$

One can choose that $x \in \mathfrak{l}$ and $s \in L$. Let $\mathscr{P} \subseteq \mathscr{B}$ be the subvariety consisting of all borel subalgebras $\mathfrak{b} \subseteq \mathfrak{p}$ (the flag variety for L). Let $\mathscr{P}^s_x := \mathscr{B}^s_x \cap \mathscr{P}$, which is not empty since it is the s and $\exp(x)$-fixed point set of the closed subset $\mathscr{P}$, and $s, \exp(x)$ generates a solvable group.

Definition 8.0.8. Let $\hat{\mathbb{O}}$ be the union of all connected components of $\widetilde{\mathscr{N}^a}$ which has nonempty intersection with $\mathscr{P}^s_x$.

It is easy to see hat $\hat{\mathbb{O}}$ is $G(s)$-stable and that the image of any connected component of $\hat{\mathbb{O}}$ contains x, which implies it contains $G(s)$-orbit $\mathbb{O}$. Checking that $\mu(\hat{\mathbb{O}}) = \mathbb{O}$ reduces to the following technical lemma.

Lemma 8.0.9. *Let $\mathfrak{b} \in \mathscr{P}^s_X$ and $\mathfrak{n}$ be its nilradical. Then $\mathrm{Ad}(G(s))(\mathfrak{n} \cap \mathfrak{g}^a) = \overline{\mathbb{O}}$.*

□

Sketch of Proof of Prop.8.0.5. Plans: $[H^\bullet(\mathscr{B}_x^s):\chi]\neq 0 \Rightarrow [H^\bullet(\mathscr{P}_x^s):\chi]\neq 0 \Rightarrow [H^\bullet(\hat{\mathscr{B}}_x^s):\chi]\neq 0$.

Let $Z=Z^0(L)$ be the identity component of the center of L, then Z is a complex torus which commutes with both x and s, since we choose $x,s\in L$. So $\mathscr{B}_x^s$ is a Z-stable variety $\mathscr{B}$. Let T be the maximal compact of Z, Then $(\mathscr{B}_x^s)^Z=(\mathscr{B}_x^s)^T$. Then Fixed-point reduction implies that

$$[H^\bullet(\mathscr{B}_x^s)]=[H^\bullet(\mathscr{B}_x^s)^{Z^0(L)}] \tag{8.8}$$

in the Grothendieck group of $L(s,x)$-modules. This is also true in the Grothendieck group of $L(s,x)/L(s,x)^\circ$-modules, and thus holds a s $C(s,x):=G(s,x)/G(s,x)^\circ$-modules. If χ is such that

$$[H^\bullet(\mathscr{B}_x^s):\chi]\neq 0\Rightarrow [H^\bullet(\mathscr{B}_s^x)^{Z^0(L)}:\chi]\neq 0 \tag{8.9}$$

then one can show that $(\mathscr{B}_x^s)^{Z^0(L)}$ is a disjoint union of pieces isomorphic to $\mathscr{B}(L)_x^s\cong\mathscr{P}_x^s$. Finally $\mathscr{O}_x^s\subseteq\hat{\mathscr{B}}_x^s$, which implies that $H^\bullet(\mathscr{P}_x^s)$ is canonically a direct summand of $H^*(\hat{\mathscr{B}}_x^s)^{Z^0(L)}$. So we know for $L(s,\chi)$-modules, and then we know

$$[H^\bullet(\mathscr{P}_x^s):\chi]\neq 0\Rightarrow [H^\bullet(\hat{\mathscr{B}}_x^s):\chi]\neq 0 \tag{8.10}$$

for $C(s,x)$-modules χ. □

Recall that

$$\mu_*C_M\cong\oplus_\phi L_\phi\otimes \mathrm{IC}_\phi. \tag{8.11}$$

Theorem 8.0.10. *The multiplicity of the simple $\mathscr{H}_a$-module L_ϕ in the standard module $H_\bullet(M_x)_\psi$ is given by the following formula*

$$[H_\bullet(M_x)_\psi:L_\phi]=\sum_k \dim H^k(i_x^!\mathrm{IC}_\phi)_\psi. \tag{8.12}$$

Proof. Recall that if $\mathbb{L}=\mu_*C_M$, this is the same as the multiplicity of the RHom$(\mathbb{L},\mathbb{L})$-module L_ϕ in the module $H^\bullet(i_x^!\mathbb{L})_\psi$. Apply the functor $H^\bullet i_x^!$ to the decomposition (8.11) and we get

$$H_\bullet^{\mathrm{BM}}(M_x)=H^\bullet(i_x^!\mu_*C_M)=\oplus_\phi L_\phi\otimes H^\bullet(i_x^!\mathrm{IC}_\phi). \tag{8.13}$$

For any j,k, we have that

$$\mathrm{Ext}^k(\mathbb{L},\mathbb{L}):\oplus_\phi L_\phi\otimes H^j(i_x^!\mu_*C_M)\to\oplus_\phi L_\phi\otimes H^{j+k}(i_x^!\mu_*C_M). \tag{8.14}$$

Define $F^pH^\bullet(i_x^!\mathbb{L})=\oplus_{j\geq p}(\oplus_\phi L_\phi\otimes H^j(i_x^!\mathrm{IC}_\phi))$. It is $\mathrm{E}xt^\bullet(\mathbb{L},\mathbb{L})$-stable. We can then consider $\mathrm{gr}^F H^\bullet(i_x^!\mathbb{L})$. Here the action factors through the projection to $\mathrm{Ext}^0(\mathbb{L},\mathbb{L})\cong\oplus_\phi\mathrm{End}(L_\phi)$. As a vector space

$$\mathrm{gr}_\bullet^F H^\bullet i_x^!\mathbb{L}\cong\oplus_\phi L_\phi\otimes H^\bullet(i_x^!\mathrm{IC}_\phi). \tag{8.15}$$

Taking ψ-components, we see that L_ϕ occurs exactly $\dim H^\bullet(i_x^!\mathrm{IC}_\phi)_\psi$. □

To wrap up, $K^{\hat{\check{G}}\times\mathbb{C}^\times}(\mathscr{Z})\cong\mathscr{H}_\mathcal{I}\cong\mathbb{C}[\mathcal{I}\backslash G(F)/\mathcal{I}]$ are two realizations of Iwahori Hecke algebras. Then one could conjecture an equivalence of derived categories

$$\mathrm{Coh}^{\check{G}}(\mathscr{Z})\cong\mathrm{Shv}_c^{\mathcal{I}}(\mathrm{FL}). \tag{8.16}$$

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