KÜNNETH FORMULA FOR ÉTALE FUNDAMENTAL GROUPS IN CHARACTERISTIC 0

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Abstract. In this note, we provide a purely algebraic proof that the étale fundamental group is invariant under base change of algebraically closed fields of characteristic 0. As a corollary, we obtain the Künneth formula for étale fundamental groups in characteristic 0. We also explain how to adapt the proof (using alterations) to apply to prime-to-p fundamental groups in characteristic $p > 0$. These results are known to experts, but proofs do not seem to be documented in the literature, so for convenient citation by others we are providing them in a written form here.

1. Introduction

In this short note, our main goal is to prove the following theorem.

Theorem 1.1 (Main Theorem)**.** *Let $k \to K$ be an extension of algebraically closed fields of characteristic 0. Let X be a k-scheme of finite type. Then*

(1) The base change functor $\mathrm{FEt}(X) \to \mathrm{FEt}(X_K)$ is an equivalence of categories.
(2) If X is connected, then for any geometric point $\overline{x}$ of X_K, the map $\pi_1^{ét}(X_K, \overline{x}) \to \pi_1^{ét}(X, \overline{x})$ is an isomorphism.

Remark 1.2. For convenience, we write

$$X_K := X \times_{\operatorname{Spec} k} \operatorname{Spec} K = X \otimes_k K$$

for short.

Remark 1.3. When X is connected, the surjectivity of

$$\pi_1^{\text{ét}}(X_K, \overline{x}) \to \pi_1^{\text{ét}}(X, \overline{x})$$

is justified as follows.

Proof. Recall the following theorem.

Theorem 1.4. [EGA IV_2, Theorem 4.4.4] Suppose that k is an algebraically closed field and X is a k-scheme. If X is irreducible (resp. connected), then X_K is irreducible (resp. connected) for any field extension K/k.

Therefore, we know that X_K is connected. Moreover, given any finite étale $E \to X$ with E connected, E_K is still connected by Theorem 1.4 above, and hence we obtain the surjectivity of $\pi_1^{\text{ét}}(X_K) \to \pi_1^{\text{ét}}(X)$ by [SGA 1, Exp. V, Proposition 6.9]. ∎

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2. Proof of the Main Theorem

In this section, we work on the proof of the main theorem.

Theorem 2.1 (Künneth formula, [SGA 1, Exp. X, Corollary 1.7])**.** *Let k be an algebraically closed field, X and Y two connected schemes over k. Suppose that X is proper over k and Y is locally Noetherian. Let $\overline{x}\colon \operatorname{Spec} K \to X$ be a geometric point of X and $\overline{y}\colon \operatorname{Spec} K \to Y$ a geometric point of Y with value in the same algebraically closed field K. Set $\overline{z} = (\overline{x}, \overline{y})$, a geometric point of $X \times_k Y$. Then the group homomorphism*

$$\pi_1^{ét}(X \times_k Y, \overline{z}) \to \pi_1^{ét}(X, \overline{x}) \times \pi_1^{ét}(Y, \overline{y}), \tag{1}$$

induced by the natural projections $p_X\colon X \times_k Y \to X$ and $p_Y\colon X \times_k Y \to Y$ is an isomorphism.

Remark 2.2. This result does not depend on the characteristic of the field k.

Remark 2.3. This result can be generalized to relax the condition that X is proper to the condition that

$$\operatorname{FEt}(X) \to \operatorname{FEt}(X_K)$$

is an equivalence of categories for any extension K/k of algebraically closed fields. See [SW20, Lemma 16.1.2].

Theorem 2.4 (proper case, [SGA 1, Exp. X, Corollary 1.8])**.** *Suppose that K/k is an extension of algebraically closed fields and X is a proper scheme over k. Then the functor $U \mapsto U_K$ is an equivalence of categories between $\operatorname{FEt}(X)$ and $\operatorname{FEt}(X_K)$. In particular, if X is connected and $\overline{x}$ is a geometric point of X_K, then the canonical group homomorphism $\pi_1^{ét}(X_K, \overline{x}) \to \pi_1^{ét}(X, \overline{x})$ is an isomorphism.*

Theorem 2.5 (Smooth case)**.** *Let k be an algebraically closed field of characteristic 0 and let X be a smooth scheme of finite type over k. Then the base change functor $\operatorname{FEt}(X) \to \operatorname{FEt}(X_{\widetilde{K}})$ is an equivalence of categories for any extension $\widetilde{K}/k$ of algebraically closed fields.*

Proof. Let $\widetilde{K}$ be an algebraically closed field extension of k. We want to show that the natural map

$$\operatorname{FEt}(X) \to \operatorname{FEt}(X_{\widetilde{K}})$$

is an equivalence of categories. Since X is Noetherian, by passing to connected components we can assume without loss of generality that X is connected. Then $X_{\widetilde{K}}$ is connected by Theorem 1.4. Firstly, we want to show that the functor $\operatorname{FEt}(X) \to \operatorname{FEt}(X_{\widetilde{K}})$ is fully faithful. Then we can work locally on X to verify that it is also essentially surjective.

The fully faithfulness follows the following Lemma 2.6.

Lemma 2.6 (descent of morphisms of étale schemes)**.** *Let Z be a scheme of finite type over an algebraically closed field k, E_1, E_2 two finite étale schemes over Z, and K/k a field extension. The natural map*

$$\operatorname{Hom}_Z(E_1, E_2) \to \operatorname{Hom}_{Z_K}((E_1)_K, (E_2)_K)$$

is bijective.

Proof of Lemma 2.6. We mimic the proof of [SGA 1, Exp. IX, Proposition 3.1]. Note that the set of morphisms $E_1 \to E_2$ over Z is in bijection with the set of sections of (the étale morphism) $E_1 \times_Z E_2 \to E_1$ over Z. By passing to connected components (which are open since Z is Noetherian), we reduce to the case where E_1 is connected. Then $(E_1)_K$ is also connected by Theorem 1.4.

Now we claim that $\mathrm{Hom}_Z(E_1, E_2)$ is in bijection with $\pi_0(E_1 \times_Z E_2)$, i.e. a section $s\colon E_1 \to E_1 \times_Z E_2$ is equivalent to choose a connected component of $E_1 \times_Z E_2$ mapping isomorphically to E_1. Note that such a section s must be finite étale due to the cancellation property of étaleness and finiteness. Moreover, s is radicial since it is a section. Being radicial and étale, s must be an open immersion by [SGA 1, Exp. I, Corollary 5.1]. By finiteness, s has closed image. Now we see that s is an isomorphism from E_1 to a connected component of $E_1 \times_Z E_2$.

Since Z_K is also Noetherian, the above argument works for Z_K, too. However,

$$\pi_0((E_1)_K \times_{Z_K} (E_2)_K) \xrightarrow{\cong} \pi_0(E_1 \times_Z E_2)$$

is bijective by Theorem 1.4, so we are done. □

Now we can work locally on X to verify that the functor is essentially surjective when X is smooth. Hence, we can assume without loss of generality that X is separated. Given any finite étale covering $E_{\widetilde{K}} \to X_{\widetilde{K}}$, we want to show that it actually comes from a finite étale cover of X. We can assume without loss of generality that $E_{\widetilde{K}}$ is connected.

Since the étale covering $E_{\widetilde{K}} \to X_{\widetilde{K}}$ is defined over a finitely generated field over k, there exists some subfield K of $\widetilde{K}$, such that

(1) K is finitely generated over k,

(2) there exists some finite étale covering $E_K \to X_K$, such that $(E_K) \otimes_K \widetilde{K} = E_{\widetilde{K}}$.

Moreover, by induction on $\mathrm{tr.deg}(\widetilde{K}/k)$ (which we can arrange to be finite by replacing $\widetilde{K}$ with K^{alg}) we can assume that K/k is of transcendence degree 1. Therefore, K is the function field of a smooth proper curve over k. By spreading out, we get $E' \to X \times_k C$ a finite étale covering, where C is an affine smooth curve with function field $k(C) = K$. Let $\overline{C}$ be the smooth compactification of C and form the normalization $\overline{E'}$ of $X \times_k \overline{C}$ in E'. Therefore, we get finite morphism $\overline{E'} \to X \times_k \overline{C}$, which is possibly ramified over the codimension 1 points corresponding to the points in $\overline{C} - C$ but at least tamely ramified by the characteristic 0 hypothesis[1]. We can find a finite surjective morphism $\overline{C'} \to \overline{C}$ sufficiently ramified at points in $\overline{C} - C$ by repeatedly applying :

Lemma 2.7 (Existence of ramified covering)**.** *Let k be an algebraically closed field and $\overline{C}$ a smooth proper curve over k. Given a rational point $x \in \overline{C}(k)$ and a non-negative integer e, there exists a finite cover $f\colon \overline{C'} \to \overline{C}$ with degree not divisible by* $\mathrm{char}(k)$ *such that f has ramification index $> e$ at the point x.*

Proof of Lemma 2.7. Note that smooth proper curves over k correspond to finitely generated field extensions of k with transcendence degree 1. Therefore, we can work on the corresponding problem

[1]This is the only place where we use the characteristic 0 condition.

about fields. Given K, a place v trivial on k, the field extension K'/K by joining a r-th root of a uniformizer at v is what we want, where we choose an integer $r > e$ and not divisible by $\mathrm{char}(k)$. $\square$

Remark 2.8. Actually in the proof of Lemma 2.6, we only need étaleness of $E_2 \to Z$.

We continue the proof of Theorem 2.5. Form the normalization $\widetilde{E}$ of $E' \times_{X \times_k C} (X \times_k C')$ over $X \times_k \overline{C'}$, where C' is the preimage of C under $\overline{C'} \to \overline{C}$. Then by Abyhankar's lemma (see [SGA 1, Exp. X, Lemma 3.6]), if $\overline{C'} \to \overline{C}$ is taken to be sufficently ramified over $\overline{C} - C$, then

$$\widetilde{E} \to X \times_k \overline{C'}$$

is unramified at all points of codimension 1. Now we can use Zariski-Nagata's theorem (also called purity of the branch locus, see [SGA 1, Exp. X, Theorem 3.1]) to get étaleness of $\widetilde{E} \to X \times_k \overline{C'}$, where we use the smoothness of X. We summarize the argument above in the following diagram

$$\begin{array}{ccccccccc}
E_K & \longrightarrow & E' & \longrightarrow & \overline{E'} & & \widetilde{E} & \longleftarrow & E' \times_{(X \times_k C)} (X \times_k C') \\
\Big\downarrow{\scriptstyle \text{étale}} & & \Big\downarrow{\scriptstyle \text{étale}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \text{étale}} & & \Big\downarrow{\scriptstyle \text{étale}} \\
X_K & \longrightarrow & X \times_k C & \hookrightarrow & X \times_k \overline{C} & \twoheadleftarrow & X \times_k \overline{C'} & \hookleftarrow & X \times_k C'
\end{array} \tag{2}$$

Since $\overline{C'}$ is proper, by Theorem 2.1, we see that $\widetilde{E} \cong E_X \times_k E_{\overline{C'}}$ for some finite étale covering E_X (resp. $E_{\overline{C'}}$) of X (resp. $\overline{C'}$). Now it remains to show that the finite étale cover $E_X \to X$ is exactly what we want.

Since $\widetilde{K}$ is algebraically closed field containing the function field of $\overline{C'}$, we can pick a morphism

$$\mathrm{Spec}\, \widetilde{K} \to \overline{C'}$$

onto the generic point of $\overline{C'}$, such that

$$\begin{array}{ccc}
\mathrm{Spec}\, \widetilde{K} & \longrightarrow & \overline{C'} \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{Spec}\, K & \longrightarrow & \overline{C}
\end{array} \tag{3}$$

commutes. Note that we have a Cartesian square

$$\begin{array}{ccc}
E_{\widetilde{K}} & \longrightarrow & E' \\
\Big\downarrow & \lrcorner & \Big\downarrow \\
X_{\widetilde{K}} & \longrightarrow & X \times_k C
\end{array} \tag{4}$$

and then obtain

$$\begin{array}{ccccc}
E_{\widetilde{K}} & \longrightarrow & E' \times_{(X \times_k C)} (X \times_k C') & \longrightarrow & E' \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
X_{\widetilde{K}} & \longrightarrow & X \times_k C' & \longrightarrow & X \times_k C
\end{array} \tag{5}$$

where each square is a Cartesian square. Since $X \times_k C' \hookrightarrow X \times_k \overline{C'}$ is an open embedding, we see that

$$\begin{array}{ccc} E_{\widetilde{K}} & \longrightarrow & \widetilde{E} = E_X \times_k E_{\overline{C'}} \\ \downarrow & & \downarrow \\ X_{\widetilde{K}} & \longrightarrow & X \times_k \overline{C'} \end{array} \tag{6}$$

is a Cartesian square. However,

$$(E_X \times_k E_{\overline{C'}}) \times_{X \times_k \overline{C'}} (X \times_k \operatorname{Spec} \widetilde{K}) \cong E_X \times_k (E_{\overline{C}} \times_{\overline{C'}} \operatorname{Spec} \widetilde{K}) \tag{7}$$

by universal property. So

$$E_X \times_k (E_{\overline{C}} \times_{\overline{C'}} \operatorname{Spec} \widetilde{K}) \cong E_{\widetilde{K}}$$

is connected. Therefore, $E_{\overline{C}} \times_{\overline{C'}} \operatorname{Spec} \widetilde{K}$ is connected finite étale cover of $\operatorname{Spec} \widetilde{K}$, and hence

$$E_{\overline{C}} \times_{\overline{C'}} \operatorname{Spec} \widetilde{K} \cong \operatorname{Spec} \widetilde{K}.$$

Therefore, $E_X \otimes_k \widetilde{K} \cong E_{\widetilde{K}}$. □

Remark 2.9. Since we only use the condition that $\operatorname{char}(k) = 0$ to obtain that show that

$$E' \to X \times_k C$$

is tamely ramified over codimension 1 points corresponding to $\overline{C} - C$, our proof works for tame fundamental groups in any characteristic (but we still have to assume that X is smooth).

Now we prove the main theorem (without the assumption of smoothness).

Corollary 2.10 (General case)**.** *Let k be an algebraically closed field of characteristic 0 and X a scheme of finite type over k. Then the base change functor $\mathrm{FEt}(X) \to \mathrm{FEt}(X_K)$ is an equivalence of categories for any extension K/k of algebraically closed fields.*

Proof. As in the proof of Theorem 2.5, we only need to work locally to show $\mathrm{FEt}(X) \to \mathrm{FEt}(X_K)$ is essentially surjective. Therefore, we can safely assume X to be separated. Then we are able to take a resolution of singularities $X' \to X$, which is surjective and proper. Given $E_K \to X_K$ a finite étale cover, we consider the base change $E'_K \to X'_K$, where $E'_K := E_K \times_{X_K} X'_K$, and by Theorem 2.5 we know it descends to some finite étale cover $E' \to X'$. Recall the following theorem in [SGA 1].

Theorem 2.11 ([SGA 1, Exp. IX, Theorem 4.12])**.** *If $f : Y \to X$ is a proper surjective morphism of finite presentation between schemes, then we have an equivalence of categories between $\mathrm{FEt}(X)$ and the category of finite étale Y-schemes equipped with a "descent datum": an isomorphism of its two pullbacks to $Y \times_X Y \rightrightarrows Y$ satisfying the cocycle condition.*

Therefore, we only need to build the descent data. Let p_1, p_2 be the two projections

$$X' \times_X X' \rightrightarrows X'$$

and $p_{1,K}, p_{2,K}$ the base changed projections

$$X'_K \times_{X_K} X'_K \rightrightarrows X'_K.$$

Note that we have canonical descent data for E'_K

$$\varphi_K \colon p_{1,K}^* E'_K \xrightarrow{\cong} p_{2,K}^* E'_K.$$

Then by Lemma 2.6, we obtain $\varphi \colon p_1^* E' \xrightarrow{\cong} p_2^* E'$ satisfying the cocycle condition. Therefore, E' descends to some finite étale E over X and this does the job. □

Combining Remark 2.3 and Corollary 2.10, we can safely drop the assumption that X is proper in Theorem 2.1 when $\mathrm{char}(k) = 0$.

Corollary 2.12 (Künneth Formula for Étale Fundamental Groups in Characteristic 0)**.** *Let X and Y be two connected schemes of finite type over an algebraically closed field k with characteristic 0. Then we have an isomorphism of topological groups*

$$\pi_1^{\acute{e}t}(X \times_k Y) \xrightarrow{\cong} \pi_1^{\acute{e}t}(X) \times \pi_1^{\acute{e}t}(Y), \tag{8}$$

where the morphism is induced by the projections $X \times_k Y \to X$ and $X \times_k Y \to Y$.

3. Generalization to Prime-to-p Fundamental Groups

Now we consider the case $\mathrm{char}(k) = p > 0$. We modify the fundamental groups to allow only coverings whose connected components have Galois closure with degree not divisible by p.

Definition 3.1. Let X be a connected k-scheme of finite type. We define $\mathrm{FEt}^{(p)}(X)$ to be the category of finite étale coverings whose connected components have Galois closure with degree over X not divisible by p.

This is a Galois category and the fiber functor at a geometric point $\overline{x}$ gives a profinite group $\pi_1^{(p)}(X, \overline{x})$, called prime-to-$p$ fundamental group (at $\overline{x}$).

Now we adapt Theorem 2.5 and Corollary 2.10 to this setting.

Theorem 3.2 (Invariance of Prime-to-p Fundamental Group under Base Change between Algebraically Closed Fields)**.** *Let $k \to K$ be an extension of algebraically closed fields. Let X be a connected k-scheme of finite type. Then we have*

1. *The base change functor $\mathrm{FEt}^{(p)}(X) \to \mathrm{FEt}^{(p)}(X_K)$ is an equivalence of categories.*
2. *For any geometric point $\overline{x}$ of X_K, the map $\pi_1^{(p)}(X_K, \overline{x}) \to \pi_1^{(p)}(X, \overline{x})$ is an isomorphism.*

Proof. Firstly, we should adapt Theorem 2.5 to this setting. Note that our proof of Theorem 2.5 only needs characteristic 0 condition to show that

$$\overline{E'} \to X \times_k \overline{C}$$

is tamely ramified over codimension 1 points corresponding to $\overline{C} - C$, which is automatically true for $E_K \to X_K$ finite Galois cover of degree not divisible by p. Therefore, Theorem 2.5 can be adapted to this case. For a finite étale covering $E_K \to X_K$ whose Galois closure $E'_K \to X_K$ is of degree prime to p, we have known that $E'_K = E' \otimes_k K$ for some finite étale covering $E' \to X$, with Galois group G. Then $E'_K \to E_K$ is Galois cover with some Galois group H. Then $E = E'/H$ does the job.

Now we deal with the general case. We can assume without loss of generality that the finite étale X_K-scheme E_K is connected. In the proof of Corollary 2.10, we can replace the resolution of singularities with a weaker version in $\mathrm{char}(k) = p$, i.e. de Jong's alteration theorem (see [Jon96, Theorem 4.1]). Then we get a proper dominant, generically étale morphism $X' \to X$, where X' is regular and hence smooth, since k is algebraically closed. Though X' might be disconnected, with connected components X'_i $(i = 1, 2, \ldots, m)$, we can focus on each X'_i. Define

$$(E'_i)_K := E'_K \times_{X'_K} (X'_i)_K.$$

Since $E_K \to X_K$ is finite étale covering whose Galois closure is of degree not divisible by p, so is

$$(E'_i)_K \to (X'_i)_K$$

for each i. Then by the smooth case, we know it descends to a k-scheme E'_i finite étale over X'_i. Set $E' := \sqcup_i E'_i$ and then descend it to $E \to X$ as in the proof of Corollary 2.10. □

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