1. Affine Grassmannian for $G_a$

In this problem we are proving:

$$\text{Gr}_{G_a} = \lim_{\rightarrow} A^n$$

with $A^n \hookrightarrow A^{n+1}$ being the inclusion of a hyperplane in the obvious way.

**Intuition.** First some intuition. We always have *to first approximation*

$$\text{Gr}_G(k) = G(k((t)))/G(k[[t]]).$$

In this case, this tells us that

$$\text{Gr}_{G_a}(k) = k((t))/k[[t]] = \lim_{\rightarrow} t^{-n}k[[t]]/k[[t]].$$

Now $t^{-n}k[[t]]/k[[t]]$ is a $k$-vector space on the coefficients of $t^{-n}, \ldots, t^{-1}$, which looks like $A^n(k)$.

It’s worth making clear why this is wrong. To promote this to a proof, we would have to make the argument with $k$ replaced by an arbitrary $k$-algebra $R$. Although $\text{Gr}_G = L^G/G/L^+G$, this does not mean that

$$\text{Gr}_G(R) = L^G(R)/L^+G(R) = G(R((t)))/G(R[[t]]).$$

Rather, the left side is the sheafification of the right side (e.g. for the étale topology). Of course, when $R$ is an *algebraically closed* field then no such sheafification is necessary.

**Onto the proof.** So what does $\text{Gr}_{G_a}$ look like anyway? By definition, its functor of point is given by $R \mapsto \{ (E, \beta) \}$ where $E$ is a $G_a$-bundle over $R[[t]]$ and $\beta$ is a trivialization of $E$ over $R((t))$.

What is a $G_a$-bundle anyway? When confused about such questions, we can always find one answer by embedding into $GL_n$, for instance as the unipotent group

$$t \mapsto \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}.$$ 

This makes it clear that a $G_a$-bundle is the same as an extension of trivial bundles:

$$1 \to \mathcal{O}_S \to E \to \mathcal{O}_S \to 1.$$
Now let’s unravel what this tells us over \( S = \text{Spec } R[[t]] \). We’re looking at pairs \((E, \beta)\) where \( E \) is a rank 2 locally free modules \( M \) over \( R[[t]] \), with an extension structure

\[
0 \rightarrow R[[t]] \rightarrow M \rightarrow R[[t]] \rightarrow 0
\]

with a trivialization \( \beta \) over \( R((t)) \). Of course all such \( M \) are abstractly trivial, but the point is that \( \beta \) rigidifies the situation. Thanks to \( \beta \), we may assume that \( M \otimes_{R[[t]]} R((t)) \cong R((t))^\oplus 2 \), with the splitting being the coordinate functions. This means that \( M \) itself is described by an \( R[[t]] \)-lattice inside \( R((t))^\oplus 2 \), which can be expressed as a matrix

\[
\begin{pmatrix}
1 & f \\
1 & 1
\end{pmatrix}.
\]

This matrix is ambiguous up to change of basis that preserves the extension structure, which amounts to multiplication by a unipotent matrix over \( R[[t]] \). This means that \( f \) (which is a priori an element of \( R((t)) \)) is well-defined up \( R[[t]] \), i.e. is determined by its class in \( R((t))/R[[t]] \). So we have concluded that indeed

\[
\text{Gr}_{G_a}(R) = R((t))/R[[t]] = \lim_{\rightarrow} A^n(R).
\]

2. Affine Grassmannian for \( G_m \)

2.1. The loop group. First we are going to investigate the loop group for \( G_m \). We have

\[
LG(R) = R((t))^*.
\]

We can canonically decompose the units as follows. Let

\[
f = \sum a_i t^i \in R((t))^*.
\]

Let \( n \) be the first index such that \( a_n \) is invertible. (This obviously exists, since the image of \( f \) in \( R/N(R)((t)) \) had better be invertible.) Multiplying by \( a_n^{-1} t^{-n} \), we can assume that this index is 0 and the corresponding coefficients is 1. Then write

\[
f = n + 1 + p
\]

where \( n \) is the “polar part” (negative powers of \( t \), with nilpotent coefficients) and \( p \in tR[[t]] \). By standard algebra, \( 1 + n \) is invertible (formally expand \( 1/(1 + n) \)). So

\[
f = (1 + n)(1 + \frac{p}{1 + n}) = (1 + n)(1 + p(1 - n + n^2 - \ldots)).
\]

We need to be careful, since the second factor need not have purely positive exponents of \( t \). Let

\[
f' = (1 + p(1 - n + n^2 - \ldots)) = (n' + 1 + p')
\]

We will argue that \( f' \) is “less bad” in some sense than \( f \), so that iterating this process eventually produces a factorization of the desired form. Note that for \( i > 0 \), the coefficient of \( t^{-i} \) in \( f' \) is a combination of coefficients for strictly less negative exponents of \( f \), or polynomials in positive degree of coefficients which are at least as negative.

Therefore, we can assign a “score” to Laurent series which measures the degree of nilpotency of the polar part. Laurent series for which the negative coefficients
are “more nilpotent” have lower scores. We leave it to the reader to formalize this idea, and verify that the operation \( f \mapsto f' \) decreases the score, so that after a finite number of iterations the nilpotent part disappears.

Since \( \frac{n}{1+n} \) has nilpotent coefficients, \( 1 + \frac{n}{1+n} \) is also invertible (again, formally expand). So we have expressed \( f \) as a product of \( f_+ \in 1 + tR[[t]] \) and \( f_- \), a polar part with nilpotent coefficients. This is obviously unique.

The conclusion is that \( R((t))^* \) can be uniquely expressed as

\[
f = r \cdot t^n \cdot f_+ \cdot f_-
\]

with \( r \in R^* \) and \( f_+, f_- \) as above.

This gives a decomposition

\[
LG_m \cong Z \times G_m \times W \times \hat{W}
\]

where \( W \) is the subscheme parametrizing possibilities for \( f_+ \), and \( W \) is the ind-subscheme parametrizing \( f_- \).

2.2. Some geometric properties. We have just seen that \( LG_m \cong Z \times \hat{W} \). The functor \( W \) assigns to \( R \) the set of finite tuples \((r_i)\) with each \( r_i \) nilpotent. This is the inductive limit of schemes \( W_{n,n} \) parametrizing tuples \((r_1, \ldots, r_n)\) with \( r_i^n = 0 \), which is represented by \( k[x_1, \ldots, x_n]/(x_1^n, \ldots, x_n^n) \), which is evidently non-reduced.

However, each \( W_n \) is formally smooth. To see this, it suffices to check that \( W_{1,n}(R) = \{ r \in R : r^n = 0 \} \) is formally smooth, i.e. if \( I \) is a nilpotent ideal in \( R \) then

\[
W_{1,n}(R) \twoheadrightarrow W_{1,n}(R/I).
\]

But this is clear: for any \( \tau \in W_{1,n}(R/I) \), choose any lift \( r \) of \( \tau \) in \( R \).

2.3. The arc group and affine Grassmannian. From this argument we deduce in particular that the arc group is

\[
L^+G_m \cong G_m \times W,
\]

embedded as you expect, so

\[
GrG_m \cong Z \times W.
\]

Note that \( \pi_0 \text{Gr}G_m = Z \), as expected!

Note that the map \( LG_m \to LG_n \) is not an open embedding. The finite layer subschemes of \( \hat{W} \) are closed, cut out by the equations satisfied by the nilpotent elements of \( t^{-n}k[[t]]/k[[t]] \).

3. Relative positions

3.1. Dominant coweights in \( GL_n \). Let’s remind ourselves about the dominant coweights of \( GL_n \), and their natural ordering.
Definition 3.1. A dominant coweight of $GL_n$ is a coweight that pairs non-negatively with any positive coroot. If $\mu, \mu'$ are coweights of $GL_n$, then we say that

$$\mu \leq \mu'$$

(sometimes denoted $\mu \prec \mu'$) if $\mu' - \mu$ is a sum of positive coroots.

Here’s the picture: each root cuts out a hyperplane in the Euclidean space $X_*(T) \otimes \mathbb{R}$. The connected components in the complement of these hyperplanes are cones called the Weyl chambers. The Weyl group acts transitively on the set of Weyl chambers.

Let’s unwind what this means for $GL_n$. We have of course $X_*(T) = \mathbb{Z}^n$, with

$$(m_1, \ldots, m_n) \leftrightarrow \text{diag}(t^{m_1}, \ldots, t^{m_n}).$$

The roots are

$$\chi^{ij} : \text{diag}(t^{m_1}, \ldots, t^{m_n}) \mapsto t^{m_i - m_j}.$$  

The standard positive roots, corresponding to the choice of upper-triangular Borel, are $\chi^{ij}$ for $i > j$. The positive coroots are then the tuples $(m_1, \ldots, m_n)$ with $m_i = 1$, $m_j = -1$ for some pair $i > j$, and all other entries 0.

The conclusion is that the order relation among the $m_i$ is exactly defined by:

$$(m'_1, \ldots, m'_n) > (m_1, \ldots, m_n) \text{ if you can "trade down" entries in } i \text{ for entries in } j, \text{ with } i > j, \text{ to get from } (m'_1, \ldots, m'_n) \text{ to } (m_1, \ldots, m_n).$$

If one wants to write this out in gory detail:

$$m'_1 \geq m_1$$
$$m'_1 + m'_2 \geq m_1 + m_2$$
$$\vdots$$
$$m'_1 + \ldots + m'_n = m_1 + \ldots + m_n.$$  

3.2. Loci of relative position. Let $X = \text{Spec } R$, and $E_1, E_2$ two vector bundles of rank $n$ over $R[[t]]$, with an identification over $R((t))$. For each $x \in \text{Spec } R$, we get by restriction two vector bundle over $\kappa(x)[[t]]$, which have some relative position $\beta_x$.

Then we claim that

$$X_{\leq \mu} := \{ x \in X : \beta_x \leq \mu \}$$

is closed. ♠♠♠ TONY: [this is trickier than it seems. TBC...]

3.3. General groups. For general $G$, for each dominant coweight $\mu$ we get a highest weight representation $V_\mu$ of $G$. This turns a principle $G$-bundle $E$ into a vector bundle $V_\mu^E$. Then we can translate the notion of relative position for $G$ into one for vector bundles (for each $\mu$):

Lemma 3.2. $Gr_{G, \leq \mu}(k)$ is the set of $(E, \beta)$ such that for each $\lambda$, the map

$$\beta_\mu : V_\mu^E \otimes_{k[[t]]} k((t)) \to V_\mu \otimes_k k((t))$$

extends to a map

$$V_\mu^E \to t^{-(\lambda, \mu)} V_\mu \otimes_k k[[t]].$$
Indeed, by choosing trivializations we can represent \( \beta \) by an element \( g \in G(k((t))) \), which is well-defined up to left and right multiplication by \( G(k[[t]]) \). The corresponding element for the vector bundle \( V^E_\mu \) is an element of \( GL_n(k((t))) \), which is the image of \( g \) by the defining map \( G \to GL(V^E_\mu) \). The “largest exponent” which appears is then \( \langle \lambda, \mu \rangle \) so that the map does extend as claimed.

4. Minuscule Schubert varieties

Let us recall the meaning of minuscule weights. We say that a dominant cocharacter \( \mu \) of \( G \) is minuscule if \( \mu \neq 0 \) and for every positive root \( \alpha \), we have \( \langle \mu, \alpha \rangle \leq 1 \).

**Example 4.1.** For \( GL_n \) the dominant minuscule coweights are

\[
\begin{align*}
\mu_1 &= (1, 0, \ldots, 0) \\
\mu_2 &= (1, 1, 0, \ldots, 0) \\
& \vdots \\
\mu_n &= (1, 1, \ldots, 1, 0) \\
\nu_1 &= (0, -1, \ldots, -1) \\
\nu_2 &= (0, 0, -1, \ldots, -1) \\
& \vdots \\
\nu_n &= (0, \ldots, 0, -1)
\end{align*}
\]

(Depending on your convention...) The Schubert cell for \( \mu_i \) parametrizes lattices \( \Lambda \) with

\[
k[[t]]^{\oplus n} \subset \Lambda \subset t^{-1}k[[t]]^{\oplus n}
\]

with \( \dim_k \Lambda/k[[t]]^{\oplus n} = i \). This is just the (normal!) Grassmannian \( Gr(i, n) \).

The Schubert cell for \( \nu_i \) parametrizes lattices \( \Lambda \supset k[[t]]^{\oplus n} \) of corank 1, i.e. such that

\[
\dim_k k[[t]]^{\oplus n}/\Lambda = 1
\]

is 1-dimensional over \( k \). (These are what we might have called “elementary upper modifications of type 1”.)

Minuscule coweights are minimal for the ordering on dominant coweights, \( Gr_{\leq \mu} = Gr_\mu \) for any such \( \mu \). Therefore, \( Gr_\mu \) is a homogeneous space under \( L^+G \), hence is smooth.

**Quasi-minuscule Schubert varieties.** A *quasi-minuscule weight* is a dominant coweight \( \mu \) such that \( \langle \mu, \alpha \rangle \leq 2 \) for all positive roots \( \alpha \). An example is the positive coroot of \( SL_2 \). Let’s examine this case. We’re supposed to see that the corresponding (closed) Schubert variety is the projective cone over a quadric curve in \( P^2 \).

The quasi-minuscule weight of \( SL_2 \) is the positive coroot

\[
\mu : t \mapsto \begin{pmatrix} t \\ t^{-1} \end{pmatrix}.
\]
Therefore, the Schubert variety parametrizes lattices $\Lambda$ such that

$$tk[[t]]^\oplus 2 \subset \Lambda \subset t^{-1}k[[t]]^\oplus 2$$

which is “commensurate” with $\mathcal{O}$, so

$$\dim_k(\Lambda/tk[[t]]^\oplus 2) = 2$$

and

$$\dim_k(t^{-1}k[[t]]^\oplus 2/\Lambda) = 2.$$  

Note that $\Lambda$ is determined by its image $\overline{\Lambda}$ in $tk[[t]]^\oplus 2$. The filtration

$$tk[[t]]^\oplus 2 \subset k[[t]] \subset t^{-1}k[[t]]^\oplus 2$$

defines a filtration on $\overline{\Lambda}$. Consider

$$\Lambda \cap k[[t]]^\oplus 2 \rightarrow k[[t]]^\oplus 2/tk[[t]]^\oplus 2.$$  

Its image is 1-dimensional, so the possibilities for this image are parametrized by a $\mathbb{P}^1$. The image of $\Lambda$ in $t^{-1}k[[t]]^\oplus 2/k[[t]]^\oplus 2$ is also 1-dimensional, and together with the choice of the first step of the filtration determines $\Lambda$. However, one line is excluded as a possible image, namely the one corresponding to the first choice under the isomorphism

$$k[[t]]^\oplus 2/tk[[t]]^\oplus 2 \rightarrow t^{-1}k[[t]]^\oplus 2/k[[t]]^\oplus 2$$

via multiplication by $t$. 