Finite State Channels with Time-Invariant Deterministic Feedback

Haim Permuter, Tsachy Weissman and Andrea Goldsmith

Abstract

We consider capacity of discrete-time channels with feedback for the general case where the feedback is a time-invariant deterministic function of the output samples. Under the assumption that the channel states take values in a finite alphabet, we find an achievable rate and an upper bound on the capacity. We further show that when the channel is indecomposable, and has no intersymbol interference (ISI), its capacity is given by the limit of the maximum of the (normalized) directed information between the input $X^N$ and the output $Y^N$, i.e. $C = \lim_{N \to \infty} \frac{1}{N} \max Q(x^N | z^{N-1})$ defined in this paper. The capacity result is used to show that the source-channel separation theorem holds for time-invariant deterministic feedback. We also show that if the state of the channel is known both at the encoder and the decoder then feedback does not increase capacity.

Index Terms

Feedback capacity, directed information, causal conditioning, code-tree, random coding, maximum likelihood, source-channel coding separation.

I. INTRODUCTION

Shannon showed in [1] that feedback does not increase the capacity of a memoryless channel, and therefore the capacity of a memoryless channel with feedback is given by maximizing the mutual information between the input $X$, and the output $Y$, i.e. $C = \max_{P(X)} I(X; Y)$. In the case that there is no feedback, and the channel is an indecomposable Finite-State Channel (FSC), the capacity was shown by Gallager [2] and by Blackwell, Breiman and Thomasian [3] to be

$$C_{NF} = \lim_{N \to \infty} \frac{1}{N} \max_{P(x^N)} I(X^N; Y^N).$$

One might be tempted to think that for an FSC with feedback all that changes for (1) to characterize capacity is the optimal input distribution, which now must depend on the feedback. However, the following simple counterexample shows that there are cases in which the mutual information $I(X^N; Y^N)$ results in a larger quantity than the capacity.

This work was supported by NSF Grant CCR-0311633, NSF CAREER grant, US Army under MURI award W911NF-05-1-0246, and the ONR under award N00014-05-1-0168.

The authors are with the Department of Electrical Engineering, Stanford University, Stanford, CA 94305, USA. (Email: {haim1, tsachy, andrea}@stanford.edu)
Let us consider the case where the channel has only one state which is a binary symmetric channel (BSC) with probability of error 0.5. It is obvious that no information can be transferred through this channel even with feedback and, therefore, the capacity of the channel is zero. However, it is easy to see that if we let the input to the channel at time $i$ equal the output of the channel at time $i - 1$, i.e. $X_i = Y_{i-1}$, for $i > 2$, which is possible in the presence of feedback, then it can be easily shown that

$$\frac{1}{N} I(X^N; Y^N) = \frac{1}{N} \sum_{i=1}^{N} I(X_i; Y^N|X^{i-1})$$

$$= \frac{1}{N} \sum_{i=2}^{N} I(Y_{i-1}; Y_i|Y^{i-2}) = \frac{N - 1}{N}. \quad (2)$$

Therefore, we see that for this channel $\lim_{N \to \infty} \max \{1/N \} I(X^N; Y^N) = 1$, while the capacity of the channel is zero. The reason such examples exist is that $I(X^N; Y^N)$ is measuring the mutual information between $X^N$ and $Y^N$, including the mutual information that is due to the feedback and not due to the channel. This example thus indicates that the capacity of the channel with feedback must involve maximization over an expression other than $I(X^N; Y^N)$.

In 1989 the directed information appeared in an implicit way in a paper by Cover and Pombra [4]. In an intermediate step [4, eq. 52] they showed that the directed information can be used to characterize the capacity of additive Gaussian noise channels with feedback. However, the term directed information was coined only a year later by Massey in a key paper [5].

In [5], Massey introduced directed information, denoted by $I(X^N \rightarrow Y^N)$, which he attributes to Marko [6]. Directed information, $I(X^N \rightarrow Y^N)$, is defined as:

$$I(X^N \rightarrow Y^N) \triangleq \sum_{i=1}^{N} I(X^i; Y_{i-1}|Y^{i-1}). \quad (3)$$

Massey showed that directed information is the same as mutual information $I(X^N; Y^N)$ in the absence of feedback and it gives a better upper bound on the information that the channel output $Y^N$ gives about the source sequence in the presence of feedback.

Tatikonda, in his Ph.D. dissertation [7], generalized the capacity formula of Verdú and Han [8] that deals with arbitrary single-user channels without feedback to the case of arbitrary single-user channels with feedback by using the directed information formula. Recently, the directed information formula was used by Yang, Kavčič and Tatikonda [9] and by Chen and Berger [10] to compute the feedback capacity for some special finite-state channels.

Directed information also appeared recently in a rate distortion problem. Following the competitive prediction of Weissman and Merhav [11], Pradhan [12], [13] formulated a problem of source coding with feed-forward and showed that directed information can be used to characterize the rate distortion function for the case of feed-forward. Another source coding context where directed information has arisen is the recent work by Zamir et. al. [14], which gives a linear prediction representation for the rate distortion function of a stationary Gaussian source.

In this paper we extend the achievability proof given by Gallager in [2] for the case of a finite-state channel (FSC) without feedback to the case of a FSC with feedback. We find an upper bound on the error of the maximum
likelihood decoder for a FSC with time invariant deterministic feedback. We develop an upper bound on the error which allows us to find an achievable rate for the channel. In addition, we state an upper bound on the capacity of the channel and show that when the state transition of the FSC does not depend on the input, the achievable rate equals the upper bound and hence equals the channel capacity. The main contribution of our work is in showing that the directed information, which was conjectured by Massey [5] to be the capacity of a channel with feedback, is achievable with a random coding scheme and maximum likelihood decoding, for any time-invariant deterministic feedback.

Time-invariant feedback includes the cases of quantized feedback, delayed feedback, and even noisy feedback where the noise is known to the encoder. In addition, it allows a unified treatment of capacity analysis for two ubiquitous cases: channels without feedback and channels with perfect feedback. These two setting are special cases of time-invariant feedback: in the first case the time-invariant function of the feedback is the null function and in the second case the time-invariant function of the feedback is the identity function.

The capacity of some channels with channel state information at the receiver and transmitter was derived by Caire and Shamai in [15]. Note that if the channel state information can be considered part of the channel output and fed back to the transmitter, then this case is a special case of a channel with time invariant feedback.

The remainder of the paper is organized as follows. Section II defines the channel setting and the notation throughout the paper. Section III provides a concise summary of the main results of the paper. Section IV introduces several properties of causal conditioning and directed information that are later used in finding an achievable rate. Section V provides the proof of achievability of capacity of FSCs with time invariant feedback. Section VI gives an upper bound on the capacity. Section VII gives the capacity of an indecomposable FSC without intersymbol interference (ISI). Section VIII considers the case of FSCs with feedback and side information and shows that if the state is known both at the encoder and decoder then feedback does not increase the capacity of the channel. Section IX shows that optimality of source-channel separation holds in the presence of time-invariant feedback. We conclude in Section X with a summary of this work and some related future directions.

II. Channel Models and Preliminaries

We use subscripts and superscripts to denote vectors in the following way: \(x^i = (x_1 \ldots x_i)\) and \(x^j_i = (x_i \ldots x_j)\) for \(i \leq j\). For \(i \leq 0\), \(x^i\) defines the null string as does \(x^j_i\) when \(i > j\). Moreover, we use lower case to denote sample values and upper case to denote random variables. Probability mass functions are denoted by \(P\) or \(Q\) when the arguments specify the distribution, e.g. \(P(x|y) = P(X=x|Y=y)\). In this paper, we consider only FSCs. The FSCs are a class of channels rich enough to include channels with memory, e.g. channels with intersymbol interference. The input of the channel is denoted by \(\{X_1, X_2, \ldots\}\), and the output of the channel is denoted by \(\{Y_1, Y_2, \ldots\}\), both taking values in a finite alphabet. In addition, the channel states take values in a finite set of possible states. The channel is stationary and is characterized by a conditional probability assignment \(P(y_i, s_i|x_i, s_{i-1})\) that satisfies

\[
P(y_i, s_i|x^i, s^{i-1}, y^{i-1}) = P(y_i, s_i|x_i, s_{i-1}).
\]
An FSC is said to be without intersymbol interference (ISI) if the input sequence does not affect the evolution of the state sequence, i.e. $P(s_i|s_{i-1},x_i) = P(s_i|s_{i-1})$.

We assume a communication setting that includes feedback as shown in Fig. 1. The transmitter (encoder) knows at time $i$ the message $m$ and the feedback samples $z_{i-1}$. The output of the encoder at time $i$ is denoted by $x_i$ and it is a function of the message and the feedback. The channel is an FSC and the output of the channel $y_i$ enters the decoder (receiver). The feedback $z_i$ is a known time-invariant deterministic function of the current output of the channel $y_i$. For example, $z_i$ could equal $y_i$ or a quantized version of it. The encoder receives the feedback sample with one unit delay.

Throughout this paper we use the *Causal Conditioning* notation $(\cdot|\cdot|\cdot)$, which was introduced and employed by Kramer [16], [17] and by Massey [18]:

$$P(y^N|x^N) \triangleq \prod_{i=1}^{N} P(y_i|x^i, y^{i-1}).$$  \hspace{1cm} (5)

In addition, we introduce the following notation:

$$P(y^N|x^{N-1}) \triangleq \prod_{i=1}^{N} P(y_i|x^{i-1}, y^{i-1}).$$  \hspace{1cm} (6)

The definition given in (6) can be considered to be a particular case of the definition given in (5) where $x_0$ is set to a dummy zero. This concept was captured by a notation of Massey in [18] via a concatenation at the beginning of the sequence $x^{N-1}$ with a dummy zero. The directed information $I(X^N \rightarrow Y^N)$ is defined in (3) and, by using the definitions, we can express directed information in terms of causal conditioning as

$$I(X^N \rightarrow Y^N) = \sum_{i=1}^{N} I(X^i; Y_i|Y^{i-1}) = \mathbb{E} \left[ \log \frac{P(Y^N||X^N)}{P(Y^N)} \right],$$  \hspace{1cm} (7)

where $\mathbb{E}$ denotes expectation. The directed information between $X^N$ and $Y^N$, conditioned on $S$, is denoted as
In this section, we state the main results of the paper.

- **Causal conditioning and directed information:** In Section IV we establish some properties of causal conditioning and directed information that are used throughout the proofs, and also provide some intuition about the meaning of these terms.

- **Achievable rate:** For any finite-state channel with an initial state denoted by \( s_0 \), and with the feedback setting of Fig. 1, and for any \( R, 0 \leq R < C \), where \( C \) is given by

\[
C = \lim_{N \to \infty} \frac{1}{N} \max_{Q(x_n || z_{n-1})} \min_{s_0} I(X^N \to Y^N | s_0) \tag{9}
\]

(a limit that can be shown to exist), and any \( \epsilon > 0 \), there exists an \((N,M)\) block code such that for all messages \( m, 1 \leq m \leq M = |2^{NR}| \) and all initial states, the decoding error is upper bounded by \( \epsilon \). This achievability result is established via analysis of a random coding scheme with maximum likelihood decoding.

- **Converse:** For any given channel with the feedback as in Fig. 1, any sequence of \((N,2^{NR})\) codes with probability of decoding error that goes to zero as \( N \to \infty \) must have

\[
R \leq \lim_{N \to \infty} \frac{1}{N} \max_{Q(x_n || z_{n-1})} I(X^N \to Y^N) \tag{10}
\]

where the limit is shown to exist.

- **Capacity:** For an indecomposable FSC without ISI, the achievable rate and the upper bound are equal. Hence the capacity, which is defined as the supremum of all achievable rates of the channel, is given by:

\[
C = \lim_{N \to \infty} \frac{1}{N} \max_{Q(x_n || z_{n-1})} I(X^N \to Y^N). \tag{11}
\]

- **State information and feedback:** Feedback does not increase the capacity of a strongly connected FSC (every state can be reached from every other state with positive probability under some input distribution) when the state of the channel is known both at the encoder and the decoder.

- **Source-channel separation** Source-channel coding separation is optimal for any channel with time-invariant deterministic feedback where the capacity is given by eq. (11).

### IV. Properties of causal conditioning and directed information

In this section we present some properties of the causal conditioning distribution and the directed information which are defined in Section II in eq. (5), (6) and (7). The properties are used throughout the proof of achievability and also help in gaining some intuition about those definitions and their role in the proof of the achievability.

**Lemma 1:** Chain rule for causal conditioning. For any random variables \((X^N,Y^N)\)

\[
P(x^N,y^N) = P(y^N || x^N) P(x^N || y^{N-1}), \tag{12}
\]
and, consequently, if \( Z^N \) is a random vector that satisfies \( P(X^N||Y^{N-1}) = P(X^N||Z^{N-1}) \) then

\[
P(X^N, Y^N) = P(Y^N||X^N)P(X^N||Z^{N-1}).
\] (13)

**Proof:**

\[
P(Y^N, X^N) = \prod_{i=1}^{N} P(y_i, x_i|X^{i-1}, y^{i-1})
\]

\[
= \prod_{i=1}^{N} P(y_i|x^i, y^{i-1})P(x_i|x^{i-1}, y^{i-1})
\]

\[
= P(Y^N||X^N)P(X^N||Z^{N-1}).
\] (14)

Note that there exists an analogy between this lemma and the chain rule \( P(X^N, Y^N) = P(Y^N|X^N)P(X^N) \). The analogy between the term \( P(Y^N|X^N) \) and the term \( P(Y^N||X^N) \), and between the term \( P(X^N) \) and the term \( P(X^N||Y^{N-1}) \), can be helpful for deriving equalities for the case of causal conditioning distributions that are analogous to the equalities that hold for regular distributions.

Let us define,

\[
P(Y^N||X^N, s) \triangleq \prod_{i=1}^{N} P(y_i|x^i, y^{i-1}, s).
\] (15)

**Lemma 2:** For any random variables \( (X^N, Y^N, Z^{N-1}, S_0) \) that satisfy \( P(X^N||Y^{N-1}, s_0) = P(X^N||Z^{N-1}) \),

\[
P(X^N, Y^N|s_0) = P(Y^N||X^N, s_0)P(X^N||Z^{N-1})
\] (16)

The proof of Lemma 2 is similar to that of Lemma 1 and therefore is omitted.

**Lemma 3:** **Causal conditioning is in the unit simplex.** For any random variables \( (X^N, Z^{N-1}) \),

\[
\sum_{x^N} P(X^N||Z^{N-1}) = 1
\] (17)

**Proof:**

\[
\sum_{x^N} P(X^N||Z^{N-1}) = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_N} \prod_{i=1}^{N} P(x_i|x^{i-1}, z^{i-1})
\]

\[
= \sum_{x_1} \sum_{x_2} \cdots \sum_{x_{N-1}} \left( \prod_{i=1}^{N-1} P(x_i|x^{i-1}, z^{i-1}) \right) \cdot P(x_N|x^{N-1}, z^{N-1})
\]

\[
= \sum_{x_1} \sum_{x_2} \cdots \sum_{x_{N-1}} P(x_N|x^{N-1}, z^{N-1}) \cdot 1
\]

\[
= \sum_{x^{N-1}} P(x^{N-1}||Z^{N-2}).
\] (18)

In addition, \( \sum_{x_i} P(x_i) = 1 \). Hence, by induction, \( \sum_{x^N} P(X^N||Z^{N-1}) = 1 \).

**Lemma 4:** There is a one to one correspondence between causal conditioning \( P(X^N||Z^{N-1}) \) and the sequence of conditional distributions \( \{ P(x_i|x^{i-1}, z^{i-1}) \}_{i=1}^{N} \).

**Proof:** It is obvious that the sequence \( \{ P(x_i|x^{i-1}, z^{i-1}) \}_{i=1}^{N} \) determines the term \( P(X^N||Z^{N-1}) \). In the other direction we can use the proof of Lemma 3, in which we showed that \( P(X^{N-1}||Z^{N-2}) \) is uniquely determined
from $P(x^N || z^{N-1})$ by a summation over $x_N$. Furthermore, by induction it can be shown that the sequence

$$P(x_i | x^{i-1}, z^{i-1}) = \frac{P(x_i | z^{i-1})}{P(x^{i+1} | z^i)}$$

(19)

to derive uniquely the sequence \( \{P(x_i | x^{i-1}, z^{i-1})\}_{i=1}^N \).

This lemma shows that the maximization in the capacity expressions can be done on the set of sequences

\( \{P(x_i | x^{i-1}, z^{i-1})\}_{i=1}^N \) or, equivalently, on the set of terms $P(x^N | z^{N-1})$. The lemma is analogous to the fact that maximization over the set $P(x^N)$ is equivalent to maximization over the set of sequences $\{P(x_i | x^{i-1})\}_{i=1}^N$.

**Lemma 5:** Let $X^N, Y^N, Z^N$ be arbitrary random vectors and $S$ a random variable taking values in an alphabet of size $|S|$. Then

$$|I(X^N \rightarrow Y^N || Z^{N-1}) - I(X^N \rightarrow Y^N || Z^{N-1}, S)| \leq H(S) \leq \log |S|.$$  

(20)

In particular, if $Z^N$ is $Y^N$, we get

$$|I(X^N \rightarrow Y^N) - I(X^N \rightarrow Y^N | S)| \leq H(S) \leq \log |S|.$$  

(21)

This lemma has an important role in the proofs for the capacity of FSCs, because it bounds by a constant the difference of directed information before and after conditioning on a state. The proof of the lemma is given in Appendix I.

The proof of the achievable rate for a channel with time-invariant feedback $z_i(y_i)$ is an extension of the proof of the achievable rate for a channel without feedback given in [2, Ch.5]. Roughly speaking, in each step we have to justify replacement of $Q(x^N)$ by $Q(x^N || z^{N-1})$ and of $P(y^N | x^N)$ by $P(y^N || x^N)$. The replacement does not work in all cases, for instance it does not work in the case of Theorem 4.6.4 in [2]. At the end of the proof we will see that the achievable rate is the same expression as the mutual information with the probability mass function $Q(x^N)$ replaced by $Q(x^N || z^{N-1})$ and $P(y^N | x^N)$ replaced by $P(y^N || x^N)$. The following lemma shows that the replacement results in directed information.

**Lemma 6:** Denote:

$$I(Q(x^N || z^{N-1}), P(y^N || x^N)) \triangleq \sum_{y^N} \sum_{x^N} Q(x^N || z^{N-1}) P(y^N || x^N) \log \frac{P(y^N || x^N)}{\sum_{x^N} Q(x^N || z^{N-1}) P(y^N || x^N)},$$

(22)

if $P(x^N || y^{N-1}) = Q(x^N || z^{N-1})$ then,

$$I(Q(x^N || z^{N-1}), P(y^N || x^N)) = I(X^N \rightarrow Y^N),$$

(23)

and similarly,

$$I(Q(x^N || z^{N-1}), P(y^N || x^N, s_0)) = I(X^N \rightarrow Y^N | s_0)$$

(24)
Proof:

\[ I(Q(x^N||z^{N-1}), P(y^N||x^N)) \triangleq \sum_{y^N} \sum_{x^N} Q(x^N||z^{N-1}) P(y^N||x^N) \log \frac{P(y^N||x^N)}{\sum_{x^N} Q(x^N||z^{N-1}) P(y^N||x^N)} \]

\[ = E \left[ \log \frac{P(Y^N||X^N)}{\sum_{x^N} Q(x^N||z^{N-1}) P(Y^N||x^N)} \right] \]

\[ = E \left[ \log P(Y^N||X^N) \right] - E \left[ \log \sum_{x^N} Q(x^N||z^{N-1}) P(Y^N||x^N) \right] \]

\[ = E \left[ \log \prod_{i=1}^{N} P(Y_i|X^i, Y^{i-1}) \right] - E \left[ \log P(Y^N) \right] \]

\[ = \sum_{i=1}^{N} I(X^i; Y_i|Y^{i-1}) \]

\[ = I(X^N \rightarrow Y^N), \quad (25) \]

Equalities (a) and (b) are due to Lemma 1.

The following lemma is an extension of the conservation law of information given by Massey in [18].

**Lemma 7: Extended conservation law.** For any random variables \((X^N, Y^N, Z^{N-1})\) that satisfy \(P(x^N||y^{N-1}) = P(x^N||z^{N-1})\),

\[ I(X^N; Y^N) = I(X^N \rightarrow Y^N) + I(\{0, Z^{N-1}\} \rightarrow X^N), \quad (26) \]

where \(\{0, Z^{N-1}\}\) is a concatenation of dummy zero to the beginning of the sequence \(Z^{N-1}\).

**Proof:**

\[ I(X^N, Y^N) \triangleq E \left[ \log \frac{P(Y^N, X^N)}{P(Y^N)P(X^N)} \right] \]

\[ = E \left[ \log \frac{P(Y^N||X^N)P(X^N||Z^{N-1})}{P(Y^N)P(X^N)} \right] \]

\[ = E \left[ \log \frac{P(Y^N||X^N)}{P(Y^N)} \right] + E \left[ \log \frac{P(X^N||Z^{N-1})}{P(X^N)} \right] \]

\[ = I(X^N \rightarrow Y^N) + I(\{0, Z^{N-1}\} \rightarrow X^N). \quad (27) \]

Equality (a) is due to the definition of mutual information. Equality (b) is due to Lemma 1, and equality (c) is due to the definition of directed information. 

The lemma was proven by induction in [18] for the case where \(z_i = y_i\). Here we see that the conservation law holds more generally when \(P(x^N||y^{N-1}) = P(x^N||z^{N-1})\). In Subsection V-A we argue that this equality holds for the setting of deterministic feedback \(z_i(y_i)\) and therefore the conservation law holds for the communication setting given in Fig 1. This lemma is not used for the proof of achievability, however, it gives a nice intuition for the relation of directed information and mutual information in the setting of deterministic feedback. In particular, the lemma implies that the mutual information between the input and the output of the channel is equal to the sum.
of directed information in the forward link and the directed information in the backward link. In addition, it is straightforward to see that in the case of no feedback, i.e. when \( z_i \) is null, then \( I(X^N;Y^N) = I(X^N \rightarrow Y^N) \).

V. PROOF OF ACHIEVABILITY

The proof of the achievable rate of a channel with feedback given here is an extension of the upper bound on the error of maximum likelihood decoding derived by Gallager in [2, Ch.5] for FSCs without feedback to the case of FSCs with feedback. The main difference is that for analyzing the new coding scheme, the feedback \( z^{i-1} \) must be taken into account.

Let us first present a short outline of the proof:

- **Encoding scheme.** We randomly generate an encoding scheme for blocks of length \( N \) by using the causal conditioning distribution \( Q(x^N||z^{N-1}) \).
- **Decoding.** We assume a maximum likelihood decoder and we denote the error probability when message \( m \) is sent and the initial state of the channel is \( s_0 \) as \( P_{e,m}(s_0) \).
- **Bounding the error probability.** We show that for each \( N > N(\epsilon) \), there exists a code for which we can bound the error probability for all messages \( 1 \leq m \leq 2^{NR} \) and all initial states \( s_0 \) by the following exponential,

\[
P_{e,m}(s_0) \leq 2^{-N[E_r(R) - \epsilon]}.
\]

In addition, we show that if \( R < C \) then \( E_r(R) \) is strictly positive and, hence, by choosing \( \epsilon < E_r(R) \), the probability of error diminishes exponentially for \( N > N(\epsilon) \).

A. Random generation of coding scheme

In the case of no feedback, a coding block of length \( N \) is a mapping of each message \( m \) to a codeword of length \( N \) and is denoted by \( x^N(m) \). In the case of feedback, a coding block is a vector function whose \( i^{th} \) component is a function of \( m \) and the first \( i-1 \) components of the received feedback. The mapping of the message \( m \) and the feedback \( z^{i-1} \) to the input of the channel \( x_i(m, z^{i-1}) \) is called a code-tree [19, Ch. 9] or strategy [20]. Figure 2 shows an example of a codeword of length \( N = 3 \) for the case of no feedback and a code-tree of depth \( N = 3 \) for the case of binary feedback.

**Randomly chosen coding scheme:** We choose the \( i^{th} \) channel input symbol \( x_i(m, z^{i-1}) \) of the codeword \( m \) by using a probability mass function (PMF) based on previous symbols of the code \( x^{i-1}(m, z^{i-2}) \) and previous feedback symbols \( z^{i-1} \). The first channel input symbol of codeword \( m \) is chosen by the probability function \( Q(x_1) \).

The second symbol of codeword \( m \) is chosen for all possible feedback observations \( z_1 \) by the probability function \( Q(x_2|x_1, z_1) \). The \( i^{th} \) bit is chosen for all possible \( z^{i-1} \) by the probability function \( Q(x_i|x^{i-1}, z^{i-1}) \). This scheme of communication assumes that the probability assignment of \( x_i \) given \( x^{i-1} \) and \( z^{i-1} \) cannot depend on \( y^{i-1} \), because it is unavailable. Therefore

\[
P(x_i|x^{i-1}, z^{i-1}(y^{i-1}), y^{i-1}) = P(x_i|x^{i-1}, z^{i-1}(y^{i-1})),
\]

(29)
We also define $Q(x^N|z^{N-1})$, similarly as in (6), to be the causal conditioning probability

$$Q(x^N|z^{N-1}) \triangleq \prod_{i=1}^{N} Q(x_i|x^{i-1}, z^{i-1}).$$  

(30)

**Encoding Scheme**: Each message $m$ has a code-tree. Therefore, for any feedback $z^{N-1}$ and message $m$ there is a unique input $x^N(m, z^{N-1})$ that was chosen randomly as described in the previous paragraph. After choosing the coding scheme, the decoder is made aware of the code-trees for all possible messages. In our coding scheme the input $x^N(m, z^{N-1})$ is always a function of the message $m$ and the feedback, but in order to make the equations shorter we also use the abbreviated notation $x^N$ for $x^N(m, z^{N-1})$.

**Decoding Scheme** The decoder in our scheme is the Maximum likelihood (ML) decoder. Since the codewords depend on the feedback, two different messages can have the same codeword for two different outputs, therefore the regular ML $\arg \max_m P(y^N|x^N)$ cannot be used for decoding the message. Instead, the ML decoder should be $\arg \max_{m} P(y^N|m)$ where $N$ is the block length. The following equation shows that finding the most likely message $m$ can be done by maximizing the causal conditioning $P(y^N|x^N)$:

$$\arg \max_{m} \log P(y^N|m) = \arg \max_{m} \log P(y^N|x^N).$$  

(31)
The equality in (31) is shown as follows:

\[
P(y^N|m) = \prod_i P(y_i|y^{i-1}, m) \\
\overset{(a)}{=} \prod_i P(y_i|y^{i-1}, m, x^i(m, z^{i-1}(y^{i-1}))) \\
\overset{(b)}{=} \prod_i P(y_i|y^{i-1}, x^i(m, z^{i-1}(y^{i-1}))) \\
\overset{(c)}{=} P(y^N|x^N).
\] (32)

Equality (a) holds because \(x^i\) is uniquely determined by the message \(m\) and the feedback \(z^{i-1}\), and the feedback \(z^{i-1}\) is a deterministic function of \(y^{i-1}\). Equality (b) holds because according to the channel structure, \(y_i\) does not depend on \(m\) given \(x^i\). Equality (c) follows from the definition of causal conditioning given in eq. (5).

B. ML decoding error bound

The next theorem, which is proved in Appendix II, is a bound on the expected ML decoding error probability with respect to the random coding. Let \(P_{e,m}\), as in [2, Ch. 5.2], denote the probability of error using the ML decoder when message \(m\) is sent. When the source produces message \(m\), there is a set of outputs denoted by \(Y_m\) that cause an error in decoding the message \(m\), i.e.,

\[
P_{e,m} = \sum_{y^N \in Y_m} P(y^N|m).
\] (33)

**Theorem 8:** Suppose that an arbitrary message \(m, 1 \leq m \leq M\), enters the encoder with feedback and that ML decoding is employed. Then the average probability of decoding error over this ensemble of codes is bounded, for any choice of \(\rho, 0 < \rho \leq 1\), by

\[
E(P_{e,m}) \leq (M - 1)\rho \sum_{y^N} \left( \sum_{z^N} Q(x^N||z^{N-1}) P(y^N||x^N)^{1+\rho} \right)^{1+\rho},
\] (34)

where the expectation is with respect to the randomness in the ensemble.

Let us define \(P_{e,m}(s_0)\) to be the probability of error given that the initial state of the channel is \(s_0\) and message \(m\) was sent. The following theorem, which is proved in Appendix III, establishes the existence of a code such that \(P_{e,m}(s_0)\) is small for all \(1 \leq m \leq M\).

**Theorem 9:** For an arbitrary finite-state channel with \(|S|\) states, for any positive integer \(N\) and any positive \(R\), there exists an \((N, M)\) code for which for all messages \(m, 1 \leq m \leq M = \lfloor 2^{NR} \rfloor\), all initial states \(s_0\), and all \(\rho, 0 \leq \rho \leq 1\), its probability of error is bounded as

\[
P_{e,m}(s_0) \leq 4|S|2^{(N-1)\rho R + F_N(\rho)}.
\] (35)

where

\[
F_N(\rho) = \frac{\rho \log |S|}{N} + \max_{Q(x^N||z^{N-1})} \left[ \min_{s_0} E_{o,N}(\rho, Q(x^N||z^{N-1}), s_0) \right],
\] (36)
\[ E_{\alpha,N}(\rho, Q(x^n \| z^{N-1}), s_0) = -\frac{1}{N} \log \sum_{y^n} \left[ \sum_{x^n} Q(x^n \| z^{N-1}) P(y^n \| x^n, s_0) \right]^{1+\rho}. \quad (37) \]

The following theorem presents a few properties of the function \( E_{\alpha,N}(\rho, Q(x^n \| z^{N-1}), s_0) \) which is defined in eq. (37), such as positivity of the function and its derivative, and convexity of the function with respect to \( \rho \).

**Theorem 10:** The term \( E_{\alpha,N}(\rho, Q(x^n \| z^{N-1}), s_0) \) has the following properties:

\[ E_{\alpha,N}(\rho, Q(x^n \| z^{N-1}), s_0) \geq 0; \quad \rho \geq 0. \quad (38) \]

\[ \frac{1}{N} T(Q(x^n \| y^{N-1}), P(y^n \| x^n, s_0)) \geq \frac{\partial E_{\alpha,N}(\rho, Q(x^n \| z^{N-1}), s_0)}{\partial \rho} > 0; \quad \rho \geq 0. \quad (39) \]

\[ \frac{\partial^2 E_{\alpha,N}(\rho, Q(x^n \| z^{N-1}), s_0)}{\partial \rho^2} > 0; \quad \rho \geq 0. \quad (40) \]

Furthermore, equality holds in (38) when \( \rho = 0 \), and equality holds on the left side of eq. (39) when \( \rho = 0 \).

The proof of the theorem is omitted because it is the same proof as Theorem 5.6.3 in [2]. Theorem 5.6.3 in [2] states these same properties with \( Q(x^n \| z^{N-1}) \) and \( P(y^n \| x^n) \) replaced by \( Q(x^n) \) and \( P(y^n \| x^n) \), respectively. The proof of those properties only requires that \( \sum_{x^n} Q(x^n \| z^{N-1}) = 1 \) and \( \sum_{x^n,y^n} Q(x^n \| z^{N-1}) P(y^n \| x^n, s_0) = 1 \), which hold according to Lemmas 3 and 1. By using Lemma 6 we can substitute \( T(Q(x^n \| y^{N-1}), P(y^n \| x^n, s_0)) \) in (39) by the directed mutual information \( I(X^N \rightarrow Y^N | s_0) \).

**Lemma 11:** Super additivity of \( F_N(\rho) \). For any given finite-state channel, \( F_N(\rho) \), as given by eq. (36), satisfies

\[ F_N(\rho) \geq \frac{n}{N} F_n(\rho) + \frac{l}{N} F_l(\rho) \quad (41) \]

for all positive integers \( n \) and \( l \) with \( N = n + l \).

The proof of the lemma is given in Appendix IV.

**Lemma 12:** Convergence of \( F_N(\rho) \). Let

\[ F_\infty(\rho) = \sup_N F_N(\rho), \quad (42) \]

then

\[ \lim_{N \rightarrow \infty} F_N(\rho) = F_\infty(\rho), \quad (43) \]

for \( 0 \leq \rho \leq 1 \). Furthermore, the convergence is uniform in \( \rho \) and \( F_\infty(\rho) \) is uniformly continuous for \( \rho \in [0,1] \).

**Proof:** Lemma 4A.2 in [2] states that if a series \( a_n \) is super additive, i.e. \( a_N \geq \frac{n}{N} a_n + \frac{N-n}{N} a_{N-n} \), then \( \lim_{N \rightarrow \infty} a_N = \sup_N a_N \). Based on Lemma 11, which states that \( \{ F_N(\rho) \} \) is super additive, we get that \( F_N(\rho) \) converges to \( \sup_N F_N(\rho) \). From Theorem 10 it follows that

\[ 0 \leq \frac{\partial E_{\alpha,N}(\rho, Q(x^n \| z^{N-1}), s_0)}{\partial \rho} \leq \frac{1}{N} I(X^N \rightarrow Y^N) \leq \log |\mathcal{Y}|, \quad (44) \]

where \( |\mathcal{Y}| \) is the size of the output alphabet. Using this bound with the definition of \( F_N \) given in eq. (36) we can bound the difference of \( F_N(\rho) \) for any \( 0 \leq \rho_1 < \rho_2 \leq 1 \) as

\[ -(\rho_2 - \rho_1) \log |\mathcal{S}| \leq F_N(\rho_2) - F_N(\rho_1) \leq (\rho_2 - \rho_1) \log |\mathcal{Y}|. \quad (45) \]
A consequence of (45) is that the function $F_N(\rho)$ and its slope are bounded independent of $N$ for each $0 \leq \rho \leq 1$. Therefore the convergence is uniform in $\rho$ and $F_\infty$ is uniformly continuous.

**Theorem 13:** Let us define

$$ C_N = \frac{1}{N} \max_{Q(x^n||z^{n-1})} \min_{s_0} I(X^N \rightarrow Y^N|s_0) $$

(46)

and

$$ C = \lim_{N \to \infty} C_N. $$

(47)

Then, for a finite state channel with $|S|$ states the limit in 47 exists and

$$ \lim_{N \to \infty} C_N = \sup_N \left[ C_N - \frac{\log|S|}{N} \right] = \sup_N C_N. $$

(48)

**Proof:**

Let us divide the input $x^N$ into two sets, $x_1 = x_1^n$ and $x_2 = x_{n+1}^N$. Similarly, let us divide the output $y^N$ into two sets $y_1 = y_1^n$ and $y_2 = y_{n+1}^N$. Let $Q_n(x_1|z_1) = \prod_{i=1}^n P(x_i|x_i^i, y_i^i)$ and $Q_l(x_2|z_2) = \prod_{i=n+1}^l P(x_i|x_i^n, y_i^n)$ be the probability assignments that achieve $C_n$ and $C_l$, respectively. Let us consider the probability assignment $Q(x^N||z^{N-1}) = Q_n(x_1|z_1)Q_l(x_2|z_2)$. Then

$$ N C_N \geq \min_{s_0} I(X^N \rightarrow Y^N|s_0) $$

$$ \geq \min_{s_0} \left[ \sum_{i=1}^n I(Y_i; X_i^i|Y_i^{i-1}, s_0) + \sum_{j=n+1}^{n+l} I(Y_j; X_j^j|Y_j^{j-1}, s_0) \right] $$

(49)

Equality (a) is due to the definition of the directed information. Inequality (b) holds because $C_n$ is the first term and for the second term we use the fact that $I(X; Y, Z) \geq I(X; Y)$ for any random variables $(X, Y, Z)$. Inequality (c) is due to Lemma 5. Rearranging the inequality we get:

$$ N \left[ C_N - \frac{\log|S|}{N} \right] \geq n \left[ C_n - \frac{\log|S|}{n} \right] + l \left[ C_l - \frac{\log|S|}{l} \right]. $$

(50)

Finally, by using the convergence of a super additive sequence, the theorem is proved.
A rate $R$ is said to be achievable if there exists a sequence of block codes $(N, [2^{NR}])$ such that the maximal probability of error $\max_m P_{e,m}(s_0)$ tends to zero as $N \to \infty$ for all initial states $s_0$ [21]. The following theorem states that any rate $R$ that satisfies $R < C$ is achievable.

**Theorem 14:** For any given finite-state channel, let

$$E_r(R) = \max_{0 \leq \rho \leq 1} [F_\infty(\rho) - \rho R].$$

(51)

Then, for any $\epsilon > 0$, there exists $N(\epsilon)$ such that for $N \geq N(\epsilon)$ there exists an $(N, M)$ code such that for all $m, 1 \leq m \leq M = [2^{NR}]$, and all initial states,

$$P_{e,m}(s_0) \leq 2^{-N[E_r(R)-\epsilon]}.$$  

(52)

Furthermore, for $0 \leq R < C$, $E_r(R)$ is strictly positive, and therefore the error can be arbitrarily small for $N$ large enough.

**Proof:** For any rate $R$, we can rewrite eq. (35) as

$$P_{e,m}(s_0) \leq 2^{-N(-\rho R + F_N(\rho) - \frac{\log 4|S|}{N})}.$$  

(53)

Because of the uniform convergence in $\rho$ proven in Lemma 12, for all $\epsilon > 0$, there exists an $N(\epsilon)$ that does not depend on $\rho$ such that, for $N \geq N(\epsilon)$,

$$F_\infty(\rho) - F_N(\rho) + \frac{\log 4|S|}{N} \leq \epsilon; \quad 0 \leq \rho \leq 1.$$  

(54)

Hence, it follows from (53) that

$$P_{e,m}(s_0) \leq 2^{-N(-\rho R + F_\infty(\rho) - \epsilon)}.$$  

(55)

If we choose the $\rho$ that maximizes $-\rho R + F_\infty(\rho)$ (note that $F_\infty(\rho)$ and therefore $-\rho R + F_\infty(\rho)$ is continuous in $\rho \in [0, 1]$, so there exists a maximizing $\rho$), then inequality (55) becomes inequality (52), proving the first part of the theorem.

Now let us show that if $R < C$, then $E_r(R) > 0$, which will prove the second part of the theorem. Let us define $\delta \triangleq C - R$. According to Theorem 13, $C_N$ converges to $\sup N C_N = C$, hence we can choose $N$ large enough so that the following inequality holds:

$$C_N \geq R + \frac{\log |S|}{N} + \frac{\delta}{2}.$$  

(56)

From Theorem 10, we have

$$\frac{\partial E_{o,N}(\rho, Q(x^N||z^{N-1}), s_0)}{\partial \rho} \leq C_N, \quad \forall s_0,$$  

(57)

where $Q(x^N||z^{N-1})$ is chosen to be the distribution that achieves $C_N$.

Note that $E_{o,N}(\rho, Q(x^N||z^{N-1}), s_0)$ is zero when $\rho = 0$, is a continuous function of $\rho$, and the derivative at zero with respect to $\rho$ is equal to $C_N \geq R + \frac{\log |S|}{N} + \frac{\delta}{2}$. Thus, for each state $s_0$ there is a range $0 < \rho < 0$ such that

$$E_{o,N}(\rho, Q(x^N||z^{N-1}), s_0) - \rho(R + \frac{\log |S|}{N}) > 0.$$  

(58)
Moreover, because the number of states is finite, there exists a $\rho^* > 0$ for which the inequality (58) is true for all $s_0$. Thus,

$$F_\infty(\rho^*) \geq F_N(\rho^*) \geq E_{0,N}(\rho, Q(x^N || z^{N-1}), s_0) - \rho^* \frac{\log |S|}{N} > \rho^* R, \quad \forall s_0,$$

and thus $E_r(R) > 0$ for $R < C$.

C. Feedback that is a deterministic function of a finite tuple of the output

We proved Theorem 14 for the case when the feedback $z_i$ is a deterministic function of the output at time $i$, i.e. $z_i = z(y_i)$. We now extend the theorem to the case where the feedback is a deterministic function of a finite tuple of the output, i.e. $z_i = z(y_{i-D}, \ldots, y_i)$.

Consider the case $D = 2$. Let us construct a new finite state channel, with input $x_i$, and output $\tilde{y}_i$ that is the tuple $\{y_{i-1}, y_i\}$. The state of the new channel $\tilde{s}_i$ is the tuple $\{s_i, y_i\}$.

Let us verify that the definition of a FSC holds for the new channel:

$$P(\tilde{y}_i, \tilde{s}_i | \tilde{y}^{i-1}, \tilde{s}^{i-1}, x^i) = P(y_i, y_{i-1}, s_i, y_i | y^{i-1}, s^{i-1}, y^{i-1}, x^i)$$

$$= P(y_i, y_{i-1}, s_i | y_{i-1}, s_{i-1}, x_i)$$

$$= P(\tilde{y}_i, \tilde{s}_i | \tilde{s}_{i-1}, x_i)$$

(60)

Both channels are equivalent, and because the feedback $z_i$ is a deterministic function of the output of the new channel, $\tilde{y}_i$, we can apply Theorem 14 and get that any achievable rate satisfies

$$R \leq C_N = \frac{1}{N} \max_{s_0} \min_{s_0} I(X^N \rightarrow \tilde{Y}^N | s_0)$$

$$= \frac{1}{N} \max_{s_0} \min_{s_0} I(X^N \rightarrow \{Y^N, Y_0^{N-1}\} | s_0, y_0)$$

$$= \frac{1}{N} \max_{s_0} \min_{s_0} \sum_{i=1}^{N} I(X^i; Y_i, Y_{i-1} | Y^{i-1}, X^{i-1}, Y_0^{i-2}, s_0, y_0)$$

$$= \frac{1}{N} \max_{s_0} \min_{s_0} \sum_{i=1}^{N} H(Y_i, Y_{i-1} | Y^{i-1}, X^{i-1}, s_0, y_0) - H(Y_i, Y_{i-1} | Y^{i-1}, X^{i-2}, s_0, y_0)$$

$$= \frac{1}{N} \max_{s_0} \min_{s_0} \sum_{i=1}^{N} H(Y_i | Y^{i-1}, s_0, y_0) - H(Y_i | Y^{i-1}, X^{i}, s_0, y_0)$$

$$= \frac{1}{N} \max_{s_0} \min_{s_0} I(X^N \rightarrow Y^N | s_0, y_0)$$

(61)

This result can be extended by induction to the general case where the feedback $z_i$ depends on a tuple of $D$ outputs, leading to the achievability of any rate smaller than $\lim_{N \rightarrow \infty} C_N$, where in this setting

$$C_N = \frac{1}{N} \max_{s_0} \min_{s_0} I(X^N \rightarrow Y^N | s_0, y_2^{-M}, \ldots, y_0).$$
VI. UPPER BOUND ON THE FEEDBACK CAPACITY

**Theorem 15:** The capacity of a channel where the input is \( x^N \) and the output is \( y^N \) and the channel has a time invariant deterministic feedback, as presented in Fig. 1, is upper bounded as

\[
C_{FB} \leq \lim_{N \to \infty} \max_{Q(x^N|z^{N-1})} I(X^N \to Y^N). \tag{63}
\]

**Proof:** Let \( W \) be the message, chosen according to a uniform distribution \( \Pr(W = w) = 2^{-NR} \). The input to the channel \( x_i \) is a function of the message \( W \) and the arbitrary deterministic feedback output \( z^{i-1}(y^{i-1}) \). We have

\[
NR = H(W) = I(W; Y^N) + H(W|Y^N)
\]

\[
\leq I(Y^N; W) + 1 + P_2^{(N)}NR
\]

\[
= H(Y^N) - H(Y^N|W) + 1 + P_2^{(N)}NR
\]

\[
= \sum_{i=1}^{N} H(Y_i|y^{i-1}) - \sum_{i=1}^{N} H(Y_i|W, y^{i-1}) + 1 + P_2^{(N)}NR
\]

\[
= \sum_{i=1}^{N} H(Y_i|y^{i-1}) - \sum_{i=1}^{N} H(Y_i|Y^{i-1}, X^i(W, z^{i-1}(y^{i-1}))) + 1 + P_2^{(N)}NR
\]

\[
= \sum_{i=1}^{N} I(Y_i; X^i|y^{i-1}) + 1 + P_2^{(N)}NR \tag{64}
\]

Inequality (a) holds because of Fano’s inequality. Equality (b) holds because of the chain rule. Equality (c) holds because \( x_i \) is a deterministic function given the message \( W \) and the feedback \( z^{i-1} \), where the feedback \( z^{i-1} \) is a deterministic function of the output. Equality (d) holds because the random variables \( W, X_i, Y^{i-1}, Y_i \) form the Markov chain \( W - (X_i, Y^{i-1}) - Y_i \). By dividing both sides of the equation by \( N \), maximizing over all possible input distributions, and letting \( N \to \infty \) we get that in order to have an error probability arbitrarily small, the rate \( R \) must satisfy:

\[
R \leq \lim_{N \to \infty} \frac{1}{N} \max_{Q(x^N|z^{N-1})} \sum_{i=1}^{N} I(Y_i; X^i|y^{i-1}) = \lim_{N \to \infty} \max_{Q(x^N|z^{N-1})} \frac{1}{N} \sum_{i=1}^{N} I(X^N \to Y^N). \tag{65}
\]

This completes the proof.

Remark: The converse proof is with respect to the average error over all messages. This, of course, implies that it is also true with respect to the maximum error over all messages. In the achievability part we proved that the maximum error over all messages goes to zero when \( R \leq C \) which, of course, also implies that the average error goes to zero. Hence, both the achievability and the converse are true with respect to average error probability and maximum error probability over all messages.
VII. INDECOMPOSABLE FSC WITHOUT ISI

In this section we assume that the channel states evolve according to a Markov chain which does not depend on the input, namely \( P(y_i, s_i|s_{i-1}, x_i) = P(s_i|s_{i-1})P(y_i|s_i, s_{i-1}, x_i) \). In addition, we assume that the Markov chain is indecomposable. Such a channel is called a Finite State Markovian indecomposable channel (FSMIC) in [22], however another suitable name which we adopt henceforth is a FSC without ISI. The difference between this channel and the indecomposable FSC defined in [2], [3] is that here we make an additional assumption that the transition probability between states is not a function of the input.

A Markov chain with transition matrix \( P(i, j) \) is indecomposable if it contains only one ergodic class [23]. An equivalent definition is that the effect of the initial state of the Markov chain dies away with time. More precisely:

**Definition 1:** A Markov Chain is indecomposable if, for every \( \epsilon > 0 \), there exists an \( N_0 \) such that for \( N > N_0 \),

\[
|P(s_N|s_0) - P(s_N|s'_0)| \leq \epsilon
\]  

for all \( s_N, s_0, s'_0 \).

In this section we prove that for a FSC without ISI the achievable rate does not depend on the initial state \( s_0 \) and therefore the lower bound and the upper bound on the capacity as given in (47) and (63) are equal.

Let us define

\[
\mathcal{C}_N = \frac{1}{N} \max_{Q(x^N|[z^{N-1}]} \max_{s_0} I(X^N \to Y^N|s_0)
\]  

and

\[
\mathcal{C} = \lim_{N \to \infty} \mathcal{C}_N,
\]

a limit that will be shown to exist. In addition let us define

\[
C = \lim_{N \to \infty} \frac{1}{N} \max_{Q(x^N|[z^{N-1}]} I(X^N \to Y^N),
\]

**Theorem 16:** For a FSC without ISI,

\[
\overline{C} = \overline{C} = C = \lim_{N \to \infty} \frac{1}{N} \max_{Q(x^N|[z^{N-1}]} I(X^N \to Y^N),
\]

where \( \overline{C} \) was defined in (68) and \( \overline{C} \) in (47).

**Proof:** For arbitrary \( N \), let \( Q_N(x^N|[z^{N-1}]} \) and \( s'_0 \) be the input distribution and the initial state that maximize \( I_Q(X^N \to Y^N|s'_0) \) and let \( s''_0 \) denote the initial state that minimizes \( I_Q(X^N \to Y^N|s''_0) \) for the same input distribution, where the subscript in \( I_Q \) is added to emphasize its dependence on \( Q_N \), though we suppress the subscript \( N \) from \( Q_N \). Thus, we have

\[
\frac{1}{N} I_Q(X^N \to Y^N|s'_0) = \mathcal{C}_N \geq \mathcal{C}_N \geq \frac{1}{N} I_Q(X^N \to Y^N|s''_0).
\]

The equation holds due to the definitions of \( \mathcal{C}_N \) and \( \mathcal{C}_N \). Next, we will prove that \( \lim_{N \to \infty} \frac{1}{N} I_Q(X^N \to Y^N|s'_0) = \lim_{N \to \infty} \frac{1}{N} I_Q(X^N \to Y^N|s''_0) \) and therefore \( \overline{C} = \overline{C} \).
Let \( n + l = N \), where \( n \) and \( l \) are positive integers and let \( s'_0 \) and \( s''_0 \) be any two initial states. Let the random variable \( S_n \) be the state at time \( n \). We would like to emphasize that the difference in the letter case notation is because \( s'_0 \) and \( s''_0 \) are specific states while \( S_n \) is a random variable. Then

\[
\frac{1}{N} I_Q(X^N - Y^N|s'_0)
\]

\[
\stackrel{(a)}{\leq} \frac{1}{N} \left[ \log |S| + I_Q(X^N - Y^N|s'_0, S_n) \right]
\]

\[
= \frac{1}{N} \left[ \log |S| + \sum_{i=1}^{n} I_Q(Y_i; X^i|Y^{i-1}, S_n, s'_0) + \sum_{i=n+1}^{N} I_Q(Y_i; X^i|Y^{i-1}, S_n, s'_0) \right]
\]

\[
\stackrel{(b)}{\leq} \frac{1}{N} \left[ \log |S| + n \log |Y| + \sum_{i=n+1}^{N} I_Q(Y_i; X^i|Y^{i-1}, S_n, s'_0) \right]
\]

\[
= \frac{1}{N} \left[ \log |S| + n \log |Y| + \sum_{i=n+1}^{N} \left[ I_Q(Y_i; X^i_{i+n+1}|Y^{i-1}, S_n, s'_0) + I_Q(Y_i; X^i_{i+n+1}|Y^{i-1}, S_n, s'_0) \right] \right]
\]

\[
\stackrel{(c)}{=} \frac{1}{N} \left[ \log |S| + n \log |Y| + \sum_{i=n+1}^{N} I_Q(Y_i; X^i_{i+n+1}|Y^{i-1}, S_n, s'_0) \right].
\]  

(72)

Inequality (a) is due to Lemma 5. Inequality (b) is due to the bound \( I_Q(Y_i; X^i|Y^{i-1}, S_n, s'_0) \leq \log |Y| \). Equality (c) holds because given the state \( S_n \) and the input after time \( n \), the output after time \( n \) does not depend on the input before time \( n \), i.e. \( P(y_i|y^{i-1}, x^i_{n+1}, s_n, x^i_n, s_0) = P(y_i|y^{i-1}, x^i_{n+1}, s_n, s_0) \), \( i > n \). By using inequality (72) we can bound the difference between the directed information starting at two different states:

\[
\frac{1}{N} |I_Q(X^N - Y^N|s'_0) - I_Q(X^N - Y^N|s''_0)|
\]

\[
\leq \frac{1}{N} \left[ \log |S| + n \log |Y| + \sum_{i=n+1}^{N} \left[ I_Q(Y_i; X^i_{i+n+1}|Y^{i-1}, S_n, s'_0) - I_Q(Y_i; X^i_{i+n+1}|Y^{i-1}, S_n, s''_0) \right] \right]
\]  

(73)

The sum in the last inequality can be bounded by using the indecomposability property of the Markov chain. For every \( i > n \) we have:

\[
I_Q(Y_i; X^i_{i+n+1}|Y^{i-1}, S_n, s'_0) - I_Q(Y_i; X^i_{i+n+1}|Y^{i-1}, S_n, s''_0)
\]

\[
\stackrel{(a)}{=} \sum_{s_n} [P(s_n|s'_0) - P(s_n|s''_0)] I_Q(Y_i; X^i_{i+n+1}|Y^{i-1}, s_n, s'_0)
\]

\[
\stackrel{(b)}{\leq} \sum_{s_n} [P(s_n|s'_0) - P(s_n|s''_0)] \log |Y|
\]

\[
\stackrel{(c)}{\leq} \epsilon_n \log |Y|.
\]  

(74)

Equality (a) is achieved by summing over all possible states \( s_n \). Inequality (b) is achieved by bounding the magnitude of each term in the sum by \( \log |Y| \). Inequality (c) holds by defining \( \epsilon_n \triangleq \max_{s_n, s'_0} \left| \sum_{s_n} P(s_n|s'_0) - P(s_n|s''_0) \right| \).
Combining eq. (73) and eq. (74) we obtain:

\[
\frac{1}{N} |I_Q(X^N \to Y^N|s'_0) - I_Q(X^N \to Y^N|s'_0')| \leq \frac{1}{N} |\log |S| + n \log |Y| + \epsilon_n \cdot |S| \cdot I \log |Y| | \leq \frac{1}{N} \log |S| + \frac{n}{N} \log |Y| + \epsilon_n \log |Y|.
\]

Since, by the indecomposability of the channel, \(\epsilon \to \infty\), and since inequality (75) holds for all \(0 \leq n \leq N\), it follows from inequality (75) (by letting \(n\) increase without bound, but sub-linearly in \(N\)) that

\[
\lim_{N \to \infty} \frac{1}{N} |I_Q(X^N \to Y^N|s'_0) - I_Q(X^N \to Y^N|s'_0')| = 0.
\] (75)

Up to now, we have proved that \(\lim_{N \to \infty} C_N = \lim_{N \to \infty} C_N'\) and this is because of eq. (75) and (71). Finally, we show that even without conditioning on \(s_0\) we get the same limit. Indeed,

\[
C_N \triangleq \frac{1}{N} \max_{Q(x^N||y^{N-1})} I(X^N \to Y^N) \overset{(a)}{=} \frac{1}{N} \max_{Q(x^N||z^{N-1})} I(X^N \to Y^N|s_0) + \frac{|S|}{N} = \frac{1}{N} \max_{Q(x^N||z^{N-1})} \sum_{s_0} P(s_0) I(X^N \to Y^N|s_0) + \frac{|S|}{N} \leq \frac{1}{N} \max_{Q(x^N||z^{N-1})} \max_{s_0} I(X^N \to Y^N|s_0) + \frac{|S|}{N} = C_N + \frac{|S|}{N}.
\] (76)

Equality (a) holds because, according to Lemma 5, the magnitude of the difference between the expression in the two sides of the equation is bounded by \(|S|/N\). In a similar way we prove that \(C_N \geq C_N' - |S|/N\) and therefore we get that \(\lim_{N \to \infty} C_N = \lim_{N \to \infty} C_N = \lim_{N \to \infty} C_N'\) which concludes the proof. 

The capacity of a channel is defined as the supremum over all achievable rates, analogous to what is done in the absence of feedback [21].

**Theorem 17:** The capacity of an Indecomposable FSC without ISI with a time invariant feedback \(z(y_i)\) is given by

\[
C = \lim_{N \to \infty} \frac{1}{N} \max_{Q(x^N||y^{N-1})} I(X^N \to Y^N),
\] (77)

where \(C\) denotes the capacity of the channel in the presence of feedback.

**Proof:** According to Theorem 14, for any given finite state channel, any rate \(R\) in the range \(0 \leq R < C\) is achievable. According to Theorem 15, the upper bound on capacity of a FSC is \(\lim_{N \to \infty} \frac{1}{N} \max_{Q(x^N||y^{N-1})} I(X^N \to Y^N)\). Hence we get that the capacity \(C\) is bounded from below and from above by:

\[
C \leq C \leq \lim_{N \to \infty} \frac{1}{N} \max_{Q(x^N||y^{N-1})} I(X^N \to Y^N).
\] (78)

Theorem 16 states that, for an indecomposable FSC without ISI, the upper bound equals the lower bound, i.e. \(C = \lim_{N \to \infty} \frac{1}{N} \max_{Q(x^N||y^{N-1})} I(X^N \to Y^N)\), and therefore the capacity is given by (77). 

\[\square\]
VIII. FEEDBACK AND SIDE INFORMATION

The results of the previous sections can be extended to the case where side information is available at the decoder that might be also fed back to the encoder. Let \( l_i \) be the side information available at the decoder and the setting of communication the one in Fig. 3. If the side information \( l_i \) satisfies
\[
P(l_i, y_i, s_i | s_{i-1}, x_i, y_{i-1}, l_{i-1}) = P(l_i, y_i, s_i | s_{i-1}, x_i)
\]
then it follows that
\[
P(\bar{y}_i, s_i | s_{i-1}, x_i, y_{i-1}, l_{i-1}) = P(\bar{y}_i, s_i | s_{i-1}, x_i),
\]
where \( \bar{y}_i = (l_i, y_i) \). We can now apply Theorem 14 and get:
\[
C_N = \frac{1}{N} \max_{Q(x^n | s_{1:N-1})} \min_{s_0} I(X^N \rightarrow Y^N, L^N | s_0),
\]
where \( z_{i-1} \) denotes the feedback available at the receiver at time \( i \) which is a time-invariant function of \( l_{i-1} \) and \( y_{i-1} \).

While many cases of side information can be studied, we are going to consider only the case in which the side information is the state of the channel, i.e. \( l_i = s_i \), which is fed back to the encoder, namely we let \( z_i(y_i, l_i) = s_i \). In this section we no longer assume that there is no ISI, instead we assume that the FSC is strongly connected, which we define as follows.

Definition 2: We say that a finite state channel is strongly connected if there exists an input distribution \( \{Q(x_t | s_{t-1})\}_{t \geq 1} \) and integer \( T \) such that
\[
\Pr\{S_t = s \text{ for some } 1 \leq t \leq T | S_0 = s'\} > 0, \ \forall s', s.
\]

Theorem 18: Feedback does not increase the capacity of a strongly connected FSC when the state of the channel is known both at the encoder and the decoder. Furthermore, the capacity of the channel under this setting is given
by
\[
C = \lim_{N \to \infty} \frac{1}{N} \max_{\mathcal{Q}(x_i|s_i-1)} \sum_{i=1}^{N} I(Y_i, S_i; X_i | S_{i-1}).
\]  
(83)

A straightforward consequence of this theorem is that feedback does not increase the capacity of a discrete memoryless channel (DMC), which Shannon proved in 1956 [1]. A DMC can be considered as an FSC with only one state, and therefore the state of the channel is known to the encoder and the decoder.

**Proof:** First, we notice that because the state of the channel is known both to the encoder and the decoder, and because the FSC is strongly connected, we can assume that with probability \(1 - \epsilon\), where \(\epsilon\) is arbitrarily small, the FSC channel can be driven, in a finite time, to the state that maximizes the achievable rate. Hence, the achievable rate does not depend on the initial state and the capacity of the channel in the presence of feedback, which we denote by \(C_{\text{SC}}\). Inequality (c) holds because conditioning reduces entropy. Equality (d) holds because maximizing the sequence of probabilities determines the channel properties, the input distribution to the channel is determined by the channel properties, the input distribution to the channel

**Proof:** First, we notice that because the state of the channel is known both to the encoder and the decoder, and because the FSC is strongly connected, we can assume that with probability \(1 - \epsilon\), where \(\epsilon\) is arbitrarily small, the FSC channel can be driven, in a finite time, to the state that maximizes the achievable rate. Hence, the achievable rate does not depend on the initial state and the capacity of the channel in the present of feedback, which we denote as \(C^{(F)}\), is given by \(\lim_{N \to \infty} C_{\text{SC}}^{(F)}\), where \(C_{\text{SC}}^{(F)}\) satisfies

\[
C_{\text{SC}}^{(F)} = \frac{1}{N} \max_{\mathcal{Q}(x_i|s_i-1)} \frac{1}{N} \sum_{i=1}^{N} I(X^N \rightarrow \{Y^N, L^N\})
\]

Equality (a) follows by replacing \(L_i\) with \(S_i\) according to the communication setting. Equality (b) follows from the FSC property. Inequality (c) holds because conditioning reduces entropy. Equality (d) holds because maximizing over the set of causal conditioning probability \(\mathcal{Q}(x_i|z_{i-1}^N)\) is the same as maximizing over the set of probabilities \(\{\mathcal{Q}(x_i|s_i-1)\}_{i=1}^{N}\), as shown in the following argument. The sum \(\sum_{i=1}^{N} I(Y_i, S_i; X_i | S_{i-1})\) is determined uniquely by the sequence of probabilities \(\{P(y_i, s_i, x_i, s_{i-1})\}_{i=1}^{N}\). Let us prove by induction that this sequence of probabilities is determined by \(\{\mathcal{Q}(x_i|x^{i-1}, y^{i-1}, s^{i-1})\}_{i=1}^{N}\) only through \(\{\mathcal{Q}(x_i|s_{i-1})\}_{i=1}^{N}\). For \(i = 1\) we have

\[
P(y_1, s_1, x_1, s_0) = P(s_0)Q(x|s_0)p(y_1, s_1 | x_1, s_0).
\]  
(85)

Since \(P(s_0)\) and \(P(y_1, s_1|x_1, s_0)\) are determined by the channel properties, the input distribution to the channel can influence only the term \(Q(x|s_0)\). Now, let us assume that the argument is true for \(i - 1\) and let us prove it for \(i\).

\[
P(y_i, s_i, x_i, s_{i-1}) = P(s_{i-1})Q(x_i | s_{i-1})P(y_i, s_i | x_i, s_{i-1}).
\]  
(86)

The term \(P(s_{i-1})\) is the same under both sequences of probabilities because of the assumption that the argument holds for \(i - 1\). The term \(P(y_i, s_i|x_i, s_{i-1})\) is determined by the channel, so the only term influenced by the input
distribution is \(Q(x_i|s_{i-1})\). This proves the validity of the argument for all \(i\) and consequently, the equality (d).

Inequality (84) proves that the achievable rate, when there is feedback and state information, cannot exceed \(\lim_{N \to \infty} \frac{1}{N} \max Q(x^N|s^{N-1}) \sum_{i=1}^{N} I(Y_i, S_i; X_i|S_{i-1})\). Now let us prove that if the state of the channel is known at the encoder and the decoder and there is no feedback, we can achieve this rate. For this setting we denote the capacity as \(C^{NF}\) and as in the case of feedback, the capacity does not depend on the initial state and is given as \(\lim_{N \to \infty} C^{NF}_N\), where \(C^{NF}_N\) satisfies

\[
C_N = \frac{1}{N} \max_{Q(x^N|z^{N-1})} I(X^N \to \{Y^N, L^N\})
\]

Equality (a) follows by replacing \(L_i\) and \(Z_i\) with \(S_i\) according to the communication setting. Equality (b) follows from the FSC property. Inequality (c) holds because we restrict the range of probabilities over which the maximization is performed. Equality (d) holds because under an input distribution \(Q(x^i|x_{i-1})\), we have the following Markov chain: \((Y_i, S_i) - S_{i-1} - (Y^{i-1}, S^{i-2})\). Inequality (e) holds due to (84).

Taking the limit \(N \to \infty\) on both sides of (87) shows that \(C^{NF} \geq C^{(F)}\). Since trivially also \(C^{NF} \leq C^{(F)}\) we have \(C^{NF} = C^{(F)}\).

IX. Source-channel separation

In this section we prove the optimality of source channel separation for the case of an ergodic source that is transmitted through a channel with a deterministic time-invariant feedback. Namely, we prove that in the communication setting presented in Fig. 5, the number of bits per channel use that can be transmitted and reconstructed within a given distortion is the same as in the communication setting of Fig 4.

Let us state the source-channel separation theorem as presented in [24, Chapter 7].

**Theorem 19**: Let \(\epsilon > 0\) and \(D \geq 0\) be given. Let \(R(\cdot)\) be the rate distortion function of a discrete, stationary, ergodic source with respect to a single letter criterion generated by a bounded distortion measure \(\rho\). Then the source
output can be reproduced with fidelity $D$ at the receiving end of any channel if $C > R(D)$. Conversely, fidelity $D$ is unattainable at the receiving end of any channel of capacity $C < R(D)$.

Remark: For the simplicity of the presentation we assumed one channel use per source symbol. Our derivation below extends to the general case where the average number of channel uses per letter is $\frac{r}{c}$, analogously as in [2, chapter 9].

The purpose of this section is to prove the theorem for a channel with time-invariant feedback, as shown in Fig. 4, for the cases where its capacity is given by

$$C = \lim_{N \to \infty} \frac{1}{N} \max_{Q(x^N | z^{N-1})} I(X^N \to Y^N).$$  \hfill (88)

In the case of no feedback the proof of separation optimality is based on data processing inequality which states that $I(U^N; V^N) \leq I(X^N; Y^N)$ because of the Markov form $U^N : X^N - Y^N - V^N$. However, the regular data processing inequality does not hold for the directed information and therefore an explicit derivation of the inequality $I(U^N; V^N) \leq I(X^N \to Y^N)$ is needed. \textit{Proof:} The direct proof, namely that if $C > R(D)$ it is possible to reproduce the source with fidelity $D$ is the same as for the case without feedback [24, Theorem 7.2.6].

For the converse, namely that $R(D)$ has to be less or equal $C$, we use the fact that for any $i$, the Markov chain

Fig. 4. Source and channel coding, where the channel has time-invariant feedback.

Fig. 5. Source and channel separation.
\( U^N - X_i(U^N, Y^{i-1}) - Y_i \) holds.

\[
N_R(D) \overset{(a)}{\leq} I(U^N; V^N)
\]
\[
\overset{(b)}{=} I(U^N; Y^N)
\]
\[
\overset{(c)}{=} \sum_{i=1}^{N} I(U^N; Y_i|Y^{i-1})
\]
\[
= \sum_{i=1}^{N} H(Y_i|Y^{i-1}) - H(Y_i|U^N, Y^{i-1})
\]
\[
\overset{(d)}{=} \sum_{i=1}^{N} H(Y_i|Y^{i-1}) - H(Y_i|U^N, Y^{i-1}, X^i)
\]
\[
\overset{(e)}{=} \sum_{i=1}^{N} H(Y_i|Y^{i-1}) - H(Y_i|Y^{i-1}, X^i)
\]
\[
= \sum_{i=1}^{N} I(Y_i; X^i|Y^{i-1})
\]
\[
= I(X^N \rightarrow Y^N)
\]
\[
\overset{(f)}{=} NC.
\]

Inequality (a) follows the converse for rate distortion [24, Theorem 7.2.5]. Inequality (b) follows the data processing inequality because \( U^N - Y^N - V^N \) form a Markov chain. Equality (c) follows the chain rule. Inequality (d) follows the fact that \( X_i \) is a deterministic function of \( (U^N, Y^{i-1}) \). Inequality (e) follows the Markov chain \( U^N - X_i(U^N, Y^{i-1}) - Y_i \). Finally, inequality (f) follows the converse of channel with feedback given in Theorem 15. □

### X. Conclusion and Future Work

We determined achievable rate and the capacity upper bound of FSCs with feedback that is a deterministic function of the channel output. The achievable rate is obtained via a random generated coding scheme that utilizes feedback, along with a ML decoder. In the case that the channel is an indecomposable FSC without ISI, the upper bound and the achievable rate coincide and, therefore, they are the capacity of the channel. One future direction is to generalize the channels for which the achievable rate equals to the upper bound on the capacity and to use this formula in order to compute the capacity in various settings involving side information and feedback.

By using the directed information formula for the capacity of FSCs with feedback developed in this work, it was shown in [25] that the feedback capacity of a channel introduced by David Blackwell in 1961 [26], also known as the trapdoor channel [27], is the logarithm of the golden ratio. The capacity of Blackwell’s channel without feedback is still unknown. Another future work is to find the capacity of additional channels with time-invariant feedback.

### ACKNOWLEDGMENT

The authors would like to thank T. Cover, G. Kramer, A. Lapidoth, T. Moon and S. Tatikonda, for helpful discussions, and are indebted to Young-Han Kim for suggesting a simple proof of Lemma 5.
REFERENCES

\[ I(X^N \rightarrow Y^N \mid Z^{N-1}) - I(X^N \rightarrow Y^N \mid Z^{N-1}, S) \]
\[
\begin{aligned}
&= \sum_{i=1}^{N} I(Y_i; X^i \mid Y^{i-1}, Z^{i-1}) - I(Y_i; X^i \mid Y^{i-1}, Z^{i-1}, S) \\
&= \sum_{i=1}^{N} H(Y_i \mid Y^{i-1}, Z^{i-1}) - H(Y_i \mid Y^{i-1}, X^i, Z^{i-1}) - H(Y_i \mid Y^{i-1}, Z^{i-1}, S) + H(Y_i \mid Y^{i-1}, X^i, Z^{i-1}, S) \\
&= \sum_{i=1}^{N} I(Y_i; S \mid Y^{i-1}, Z^{i-1}) - I(Y_i; S \mid Y^{i-1}, X^i, Z^{i-1}) \\
&\leq \max \left( \sum_{i=1}^{N} I(Y_i; S \mid Y^{i-1}, Z^{i-1}), \sum_{i=1}^{N} I(Y_i; S \mid Y^{i-1}, X^i, Z^{i-1}) \right) \\
&\leq \max \left( \sum_{i=1}^{N} I(Y_i, Z_i; S \mid Y^{i-1}, Z^{i-1}), \sum_{i=1}^{N} I(Y_i, Z_i, X_{i+1}; S \mid Y^{i-1}, X^i, Z^{i-1}) \right) \\
&= \max \left( I(Y^N, Z^N; S), I(Y^N, Z^N, X_2^N; S) \right) \\
&\leq \log |S| 
\end{aligned}
\] (90)

Equality (a) is due to the definition of the directed information. Inequality (b) holds because the magnitude of the difference between two positive numbers is smaller than the maximum of the numbers. Inequality (c) is due to the fact that \( I(X; Y) \leq I(X, Z; Y) \) for any random variables \( X, Y, Z \). Equality (d) is due to the chain rule of mutual information. Inequality (e) is due to the fact that mutual information of two variables is smaller than the entropy of each variable, and the last inequality holds because the cardinality of the alphabet of \( S \) is \(|S|\). ■

**APPENDIX II**

**Proof of Theorem 8**

\[
\mathbb{E}(P_{e,m}) = \sum_{y^N} \sum_{x^N} P(x^N, y^N) P[error \mid m, x^N, y^N] \\
= \sum_{y^N} \sum_{x^N} Q(x^N \mid z^{N-1}) P(y^N \mid x^N) P[error \mid m, x^N, y^N], 
\] (91)

where \( P[error \mid m, x^N, y^N] \) is the probability of decoding error conditioned on the message \( m \), the output \( y^N \) and the input \( x^N \). The second equality is due to Lemma 1. Throughout the reminder of the proof we fix the message \( m \). For a given tuple \((m, x^N, y^N)\) define the event \( A_{m'} \), for each \( m' \neq m \), as the event that the message \( m' \) is selected in such
a way that \( P(y^N|m') > P(y^N|m) \) which, according to eq. (32), is the same as \( P(y^N|x'^N) > P(y^N|x^N) \) where \( x'^N \) is a shorthand notation for \( x^N(m', z^{N-1}(y^{N-1})) \) and \( x^N \) is a shorthand notation for \( x^N(m, z^{N-1}(y^{N-1})) \).

From the definition of \( A_{m'} \) we have

\[
P(A_{m'}|m, x^N, y^N) = \sum_{x'^N} Q(x'^N||z^{N-1}) \cdot \text{I}[P(y^N||x'^N) > P(y^N||x^N)]
\]

\[
\leq \sum_{x'^N} Q(x'^N||z^{N-1}) \left[ \frac{P(y^N||x'^N)}{P(y^N||x^N)} \right]^s, \quad \text{any } s > 0
\]

(92)

where \( \text{I}(x) \) denotes the indicator function.

\[
P[\text{error}|m, x^N, y^N] = P\left( \bigcup_{m' \neq m} A_{m'}|m, x^N, y^N \right)
\]

\[
\leq \min \left\{ \sum_{m' \neq m} P(A_{m'}|m, x^N, y^N), 1 \right\}
\]

\[
\leq \left[ \sum_{m' \neq m} P(A_{m'}|m, x^N, y^N) \right]^\rho, \quad \text{any } 0 \leq \rho \leq 1
\]

\[
\leq \left( M - 1 \right) \sum_{x'^N} Q(x'^N||z^{N-1}) \left[ \frac{P(y^N||x'^N)}{P(y^N||x^N)} \right]^s, \quad 0 \leq \rho \leq 1, s > 0,
\]

(93)

where the last inequality is due to inequality (92). By substituting inequality (93) in eq. (91) we get:

\[
\mathbb{E}[P_{e,m}] \leq (M - 1)^\rho \sum_{y^N} \left[ \sum_{x'^N} Q(x'^N||z^{N-1}) P(y^N||x^N)^{1-s\rho} \right] \left[ \sum_{x'^N} Q(x'^N||z^{N-1}) P(y^N||x'^N)^s \right]^\rho.
\]

(94)

By substituting \( s = 1/(1 + \rho) \), and recognizing that \( x' \) is a dummy variable of summation, we obtain eq. (34) and complete the proof. ■

### Appendix III

**Proof of Theorem 9**

Theorem 8 holds for any distribution of the initial states \( S_0 \). In particular, it holds for the case that \( P(s_0) = \frac{1}{|S|} \), namely, the uniform distribution. By assuming a uniform distribution on the initial state, we get that the likelihood
function satisfies

\[ P(y^N|\{x^N\}) \overset{(a)}{=} P(y^N|m) = \sum_{s_0} P(y^N, s_0|m) \]

\[ \overset{(b)}{=} \sum_{s_0} P(s_0)P(y^N|m, s_0) \]

\[ = \sum_{s_0} \frac{1}{|S|} \prod_{i=1}^{N} P(y_i|y^{i-1}, m, s_0) \]

\[ = \sum_{s_0} \frac{1}{|S|} \prod_{i=1}^{N} P(y_i|y^{i-1}, m, x^i, s_0) \]

\[ = \sum_{s_0} \frac{1}{|S|} P(y^N||x^N, s_0). \quad (95) \]

Equality (a) is shown in eq. (32) and Equality (b) holds due to the assumption that the initial state \( S_0 \) and the message \( m \) are independent. Thus, assuming that \( s_0 \) is uniformly distributed the bound on error probability under ML decoding given in Theorem 8 becomes

\[ \sum_{s_0} \frac{1}{|S|} E(P_{e,m}(s_0)) \leq (M - 1)^\rho \sum_{y^N} \left\{ \sum_{x^N} Q(x^N||z^{N-1}) \left[ \sum_{s_0} \frac{1}{|S|} P(y^N||x^N, s_0) \right]^{\frac{1}{(1+\rho)}} \right\}^{1+\rho}, \quad 0 \leq \rho \leq 1 \]

(96)

And therefore, for any initial state \( s_0 \)

\[ E(P_{e,m}(s_0)) \leq |S|(M - 1)^\rho \sum_{y^N} \left\{ \sum_{x^N} Q(x^N||z^{N-1}) \left[ \sum_{s_0} \frac{1}{|S|} P(y^N||x^N, s_0) \right]^{\frac{1}{(1+\rho)}} \right\}^{1+\rho}, \quad 0 \leq \rho \leq 1 \]

(97)

Since \( m \) was arbitrary, we obtain a fortiori

\[ E(P_e(s_0)) \leq |S|(M - 1)^\rho \sum_{y^N} \left\{ \sum_{x^N} Q(x^N||z^{N-1}) \left[ \sum_{s_0} \frac{1}{|S|} P(y^N||x^N, s_0) \right]^{\frac{1}{(1+\rho)}} \right\}^{1+\rho}, \quad 0 \leq \rho \leq 1 \]

(98)

where \( P_e(s_0) \) is the probability of error over all messages given that the initial state is \( s_0 \) and the expectation is w.r.t the random generation of the code. It is possible to construct a code for \( 2M \) messages that this inequality holds for the average and then to pick the best \( M \) messages such that the bound holds for each message within a factor of 4. I.e., we get that for every \( 1 \leq m \leq M \),

\[ P_{e,m}(s_0) \leq 4|S|(M - 1)^\rho \sum_{y^N} \left\{ \sum_{x^N} Q(x^N||z^{N-1}) \left[ \sum_{s_0} \frac{1}{|S|} P(y^N||x^N, s_0) \right]^{\frac{1}{(1+\rho)}} \right\}^{1+\rho}, \quad 0 \leq \rho \leq 1 \]

(99)

By using the inequality \( \sum_i a_i^r \leq \left( \sum_i (a_i)^r \right)^{1/r} \) for \( 0 \leq r \leq 1 \) we can move the sum over \( s_0 \), yielding

\[ P_{e,m}(s_0) \leq 4|S|(M - 1)^\rho \sum_{y^N} \left\{ \sum_{x^N} \sum_{s_0} Q(x^N||z^{N-1}) \left[ \frac{1}{|S|} P(y^N||x^N, s_0) \right]^{\frac{1}{(1+\rho)}} \right\}^{1+\rho}, \quad 0 \leq \rho \leq 1 \]

(100)
Furthermore, we can move the sum over \( s_0 \) once again by rearranging the sum and then using the Jensen’s inequality:

\[
P_{e,m}(s_0) \leq (a) 4|\mathcal{S}|(M - 1)^\rho \sum_{y^n} \left\{ |\mathcal{S}|^\frac{1}{|\mathcal{S}|} \sum_{x^n} Q(x^n || z^{N-1}) \left[ P(y^n || x^n, s_0) \right]^{\frac{1}{1+\rho}} \right\}^{1+\rho}
\]

\[
= (b) 4|\mathcal{S}|(M - 1)^\rho |\mathcal{S}|^\rho \sum_{y^n} \left\{ \sum_{s_0} \frac{1}{|\mathcal{S}|} \sum_{x^n} Q(x^n || z^{N-1}) \left[ P(y^n || x^n, s_0) \right]^{\frac{1}{1+\rho}} \right\}^{1+\rho}
\]

\[
= (c) 4|\mathcal{S}|(M - 1)^\rho \sum_{y^n} \left\{ \sum_{s_0} \sum_{x^n} Q(x^n || z^{N-1}) \left[ P(y^n || x^n, s_0) \right]^{\frac{1}{1+\rho}} \right\}^{1+\rho}
\]

\[
\leq (d) 4(M - 1)^\rho \sum_{y^n} \left\{ \sum_{x^n} Q(x^n || z^{N-1}) \left[ P(y^n || x^n, s_0) \right]^{\frac{1}{1+\rho}} \right\}^{1+\rho}
\]

\[
\leq 4(M - 1)^\rho \sum_{y^n} \left\{ \sum_{x^n} Q(x^n || z^{N-1}) \left[ P(y^n || x^n, s_0) \right]^{\frac{1}{1+\rho}} \right\}^{1+\rho}
\]

\[
\leq 4(M - 1)^\rho |\mathcal{S}|^\rho \max_{s_0} \sum_{y^n} \left\{ \sum_{x^n} Q(x^n || z^{N-1}) \left[ P(y^n || x^n, s_0) \right]^{\frac{1}{1+\rho}} \right\}^{1+\rho}
\]

\[
(101)
\]

Inequalities (a) and (b) are achieved by moving the the term \( \frac{1}{|\mathcal{S}|} \) outside the sums. Inequality (c) is achieved by applying Jensen’s inequality \((\sum_i P(a_i)^\rho \leq \sum_i P(a_i)^r)\). Inequality (d) holds because the number of elements multiplied by the maximum element is larger than the sum of elements. Because the inequality holds for all \( Q(x^n || z^{N-1}) \),

\[
P_{e,m}(s_0) \leq 4|\mathcal{S}|(M - 1)^\rho |\mathcal{S}|^\rho \min_{Q(x^n || z^{N-1})} \max_{s_0} \sum_{y^n} \left\{ \sum_{x^n} Q(x^n || z^{N-1}) \left[ P(y^n || x^n, s_0) \right]^{\frac{1}{1+\rho}} \right\}^{1+\rho}
\]

\[
(102)
\]

By substituting \( M = 2^{NR} \) and eq. (36) and (37) into (102), we prove the theorem. ■

**APPENDIX IV**

**PROOF OF LEMMA 11**

Let us divide the input \( x^n \) into two sets \( x_1 = x^n_1 \) and \( x_2 = x^n_{n+1} \). Similarly, let us divide the output \( y^n \) into two sets \( y_1 = y^n_1 \) and \( y_2 = y^n_{n+1} \) and the feedback \( z^n \) into \( z_1 = z^n_1 \) and \( z_2 = z^n_{n+1} \). Let \( Q_n(x_1 || z_1) = \prod_{i=1}^{n} P(x_i || x^{i-1}) \) and \( Q_l(x_2 || z_2) = \prod_{i=1}^{l} P(x_{n+i+1} || x_{n+i+1}^{n+i-1}) \) be the probability assignments that achieve the maxima \( F_n(\rho) \) and \( F_l(\rho) \), respectively. Let us consider the probability assignment \( Q(x^n || z^{N-1}) = Q_n(x_1 || z_1)Q_l(x_2 || z_2) \). Then

\[
F_N \geq \frac{\rho \log |\mathcal{S}|}{N} + E_{o,N}(\rho, Q(x^n || z^{N-1}, s'_0)
\]

where \( s'_0 \) is the state that minimizes \( E_{o,N}(\rho, Q(x^n || z^{N-1}), s'_0) \).

Now,

\[
P(y^n || x^n, s'_0) \overset{(a)}{=} P(y^n | m, s'_0)
\]

\[
= \sum_{s_n} P(y^n, s_n | m, s'_0)
\]

\[
= \sum_{s_n} P(y_1, s_n | m, s'_0)P(y_2 | m, s_n, y_1, s'_0)
\]

\[
= \sum_{s_n} P(y_1, s_n | m, s'_0)P(y_2 | x_2, s_n)
\]

(104)
Equality (a) can be proved in the same way as eq. (32) was proved. The term \( P(y_1, s_n|m, s_0') \) can be also expressed in terms of \( y_1, x_1 \) in the following way:

\[
P(y_1, s_n|m, s_0') = P(s_n|m, y_1, s_0')P(y_1|m, s_0') = P(s_n|x_1, y_1, s_0')P(y_1|x_1, s_0')
\]

hence we obtain:

\[
P(y^N|x^N, s_0') = \sum_{s_n} P(y_2|x_2, s_n)P(s_n|x_1, y_1, s_0')P(y_1|x_1, s_0')
\]  

(106)

Consequently,

\[
2^{-nF_N(\rho)} 
\]

\[
\leq |S|^p \sum_{y_1} \left[ \sum_{x^N} Q(x^N||z^N-1)P(y^N|x^N, s_0')\right]^{1+\rho}(105)
\]

\[
= |S|^p \sum_{y_1} \left\{ \sum_{x_1} P(y_2|x_2, s_n)P(s_n|x_1, y_1, s_0')P(y_1|x_1, s_0') \right\}^{1+\rho}(106)
\]

\[
\leq |S|^p \sum_{y_1} \left[ \sum_{x_1} Q(x_1||z_1)P(s_n|x_1, y_1, s_0')P(y_1|x_1, s_0') \right]^{1+\rho}(107)
\]

(107)

Inequality (a) is due to inequality 103. Equality (b) is due to eq. (106). Inequality (c) holds because of the same reason as given in eq. (100), namely \( \left( \sum_i a_i \right)^r \leq \sum_i (a_i)^r \). Inequality (d) is due to Minkowski’s inequality

\[
\left[ \sum_j P_j \left( \sum_k a_{jk} \right)^{1/r} \right]^r \geq \sum_k \left( \sum_j P_j a_{jk}^{1/r} \right)^r \text{ for } r > 1.
\]