ORIE 6326: Convex Optimization

Subgradients

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Some slides adapted from Stanford EE364b

Outline

Subgradients

Properties

Subgradient calculus

Optimality

Basic inequality

recall basic inequality for convex differentiable f:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

- first-order approximation of f at x is global underestimator
- $(\nabla f(x), -1)$ supports $\mathbf{epi}\, f$ at $(x, f(x))$

what if f is not differentiable?

Non-differentiable functions

are these functions differentiable?

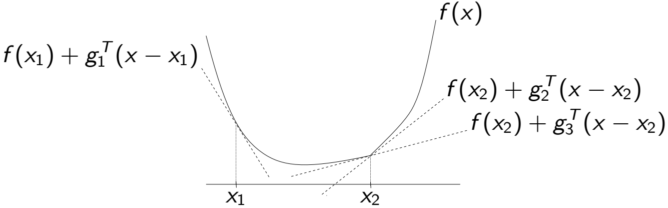
- $|t|$ for $t \in \mathbf{R}$
- $\|x\|_1$ for $x \in \mathbf{R}^n$
- $\|X\|_*$ for $X \in \mathbf{R}^{n \times n}$
- $\max_i a_i^T x + b_i$ for $x \in \mathbf{R}^n$
- $\lambda_{\max}(X)$ for $X \in \mathbf{R}^{n \times n}$
- indicators of convex sets $\mathcal{C}$

if not, where? can we find underestimators for them?

Subgradient of a function

g is a **subgradient** of f (not necessarily convex) at x if

$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$



g_2, g_3 are subgradients at x_2; g_1 is a subgradient at x_1

Subgradients and convexity

- g is a subgradient of f at x iff $(g, -1)$ supports **epi** f at $(x, f(x))$
- g is a subgradient iff $f(x) + g^T(y - x)$ is a global (affine) underestimator of f
- if f is convex and differentiable, $\nabla f(x)$ is a subgradient of f at x

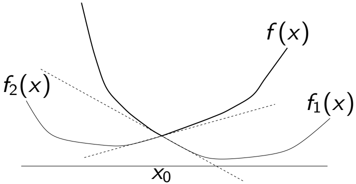
subgradients come up in several contexts:

- algorithms for nondifferentiable convex optimization
- convex analysis, *e.g.*, optimality conditions, duality for nondifferentiable problems

(if $f(y) \leq f(x) + g^T(y - x)$ for all y, then g is a **supergradient**)

Example

$f = \max\{f_1, f_2\}$, with f_1, f_2 convex and differentiable



- $f_1(x_0) > f_2(x_0)$: unique subgradient $g = \nabla f_1(x_0)$
- $f_2(x_0) > f_1(x_0)$: unique subgradient $g = \nabla f_2(x_0)$
- $f_1(x_0) = f_2(x_0)$: subgradients form a line segment $[\nabla f_1(x_0), \nabla f_2(x_0)]$

Subdifferential

set of all subgradients of f at x is called the **subdifferential** of f at x, denoted $\partial f(x)$

$$\partial f(x) = \{g : f(y) \geq f(x) + g^T(y - x) \quad \forall y\}$$

for any f,

- $\partial f(x)$ is a closed convex set (can be empty)
- $\partial f(x) = \emptyset$ if $f(x) = \infty$

proof: use the definition

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proof: use the definition

if f is convex,

- $\partial f(x)$ is nonempty, for $x \in \textbf{relint dom}\, f$
- $\partial f(x) = \{\nabla f(x)\}$, if f is differentiable at x
- if $\partial f(x) = \{g\}$, then f is differentiable at x and $g = \nabla f(x)$

Compute subgradient via definition

$g \in \partial f(x)$ iff

$$f(y) \geq f(x) + g^T(y - x) \quad \forall y \in \textbf{dom}(f)$$

example. let $f(x) = |x|$ for $x \in \mathbf{R}$. suppose $s \in \textbf{sign}(x)$, where

$$\textbf{sign}(x) = \begin{cases} \{1\} & x > 0 \\ [-1, 1] & x = 0 \\ -\{1\} & x < 0. \end{cases}$$

then

$$f(y) = \max(y, -y) \geq sy = s(x + y - x) = |x| + s(y - x)$$

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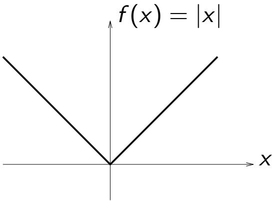
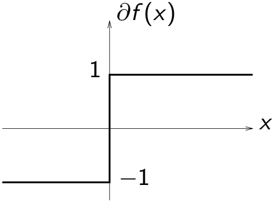
$$f(y) = \max(y, -y) \geq sy = s(x + y - x) = |x| + s(y - x)$$

so $\mathbf{sign}(x) \subseteq \partial f(x)$ (in fact, holds with equality)

Subgradient of $|x|$

$f(x) = |x|$ for $x \in \mathbf{R}$





righthand plot shows $\bigcup \{(x, g) \mid x \in \mathbf{R},\ g \in \partial f(x)\}$

Compute subgradient via definition

$g \in \partial f(x)$ iff

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example. let $f(x) = \max_i a_i^T x + b_i$.

Compute subgradient via definition

$g \in \partial f(x)$ iff

$$f(y) \geq f(x) + g^T(y - x) \quad \forall y \in \mathbf{dom}(f)$$

example. let $f(x) = \max_i a_i^T x + b_i$. then for any i,

$$\begin{aligned} f(y) &= \max_i a_i^T y + b_i \\ &\geq a_i^T y + b_i \\ &= a_i^T(x + y - x) + b_i \\ &= a_i^T x + b_i + a_i^T(y - x) \\ &= f(x) + a_i^T(y - x), \end{aligned}$$

where the last line holds for $i \in \operatorname{argmax}_j a_j^T x + b_j$. so

- $a_i \in \partial f(x)$ for each $i \in \operatorname{argmax}_j a_j^T x + b_j$
- $\partial f(x)$ is convex, so

$$\mathbf{conv}\{a_i : i \in \operatorname*{argmax}_j a_j^T x + b_j\} \subseteq \partial f(x)$$

Compute subgradient via definition

$g \in \partial f(x)$ iff

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example. let $f(X) = \lambda_{\max}(X)$.

Compute subgradient via definition

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$$f(y) \geq f(x) + g^T(y - x) \quad \forall y \in \textbf{dom}(f)$$

example. let $f(X) = \lambda_{\max}(X)$. then

$$\begin{aligned}
f(Y) &= \sup \|v\| \leq 1 v^T Y v \\
&= \sup \|v\| \leq 1 v^T (X + Y - X) v, \quad \|v\| \leq 1 \\
&= \sup \|v\| \leq 1 \left(v^T X v + v^T (Y - X) v \right), \quad \|v\| \leq 1 \\
&= v^T X v + \textbf{tr}(v v^T (Y - X)), \quad v \in \underset{\|v\| \leq 1}{\operatorname{argmax}} v^T X v \\
&= \lambda_{\max}(X) + \textbf{tr}(v v^T (Y - X)), \quad v \in \underset{\|v\| \leq 1}{\operatorname{argmax}} v^T X v
\end{aligned}$$

so

- $vv^T \in \partial f(x)$ for each $v \in \operatorname{argmax}_{\|v\| \leq 1} v^T X v$
- $\partial f(x)$ is convex, so

$$\textbf{conv}\{vv^T : v \in \underset{\|v\| \leq 1}{\operatorname{argmax}} v^T X v\} \subseteq \partial f(x)$$

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Optimality

Properties of subgradients

subgradient inequality:

$$g \in \partial f(x) \iff f(y) \geq f(x) + g^T(y - x) \quad \forall y \in \mathbf{dom}(f)$$

for convex f, we'll show

- subgradients are monotone: for any $x, y \in \mathbf{dom}\, f$, $g_y \in \partial f(y)$, and $g_x \in \partial f(x)$,
$$(g_y - g_x)^T(y - x) \geq 0$$
- $\partial f(x)$ is continuous: if f is (lower semi-)continuous, $x^{(k)} \to x$, $g^{(k)} \to g$, and $g^{(k)} \in \partial f(x^{(k)})$ for each k, then $g \in \partial f(x)$
- $\partial f(x) = \mathrm{argmax}\, g^T x - f(x)$

these will help us compute subgradients

Subgradients are monotone

fact. for any $x, y \in \mathbf{dom}\, f$, $g_y \in \partial f(y)$, and $g_x \in \partial f(x)$,

$$(g_y - g_x)^T (y - x) \geq 0$$

proof. same as for differentiable case:

$$f(y) \geq f(x) + g_x^T (y - x) \qquad f(x) \geq f(y) + g_y^T (x - y)$$

add these to get

$$(g_y - g_x)^T (y - x) \geq 0$$

Subgradients are preserved under limits

subgradient inequality:

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fact. if f is (lower semi-)continuous, $x^{(k)} \to x$, $g^{(k)} \to g$, and $g^{(k)} \in \partial f(x^{(k)})$ for each k, then $g \in \partial f(x)$

proof.

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proof. For each k and for every y,

$$\begin{aligned} f(y) &\geq f(x^{(k)}) + (g^{(k)})^T(y - x^{(k)}) \\ \lim_{k\to\infty} f(y) &\geq \lim_{k\to\infty} f(x^{(k)}) + (g^{(k)})^T(y - x^{(k)}) \\ f(y) &\geq f(x) + g^T(y - x) \end{aligned}$$

moral. To find a subgradient $g \in \partial f(x)$, find points $x^{(k)} \to x$ where f is differentiable, and let $g = \lim_{k\to\infty} \nabla f(x^{(k)})$.

Subgradients are preserved under limits: example

consider $f(x) = |x|$. we know

$$\partial f(x) = \begin{cases} \{-1\} & x < 0 \\ ? & x = 0 \\ \{1\} & x > 0 \end{cases}$$

so

- $\lim_{x \to 0^+} \nabla(x) = 1$
- $\lim_{x \to 0^-} \nabla(x) = -1$

hence

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- $\lim_{x \to 0^+} \nabla(x) = 1$
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hence

- $-1 \in \partial f(0)$ and $-1 \in \partial f(0)$
- $\partial f(0)$ is convex, so $[-1, 1] \subseteq \partial f(0)$
- and $\partial f(0)$ is monotone, so $[-1, 1] = \partial f(0)$

Convex functions can't be very non-differentiable

Theorem. (Rockafellar, Convex Analysis, Thm 25.5)
a convex function is differentiable almost everywhere on the interior of its domain.

In other words, if you pick $x \in \textbf{dom}\, f$ uniformly at random, then with probability 1, f is differentiable at x.

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$$\tilde{\nabla} f(y)^T(y-x) \geq \tilde{\nabla} f(x)^T(y-x);$$

notice $(y-x)$ is scalar to see $\tilde{\nabla} f(y) \geq \tilde{\nabla} f(x)$ if $y \geq x$.) At each nondifferentiable point x, $\tilde{\nabla} f(y)$ jumps up by some finite amount! It can't do that too often.

More formally, $|\partial f(x)|$ is strictly positive for each x where f is nondifferentiable; and the sum of uncountably many positive numbers is infinite. So the number of x's where f is not differentiable must be countable over the interior of the domain of f; and hence, f is a.e. differentiable on the interior of its domain.

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Subgradients and fenchel conjugates

fact. $g \in \partial f(x) \iff f^*(g) + f(x) = g^T x$

(recall the conjugate function $f^*(g) = \sup_x g^T x - f(x)$.)

Subgradients and fenchel conjugates

proof. if $f^*(g) + f(x) = g^T x$,

$$\begin{array}{rcl} f^*(g) & = & \sup_y g^T y - f(y) \\ & \geq & g^T y - f(y) \quad \forall y \\ f(y) & \geq & g^T y - f^*(g) \quad \forall y \\ & = & g^T y - g^T x + f(x) \quad \forall y \\ & = & g^T (y - x) + f(x) \quad \forall y \end{array}$$

so $g \in \partial f(x)$. conversely, if $g \in \partial f(x)$,

Subgradients and fenchel conjugates

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so $g \in \partial f(x)$. conversely, if $g \in \partial f(x)$,

$$\begin{array}{rcl} f(y) &\geq& g^T(y-x) + f(x) \\ g^T x - f(x) &\geq& g^T y - f(y) \\ \sup_y g^T x - f(x) &\geq& \sup_y g^T y - f(y) \\ g^T x - f(x) &\geq& f^*(g) \end{array}$$

so $f^*(g) + f(x) = g^T x$.

Subgradients and fenchel conjugates

Conclusion.

$$\begin{aligned} g \in \partial f(x) \quad &\iff \quad f^*(g) + f(x) = g^T x \\ &\iff \quad x \in \operatorname*{argmax}_x g^T x - f(x) \end{aligned}$$

consider the same implications for the function f^*:

$$\begin{aligned} x \in \partial f^*(g) \quad &\iff \quad f(x) + f^*(g) = x^T g \\ &\iff \quad g \in \operatorname*{argmax}_g g^T x - f^*(g) \end{aligned}$$

so all these conditions are equivalent, and
$g \in \partial f(x) \iff x \in \partial f^*(g)$!

Compute subgradient via fenchel conjugate

$$\partial f(x) = \operatorname*{argmax}_g g^T x - f^*(g)$$

example. let $f(x) = \|x\|_1$. compute

$$\begin{aligned} f^*(g) &= \sup_x g^T x - \|x\|_1 \\ \\ &= \end{aligned}$$

Compute subgradient via fenchel conjugate

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example. let $f(x) = \|x\|_1$. compute

$$\begin{aligned} f^*(g) &= \sup_x g^T x - \|x\|_1 \\ &= \begin{cases} 0 & \|g\|_\infty \leq 1 \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

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given x,

$$\begin{aligned} \partial f(x) &= \operatorname*{argmax}_g g^T x - f^*(g) \\ &= \operatorname*{argmax}_{\|g\|_\infty \le 1} g^T x \\ &= \mathbf{sign}(x) \end{aligned}$$

where **sign** is computed elementwise.

Compute subgradient via fenchel conjugate

$$\partial f(x) = \operatorname*{argmax}_{g} g^T x - f^*(g)$$

example. let $f(X) = \|X\|_*$. compute

$$\begin{aligned} f^*(G) &= \sup_X \mathbf{tr}(G, X) - \|X\|_* \\ \\ &= \end{aligned}$$

Compute subgradient via fenchel conjugate

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where $\|G\| = \sigma_1(G)$ is the operator norm of G.

Compute subgradient via fenchel conjugate

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example. let $f(X) = \|X\|_*$. compute

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where $\|G\| = \sigma_1(G)$ is the operator norm of G.

given $X = U \,\mathbf{diag}(\sigma) V^T$,

$$\begin{aligned} \partial f(x) &= \operatorname*{argmax}_G \mathbf{tr}(G, X) - f^*(G) \\ &= \operatorname*{argmax}_{\|G\| \le 1} \mathbf{tr}(G, X) \\ &= U \,\mathbf{diag}(\mathbf{sign}(\sigma)) V^T \end{aligned}$$

where **sign** is computed elementwise.

Outline

Subgradients

Properties

Subgradient calculus

Optimality

Subgradient calculus

- **weak subgradient calculus**: formulas for finding **one** subgradient $g \in \partial f(x)$
- **strong subgradient calculus**: formulas for finding the whole subdifferential $\partial f(x)$, *i.e.*, **all** subgradients of f at x
- many algorithms for nondifferentiable convex optimization require only **one** subgradient at each step, so weak calculus suffices
- some algorithms, optimality conditions, etc., need whole subdifferential
- roughly speaking: if you can compute $f(x)$, you can usually compute a $g \in \partial f(x)$
- we'll assume that f is convex, and $x \in \mathbf{relint\,dom}\, f$

Some basic rules

- $\partial f(x) = \{\nabla f(x)\}$ if f is differentiable at x
- **scaling:** $\partial(\alpha f) = \alpha \partial f$ (if $\alpha > 0$)
- **addition:** $\partial(f_1 + f_2) = \partial f_1 + \partial f_2$ (RHS is addition of point-to-set mappings)
- **affine transformation of variables:** if $g(x) = f(Ax + b)$, then $\partial g(x) = A^T \partial f(Ax + b)$
- **finite pointwise maximum:** if $f = \max_{i=1,\ldots,m} f_i$, then

$$\partial f(x) = \mathbf{conv} \bigcup \{\partial f_i(x) \mid f_i(x) = f(x)\},$$

  i.e., convex hull of union of subdifferentials of 'active' functions at x

Minimization

define $g(y)$ as the optimal value of

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq y_i, \quad i = 1, \dots, m \end{array}$$

(f_i convex; variable x)

with $\lambda^\star$ an optimal dual variable, we have

$$g(z) \geq g(y) - \sum_{i=1}^{m} \lambda_i^\star (z_i - y_i)$$

i.e., $-\lambda^\star$ is a subgradient of g at y

Composition

- $f(x) = h(f_1(x), \ldots, f_k(x))$, with h convex nondecreasing, f_i convex
- find $q \in \partial h(f_1(x), \ldots, f_k(x))$, $g_i \in \partial f_i(x)$
- then, $g = q_1 g_1 + \cdots + q_k g_k \in \partial f(x)$
- reduces to standard formula for differentiable h, f_i

proof:

$$\begin{aligned} f(y) &= h(f_1(y), \ldots, f_k(y)) \\ &\geq h(f_1(x) + g_1^T(y-x), \ldots, f_k(x) + g_k^T(y-x)) \\ &\geq h(f_1(x), \ldots, f_k(x)) + q^T(g_1^T(y-x), \ldots, g_k^T(y-x)) \\ &= f(x) + g^T(y-x) \end{aligned}$$

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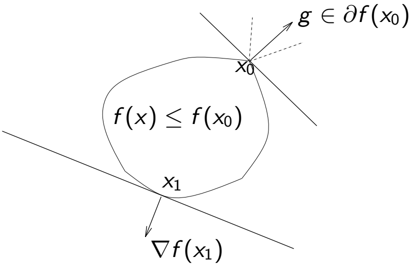
Subgradients and sublevel sets

g is a subgradient at x means $f(y) \geq f(x) + g^T(y - x)$

hence $f(y) \leq f(x) \Longrightarrow g^T(y - x) \leq 0$



- f differentiable at x_0: $\nabla f(x_0)$ is normal to the sublevel set $\{x \mid f(x) \leq f(x_0)\}$
- f nondifferentiable at x_0: subgradient defines a supporting hyperplane to sublevel set through x_0

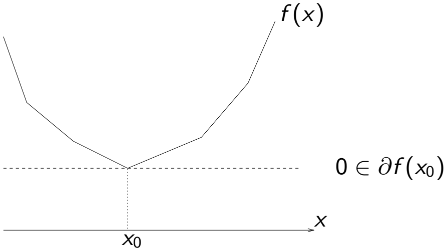
Optimality conditions — unconstrained

recall for f convex, differentiable,

$$f(x^\star) = \inf_x f(x) \iff 0 = \nabla f(x^\star)$$

generalization to nondifferentiable convex f:

$$f(x^\star) = \inf_x f(x) \iff 0 \in \partial f(x^\star)$$



proof.

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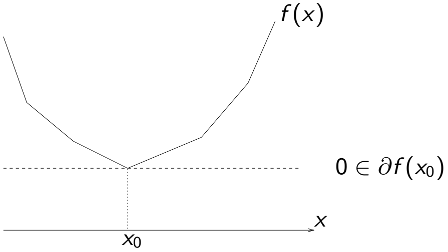


proof. by definition (!)

$$f(y) \geq f(x^\star) + 0^T(y - x^\star) \text{ for all } y \quad \iff \quad 0 \in \partial f(x^\star)$$

...seems trivial but isn't

Example: piecewise linear minimization

$f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$

$x^\star$ minimizes $f \iff 0 \in \partial f(x^\star) = \mathbf{conv}\{a_i \mid a_i^T x^\star + b_i = f(x^\star)\}$

$\iff$ there is a λ with

$$\lambda \succeq 0, \qquad \mathbf{1}^T\lambda = 1, \qquad \sum_{i=1}^m \lambda_i a_i = 0$$

where $\lambda_i = 0$ if $a_i^T x^\star + b_i < f(x^\star)$

. . . but these are the KKT conditions for the epigraph form

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \end{array}$$

with dual

$$\begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & \lambda \succeq 0, \qquad A^T \lambda = 0, \qquad \mathbf{1}^T \lambda = 1 \end{array}$$

Optimality conditions — constrained

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \;\; i = 1, \ldots, m \end{array}$$

we assume

- f_i convex, defined on $\mathbf{R}^n$ (hence subdifferentiable)
- strict feasibility (Slater's condition)

$x^\star$ is primal optimal ($\lambda^\star$ is dual optimal) iff

$$\begin{array}{l} f_i(x^\star) \leq 0, \;\; \lambda_i^\star \geq 0 \\ 0 \in \partial f_0(x^\star) + \sum_{i=1}^m \lambda_i^\star \partial f_i(x^\star) \\ \lambda_i^\star f_i(x^\star) = 0 \end{array}$$

... generalizes KKT for nondifferentiable f_i

Directional derivative

directional derivative of f at x in the direction δx is

$$f'(x; \delta x) \stackrel{\Delta}{=} \lim_{h \searrow 0} \frac{f(x + h\delta x) - f(x)}{h}$$

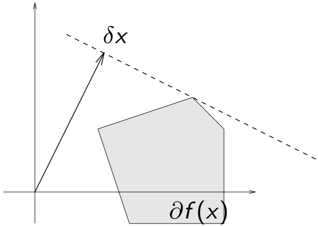
can be $+\infty$ or $-\infty$

- f convex, finite near $x \Longrightarrow f'(x; \delta x)$ exists

- f differentiable at x if and only if, for some g $(= \nabla f(x))$ and all δx, $f'(x; \delta x) = g^T \delta x$ (*i.e.*, $f'(x; \delta x)$ is a linear function of δx)

Directional derivative and subdifferential

general formula for convex f: $f'(x;\delta x) = \sup_{g \in \partial f(x)} g^T \delta x$



Descent directions

δx is a **descent direction** for f at x if $f'(x; \delta x) < 0$

for differentiable f, $\delta x = -\nabla f(x)$ is always a descent direction (except when it is zero)

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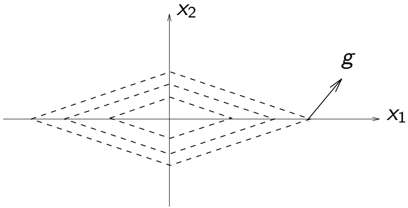
warning: for nondifferentiable (convex) functions, $\delta x = -g$, with $g \in \partial f(x)$, need not be descent direction

example: $f(x) = |x_1| + 2|x_2|$



Subgradients and distance to sublevel sets

if f is convex, $f(z) < f(x)$, $g \in \partial f(x)$, then for small $t > 0$,

$$\|x - tg - z\|_2 < \|x - z\|_2$$

thus $-g$ is descent direction for $\|x - z\|_2$, for **any** z with $f(z) < f(x)$ (*e.g.*, $x^\star$)

negative subgradient is descent direction for distance to optimal point

$$\text{proof:} \quad \begin{aligned} \|x - tg - z\|_2^2 &= \|x - z\|_2^2 - 2tg^T(x - z) + t^2\|g\|_2^2 \\ &\leq \|x - z\|_2^2 - 2t(f(x) - f(z)) + t^2\|g\|_2^2 \end{aligned}$$

Descent directions and optimality

fact: for f convex, finite near x, either

- $0 \in \partial f(x)$ (in which case x minimizes f), or
- there is a descent direction for f at x

i.e., x is optimal (minimizes f) iff there is no descent direction for f at x

proof: define $\delta x_{\mathsf{sd}} = -\operatorname*{argmin}_{z \in \partial f(x)} \|z\|_2$

if $\delta x_{\mathrm{sd}} = 0$, then $0 \in \partial f(x)$, so x is optimal; otherwise $f'(x; \delta x_{\mathsf{sd}}) = -\left(\inf_{z \in \partial f(x)} \|z\|_2\right)^2 < 0$, so δx_{sd} is a descent direction