

# The Stationary Prophet Inequality Problem

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## Abstract

We study a continuous and infinite time horizon counterpart to the classic prophet inequality, which we term the stationary prophet inequality problem. Here, copies of a good arrive and perish according to Poisson point processes. Buyers arrive similarly and make take-it-or-leave-it offers for unsold items. The objective is to maximize the (infinite) time average revenue of the seller.

Our main results are pricing-based policies which (i) achieve a  $1/2$ -approximation of the optimal offline policy, which is best possible, and (ii) achieve a better than  $(1 - 1/e)$ -approximation of the optimal online policy. Result (i) improves upon bounds implied by recent work of Collina et al. (WINE'20), and is the first optimal prophet inequality for a stationary problem. Result (ii) improves upon a  $1 - 1/e$  bound implied by recent work of Aouad and Saritaç (EC'20), and shows that this prevalent bound in online algorithms is not optimal for this problem.

# 1 Introduction

A ubiquitous challenge in market economics is decision making under uncertainty, addressed by the area of online algorithms. Should a firm sell an item to a buyer now, or reject their bid in favor of possibly higher future bids (at the risk of no such higher future bids arriving)? Such dynamics were studied by probabilists in the area of optimal stopping theory as early as the 60s and 70s [13, 21, 22], and have regained renewed interest in recent years in the online algorithms community, in large part due to their relevance to mechanism design.

A classic problem in the area is the single-item prophet inequality problem. Here, a buyer wishes to sell a single item, and buyers arrive in some order, with buyer  $i$  making a take-it-or-leave-it bid  $v_i$  drawn from a (known) distribution  $\mathcal{D}_i$ . The classic result of Krenkel and Sucheston [21, 22] asserts that there exists a  $1/2$ -competitive algorithm, i.e., an algorithm whose expected value is at least  $1/2$  of the value  $\mathbb{E}[\max_i v_i]$  obtained by a “prophet” who knows the future. This is optimal—no online algorithm has higher competitive ratio. Shortly after, Samuel-Cahn [28] presented a  $1/2$ -competitive *posted-price* policy (i.e., selling the item to the first buyer bidding above some fixed threshold), foreshadowing a long line of work on such pricing-based policies.

The classic single-item prophet inequality problem has been generalized to selling more complicated combinatorial structures, including, e.g., multiple items [1, 7, 18], knapsacks [12, 16], matroids and their intersections [4, 8, 12, 16, 20], matchings [12, 14, 15, 16, 17], and arbitrary downward-closed families [27]. Most of this work has focused on approximating the offline optimum algorithm, with recent work also studying the (in)approximability of the optimal online algorithm by poly-time algorithms [2, 26] and particularly by posted-price policies [25]. Much of the interest in prophet inequality problems, and specifically pricing-based policies, has been fueled by their implication of truthful mechanisms which approximately maximize social welfare and revenue, first observed in [8] (see the surveys [10, 19, 23], and [11] for the “opposite” direction).

Despite this rich line of work on prophet inequalities and their use in online markets, one salient feature of motivating markets is missing in these problems’ formulations: the repeated nature of the dynamics of such markets. In such markets, companies care less about their immediate returns than their average long-term rewards. Over such long time horizons, companies produce additional goods, while items which are not sold fast enough may expire. Neither the long-term objective nor the dynamic nature of goods to sell is captured by traditional prophet inequality problems.

We introduce a continuous-time, infinite time horizon counterpart to the classic prophet inequality problem, which we term the *stationary prophet inequality* problem. Here, goods are produced over time, where items of each good arrive according to Poisson processes, and, if unsold, perish according to Poisson processes. Buyers with different valuations for the different goods similarly enter the market according to Poisson processes. When a buyer  $b$  arrives, a policy determines immediately which item (if any) to sell to  $b$ . The objective is to maximize the infinite time horizon average reward of the policy, compared to the optimal offline or online policies. (See Section 2 for precise problem formulation.)

A closely related problem of dynamic weighted matching was recently introduced by [3, 9]. In their problem, the market is not bipartite, and agents arrive over time and can be matched at any time before their (sudden and unpredictable) departure from the market. (Our problem is the special case of theirs where buyers’ departure rate is infinite.) The authors of [3, 9] present algorithms which give  $\frac{1}{4}(1 - 1/e)$ - and  $1/8$ -approximations of the optimal online and offline policies, respectively. For our single-good problem, their approaches yield  $(1 - 1/e)$ - and  $(1 - 1/(e-1)) \approx 0.418$ -approximations of the optimal online and offline policies, respectively (see Appendix B).

We ask to what extent these bounds can be improved, in particular, using *posted-price* policies.

## 1.1 Our contributions

Our main results concern the single-good stationary prophet inequality problem.

Our first result is a pricing-based policy which achieves a competitive ratio of  $1/2$ . That is, this policy achieves an approximation of  $1/2$  of the value gained by the optimal offline policy for this problem—a bound we show is optimal for any policy (pricing-based or otherwise).

**Theorem 1.1.** *There exists a  $1/2$ -competitive posted-price policy for the single-good stationary prophet inequality problem. No online policy has competitive ratio greater than  $1/2$ .*

Theorem 1.1 is the first optimally competitive policy for a stationary prophet inequality problem. Our policy follows the approach given by [9], whose analysis implies a  $1/3$ -competitive ratio by comparing to a natural LP benchmark (see Appendix B). Our first technical contribution is a more queuing-theoretic analysis, via which we show that their policy is in fact  $1 - 1/(e-1) \approx 0.418$ -competitive. Unfortunately, this is the best bound achievable using their approach; we show that for their LP benchmark, the above  $1 - 1/(e-1)$  bound is tight (see Section 3). Our second contribution is a new constraint, relying on another fundamental result in queuing theory, namely that Poisson arrivals “see” time averages (PASTA, [29]). These queuing theoretic and approximation algorithmic ideas combined yield our optimally-competitive policy.

We next turn to the approximability of the optimal *online* policy, where we might hope to achieve higher approximation guarantees. For the classic prophet inequality problem, Niazadeh et al. [25] show that pricing-based policies yield no better approximation of the optimal online policy than they do of the optimal offline policy. For the stationary prophet inequality problem, the same is not true; while our inapproximability result of Theorem 1.1 implies that no competitive ratio beyond  $1/2$  is possible, an algorithm of [3] yields a  $1 - 1/e \approx 0.632$  approximation of the optimal online policy. We prove that this latter natural bound, prevalent in the online algorithms literature, is not optimal for our problem, and present a pricing-based policy which breaks this bound.

**Theorem 1.2.** *There exists a posted-price policy for the single-good stationary prophet inequality problem which is a 0.656-approximation of the optimal online policy in expectation.*

Our analysis compared to the LP benchmark of [3] gives a simple  $(1 - 1/e)$ -approximation of the optimal online policy, which is tight for their LP. Here, we show that our new PASTA constraint and analysis allow us to break this ubiquitous bound.

**Mechanism design implications.** By standard connections to mechanism design, our pricing-based policies immediately imply truthful mechanisms which approximate the social-welfare and revenue maximizing offline and online mechanisms (see Appendix E).

Finally, using the same algorithmic and analytic ideas, together with additional stochastic dominance results, we extend our approach from the single-good to the multi-good problem. For this natural generalization, we present a  $15/56 \approx 0.267$ -competitive policy, improving on the  $1/8 = 0.125$ -competitive policy of [9].

**Theorem 1.3.** *There exists a  $15/56$ -competitive policy for the multi-good problem.*

**Bounded inventory.** Surprisingly, all of our policies retain their approximation guarantees even when sellers have small inventory sizes, and must discard items of goods when more than some (small) number of items of said good are already available. In contrast, in Lemma 4.6 we show that sufficiently small inventory size does, however, limit achievable approximation (of any policy).

## 2 Preliminaries

**Problem statement.** In the stationary prophet inequality problem, a seller wishes to sell items of  $n$  types of goods  $\mathcal{G}$ , while (approximately) maximizing the seller’s average gain over an infinite time horizon. Items of good  $i \in \mathcal{G}$  are homogeneous and are supplied according to a Poisson process with rate  $\lambda_i \in (0, \infty)$  and perish at an exponential rate  $\mu_i \in (0, \infty)$ . An item is *present* if it has been supplied but has not yet perished or been discarded, whereas it is *available* if it is both present and has not yet been sold. An item is discarded on arrival if the number of available items of the same good equals the seller’s inventory capacity  $C$  (unless otherwise specified,  $C \rightarrow \infty$ ). Buyers are unit-demand and arrive according to a Poisson process with rate  $\gamma > 0$ , with their types drawn i.i.d. from a distribution  $\mathcal{D}$  over a set of  $m$  types  $\mathcal{B}$ , with type  $j \in \mathcal{B}$  bidding values  $\mathbf{v}_j = (v_{ij})_{i \in \mathcal{G}}$ . Thus, buyers of type  $j \in \mathcal{B}$  arrive according to a Poisson process with rate  $\gamma_j \triangleq \gamma \cdot \mathbb{P}_{\mathbf{v} \sim \mathcal{D}}[\mathbf{v} = \mathbf{v}_j]$ . Upon the arrival of a buyer of type  $j \in \mathcal{B}$ , the seller must irrevocably decide whether to sell an available item of at most one good  $i \in \mathcal{G}$  to the buyer at their bid price,  $v_{ij}$ , and the buyer immediately departs after the seller’s decision.

An offline policy for an instance  $\mathcal{I}$  of the stationary prophet inequality problem knows in advance the realization of all the randomness of the input, i.e., it knows the times at which items of goods are supplied and perish, and the times at which different buyers arrive. An online policy, on the other hand, knows  $\{\lambda_i, \mu_i\}_{i \in \mathcal{G}}$  and  $\{\mathbf{v}_j, \gamma_j\}_{j \in \mathcal{B}}$  a priori, but does not know the realization of future randomness of the input. An example of online single-good policies are *posted-price policies*, which set a pair  $(\bar{v}, \bar{p})$  and accept all bids strictly greater than  $\bar{v}$ , accept bids equal to  $\bar{v}$  with probability  $\bar{p}$ , and reject all bids strictly less than  $\bar{v}$ . The optimal expected average reward of an unbounded-capacity offline (resp., online) policy for instance  $\mathcal{I}$  is denoted by  $\text{OPT}_{\text{off}}(\mathcal{I})$  (resp.,  $\text{OPT}_{\text{on}}(\mathcal{I})$ ). We measure online policies’ average reward in terms of their approximation of  $\text{OPT}_{\text{off}}(\mathcal{I})$  and  $\text{OPT}_{\text{on}}(\mathcal{I})$ .

### 2.1 Prior LP benchmarks and a natural algorithm

Collina et al. [9] and Aouad and Saritaç [3] present the following LP benchmarks, which upper bound the average gain of *any* offline or online policy, respectively.

**Lemma 2.1** ([3, 9]). *Let  $x_{ij}$  be the rate at which an offline policy sells items of good  $i \in \mathcal{G}$  to buyers of type  $j \in \mathcal{B}$ . Then  $\mathbf{x} = (x_{ij})_{i \in \mathcal{G}, j \in \mathcal{B}}$  satisfies the following constraints:*

$$\sum_{j \in \mathcal{B}} x_{ij} \leq \lambda_i \quad \forall i \in \mathcal{G} \quad (1)$$

$$\sum_{i \in \mathcal{G}} x_{ij} \leq \gamma_j \quad \forall j \in \mathcal{B} \quad (2)$$

$$x_{ij} \leq \gamma_j \cdot \frac{\lambda_i}{\mu_i} \quad \forall i \in \mathcal{G}, j \in \mathcal{B} \quad (3)$$

$$x_{ij} \geq 0. \quad (4)$$

If  $\mathbf{x}$  is the vector of rates derived by an online policy, then  $\mathbf{x}$  also satisfies the following constraint:

$$x_{ij} \leq \gamma_j \cdot \left( \frac{\lambda_i - \sum_{\ell \in \mathcal{B}} x_{i\ell}}{\mu_i} \right). \quad (5)$$

**Corollary 2.2.** *For all instances  $\mathcal{I}$  of the stationary prophet inequality problem,*

1.  $\text{RB}_{\text{off}}(\mathcal{I}) \triangleq \max \left\{ \sum_{i,j} v_{ij} \cdot x_{ij} \mid \mathbf{x} \text{ satisfies (1) - (4)} \right\}$  satisfies  $\text{RB}_{\text{off}}(\mathcal{I}) \geq \text{OPT}_{\text{off}}(\mathcal{I})$ , and

$$2. \text{RB}_{\text{on}}(\mathcal{I}) \triangleq \max \left\{ \sum_{i,j} v_{ij} \cdot x_{ij} \mid \mathbf{x} \text{ satisfies (1) - (5)} \right\} \text{ satisfies } \text{RB}_{\text{on}}(\mathcal{I}) \geq \text{OPT}_{\text{on}}(\mathcal{I}).$$

Collina et al. [9] present a simple non-adaptive algorithm for the stationary prophet inequality problem, which attempts to (approximately) follow the sale rates prescribed by a solution  $\mathbf{x}^*$  to the offline benchmark  $\text{RB}_{\text{off}}$ , with additional parameters  $\alpha \in [0, 1]$  and  $w_i$  satisfying  $x_{ij} \leq \gamma_j \cdot w_i$ . Their algorithm, generalized to take as input any  $\mathbf{x}^* \in \mathbb{R}^{|\mathcal{G}| \times |\mathcal{B}|}$ , is given in Algorithm 1.<sup>1</sup> Intuitively, Algorithm 1 attempts to match good  $i \in \mathcal{G}$  and buyer type  $j \in \mathcal{B}$  at a rate close to  $x_{ij}^*$ . The role of high  $\alpha$  is to increase the probability of a sale of good  $i$  to buyer type  $j$ , conditioned on an item of good  $i$  being available, while the role of a lower  $\alpha$  is to increase the probability of items being available. The crux of the analysis is in bounding this probability, for which we rely on machinery from the queuing theory literature, described below.

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### Algorithm 1

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- 1: **for** arrival of buyer of type  $j \in \mathcal{B}$  **do**
  - 2:     **for** each good  $i \in \mathcal{G}$  in a uniform random order **do**
  - 3:         **if** buyer unmatched and at least one item of good  $i$  is available **then**
  - 4:             sell with probability  $p_{ij} \triangleq \alpha \cdot \frac{x_{ij}^*}{\gamma_j \cdot w_i}$
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## 2.2 Queuing theory background

Throughout, we will want to bound the probability of there being an available (i.e., present and unsold) item of good  $i \in \mathcal{G}$ . We denote this probability by  $\mathbb{P}_C [A_i \geq 1]$ , where  $A_i$  denotes the number of available items of good  $i$ .

**Lemma 2.3.** *For any online policy with inventory capacity  $C \in \mathbb{Z}_{>0}$  and which sells any available item of good  $i \in \mathcal{G}$  to buyers which arrive at rate  $\gamma^*$ , the stationary probability of an item of good  $i \in \mathcal{G}$  being available satisfies*

$$\mathbb{P}_C [A_i \geq 1] = 1 - \left( 1 + \sum_{q=1}^C \prod_{r=1}^q \frac{\lambda_i}{r \cdot \mu_i + \gamma^*} \right)^{-1}.$$

As a corollary of the previous lemma, we have the following.

**Corollary 2.4.** *For any online policy with inventory capacity  $C \in \mathbb{Z}_{>0}$  and which sells any available item of good  $i \in \mathcal{G}$  to buyers which arrive at rate  $\gamma^*$ , the stationary probability of an item of good  $i \in \mathcal{G}$  being available satisfies*

$$\mathbb{P}_C [A_i \geq 1] \in \left[ 1 - \left( \sum_{q=0}^C \frac{1}{q!} \left( \frac{\lambda_i}{\mu_i + \gamma^*} \right)^q \right)^{-1}, 1 - \left( \sum_{q=0}^C \frac{1}{q!} \left( \frac{\lambda_i}{\mu_i} \right)^q \right)^{-1} \right].$$

See Appendix A for the proofs of Lemma 2.3 and Corollary 2.4. Notice that as  $C$  approaches infinity (i.e., the seller has unbounded inventory capacity), the lower and upper bounds on  $\mathbb{P}_C [A_i \geq 1]$  approach  $1 - \exp(-\lambda_i/(\mu_i + \gamma^*))$  and  $1 - \exp(-\lambda_i/\mu_i)$ , respectively.

Finally, we recall the following fundamental PASTA property, due to Wolff [29].

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<sup>1</sup>The algorithm of [9] is Algorithm 1 with  $\alpha = 1$ , applied to  $\mathbf{x}^*$  an optimal solution to  $\text{RB}_{\text{off}}$ , with  $w_i = \min\{1, \lambda_i/\mu_i\}$ . The analysis of [9] implies that this algorithm is  $1/3$ -competitive in the single-good setting, while we show that this algorithm is in fact  $(1 - 1/(e-1)) \approx 0.418$ -competitive algorithm (see Lemma B.4).

**Lemma 2.5** (PASTA [29]). *The fraction of Poisson arrivals who observe a stochastic process in a state is equal to the fraction of time the stochastic process is in this state, provided that the Poisson arrivals and the history of the stochastic process are independent.*

In our analysis of the multi-good problem we will need to prove stochastic dominance between two processes, for which the following lemma will prove useful. For this lemma, we recall that a set  $S \subseteq \mathcal{Y} \subseteq \mathbb{R}^n$  is or upward closed if for every  $y \geq \tilde{y}$  with  $\tilde{y} \in S$  and  $y \in \mathcal{Y}$ , we have that  $y \in S$ .

**Lemma 2.6** ([6]). *Let  $Y, \tilde{Y}$  be two stochastic processes taking values in  $\mathcal{Y} \subseteq \mathbb{R}^n$ , with time-homogeneous intensity matrices  $Q, \tilde{Q}$ . Then,  $Y$  stochastically dominates  $\tilde{Y}$  ( $\Pr[Y \geq y] \geq \Pr[\tilde{Y} \geq y]$  for all  $y \in \mathcal{Y}$ ) if and only if the following holds: for every  $y, \tilde{y} \in \mathcal{Y}$  and upward closed set  $S \subseteq \mathcal{Y}$ , if  $y \geq \tilde{y}$ , and either  $y, \tilde{y} \in S$  or  $y, \tilde{y} \notin S$ , then*

$$\sum_{z \in S} Q(y, z) \geq \sum_{z \in S} \tilde{Q}(\tilde{y}, z).$$

### 3 Tighter LP benchmarks

In this section, we present our tighter LP benchmarks for the optimal offline and online policies. We start by noting that tighter benchmarks are needed than those considered in [3, 9] in order to improve on prior approximations. We then propose a new constraint which in fact tightens these benchmarks and enables us to obtain improved bounds in Section 4.

**Limitations of prior LP benchmarks.** The algorithms of Collina et al. [9] and Aouad and Saritaç [3] yield  $(1 - 1/(e-1))$ -competitive and  $(1 - 1/e)$ -approximate online policies in the single-good problem by relying on LP benchmarks  $\text{RB}_{\text{off}}$  and  $\text{RB}_{\text{on}}$ . Unfortunately, improving on the bounds implied by [3, 9] for the stationary prophet inequality problem requires tighter benchmarks, since these bounds are tight versus these LP benchmarks. (See proofs in Appendix C.)

**Observation 3.1.** *There exist instances  $\mathcal{I}$  of the single-good stationary prophet inequality problem for which  $\text{OPT}_{\text{off}}(\mathcal{I}) \leq (1 - 1/(e-1)) \cdot \text{RB}_{\text{off}}(\mathcal{I})$ .*

**Observation 3.2.** *There exist instances  $\mathcal{I}$  of the single-good stationary prophet inequality problem for which  $\text{OPT}_{\text{on}}(\mathcal{I}) \leq (1 - 1/e) \cdot \text{RB}_{\text{on}}(\mathcal{I})$ .*

#### 3.1 A new constraint via the PASTA property & new LP benchmarks

We introduce an additional constraint previously overlooked in the literature, which follows from the PASTA property, Lemma 2.5.<sup>2</sup>

**Lemma 3.3.** *Let  $x_{ij}$  be the rate at which an offline (or online) policy sells items of good  $i \in \mathcal{G}$  to buyers of type  $j \in \mathcal{B}$ . Then  $x_{ij}$  satisfies the following constraint:*

$$x_{ij} \leq \gamma_j \cdot (1 - \exp(-\lambda_i/\mu_i)). \tag{6}$$

*Proof.* The rate at which a policy sells items of good  $i$  to buyers of type  $j$  is trivially upper bounded by the rate at which such buyers arrive and inspect at least one present item. The bound therefore follows from the PASTA property (Lemma 2.5), together with the upper bound of Corollary 2.4.  $\square$

<sup>2</sup>In the third and latest arxiv version of [9] (uploaded January, 2021), the authors of that paper observe this constraint too, though they do not make use of it, instead replacing it by the potentially looser Constraint (3).

Combining Lemmas 2.1 and 3.3 yields the following tighter bounds on  $\text{OPT}_{\text{off}}(\mathcal{I})$  and  $\text{OPT}_{\text{on}}(\mathcal{I})$ .

**Corollary 3.4.** *For all instances  $\mathcal{I}$  of the stationary prophet inequality problem,*

1.  $\text{LP}_{\text{off}}(\mathcal{I}) \triangleq \max \left\{ \sum_{i,j} v_{ij} \cdot x_{ij} \mid \mathbf{x} \text{ satisfies (1) - (4), (6)} \right\}$  satisfies  $\text{LP}_{\text{off}}(\mathcal{I}) \geq \text{OPT}_{\text{off}}(\mathcal{I})$ , and
2.  $\text{LP}_{\text{on}}(\mathcal{I}) \triangleq \max \left\{ \sum_{i,j} v_{ij} \cdot x_{ij} \mid \mathbf{x} \text{ satisfies (1) - (6)} \right\}$  satisfies  $\text{LP}_{\text{on}}(\mathcal{I}) \geq \text{OPT}_{\text{on}}(\mathcal{I})$ .

We note that Constraint (6) subsumes Constraint (3) since  $1 - \exp(-z) \leq z$  for all  $z \in \mathbb{R}$ . As we show in Section 4, this tighter constraint and the derived tighter LP bounds on  $\text{OPT}_{\text{off}}(\mathcal{I})$  and  $\text{OPT}_{\text{on}}(\mathcal{I})$  allow for better approximations of the optimal offline and online policies for the single-good problem and a better competitive ratio in the multi-good problem.

## 4 Improved approximation ratios

In this section we present the proofs of our algorithmic results. In particular, we show that using the solutions to our tighter LP benchmarks  $\text{LP}_{\text{off/on}}$  and an appropriate choice of  $\alpha$  and  $w_i$  for all  $i \in \mathcal{G}$  in Algorithm 1, we achieve all of our improved approximation guarantees.

### 4.1 Single-good problem

When the seller has only one good  $i$  for sale, we drop the subscript  $i$  for notational convenience, using, e.g.,  $\lambda$  and  $\mu$  as shorthand for  $\lambda_i$  and  $\mu_i$ , and using  $v_j$  and  $x_j$  as shorthand for  $v_{ij}$  and  $x_{ij}$ . We assume, without loss of generality (due to rate re-scaling) that  $\mu = 1$  and buyer types are sorted such that  $v_1 > v_2 > \dots > v_m$ .

We first note that for the single-good problem, Algorithm 1 with  $\alpha = 1$  (as we will use it) yields a *posted-price* policy.

**Observation 4.1.** *For any instance  $\mathcal{I}$  of the single-good stationary prophet inequality problem, Algorithm 1 with  $\alpha = 1$  is a posted-price policy if*

1.  $\mathbf{x}^* \triangleq \arg \max_{\mathbf{x}} \text{LP}_{\text{off}}(\mathcal{I})$  and  $w \triangleq 1 - \exp(-\lambda)$ , or
2.  $\mathbf{x}^* \triangleq \arg \max_{\mathbf{x}} \text{LP}_{\text{on}}(\mathcal{I})$  and  $w \triangleq \min \left\{ 1 - \exp(-\lambda), \lambda - \sum_{j \in \mathcal{B}} x_j^* \right\}$ .

*Proof.* By constraints (4), (5) and (6), we have that  $0 \leq x_j^* \leq \gamma_j \cdot w$ , and so these choices of  $\mathbf{x}^*$  and  $w$  guarantee that  $p_j$  is a valid probability for all  $j \in \mathcal{B}$ . Furthermore, by local exchange arguments and the strict inequalities  $v_1 > v_2 > \dots > v_m$ , combined with  $x_j^* \leq \gamma_j \cdot w$ , an optimal solution  $\mathbf{x}^*$  to  $\text{LP}_{\text{off}}$  for instance  $\mathcal{I}$  is in some sense greedy, and has the following form for some  $\ell \in [m]$ :

$$x_j^* = \begin{cases} \gamma_j \cdot w & \text{for } j \leq \ell \\ p_{\ell+1} \cdot \gamma_{\ell+1} \cdot w & \text{for } j = \ell + 1 \text{ and some } p_{\ell+1} \in [0, 1] \\ 0 & \text{for } j > \ell + 1. \end{cases}$$

Therefore, Algorithm 1 with  $\alpha = 1$  and the above choices of  $\mathbf{x}^*$  and  $w$  is a posted-price policy characterized by  $(v_{\ell+1}, p_{\ell+1})$ .  $\square$

**The sale rate.** By definition of Algorithm 1, it is clear that an item of the single good  $i$  is sold to a buyer of type  $j$  if at least one item of good  $i$  is available and the probabilistic test in Line 4 passes (which is independent of whether or not an item is available). We say a buyer who passes this probabilistic test of Line 4 has his bid *permitted* by the seller. By standard Poisson splitting, buyers of type  $j$  whose bid is permitted arrive at rate  $\gamma_j \cdot p_j = x_j^*/w$ , and by Poisson merging, the arrival rate of buyers (of any type) whose bid the seller permits is  $\gamma^* \triangleq \sum_{j \in \mathcal{B}} \gamma_j \cdot p_j = \sum_{j \in \mathcal{B}} x_j^*/w$ . From the above, by the PASTA property (Lemma 2.5), we have the following expression for the selling rate to buyers of type  $j$  for our policy with capacity  $C$ , which we denote by  $s_j^C$ :

$$s_j^C = \gamma_j \cdot p_j \cdot \mathbb{P}_C[A \geq 1] = \frac{\mathbb{P}_C[A \geq 1]}{w} \cdot x_j^*, \quad (7)$$

where  $A$  is the number of available items on arrival of the buyer. In what follows, we will obtain our algorithmic results of Theorems 1.1 and 1.2 by lower bounding  $\mathbb{P}_C[A \geq 1]/w$ , from which our approximation ratios follow by linearity of expectation.

#### 4.1.1 Proof of Theorem 1.1 (algorithmic result)

In this section we prove the algorithmic result of Theorem 1.1, restated below.

**Lemma 4.2.** *There exists a  $1/2$ -competitive posted-price policy with capacity  $C = 2$  for the single-good stationary prophet inequality problem.*

*Proof.* Fix an instance  $\mathcal{I}$  of the single-good stationary prophet inequality problem where the seller has inventory capacity  $C = 2$ . Consider Algorithm 1 where  $\mathbf{x}^* = \arg \max_{\mathbf{x}} \text{LP}_{\text{off}}(\mathcal{I})$ ,  $w \triangleq 1 - \exp(-\lambda)$ , and  $\alpha = 1$ , which is indeed a posted-price policy by Observation 4.1. By definition of  $w$  and constraint (1), we have that

$$\gamma^* = \sum_{j \in \mathcal{B}} p_j \cdot \gamma_j = \frac{\sum_{j \in \mathcal{B}} x_j^*}{1 - \exp(-\lambda)} \leq \frac{\lambda}{1 - \exp(-\lambda)}. \quad (8)$$

By Lemma 2.3 and Equation (8), we therefore have that

$$\frac{\mathbb{P}_2[A \geq 1]}{w} = \frac{1 - \left(1 + \sum_{q=1}^2 \prod_{r=1}^q \frac{\lambda}{r + \gamma^*}\right)^{-1}}{1 - \exp(-\lambda)} \geq \frac{1 - \left(1 + \sum_{q=1}^2 \prod_{r=1}^q \frac{\lambda}{r + \frac{\lambda}{1 - \exp(-\lambda)}}\right)^{-1}}{1 - \exp(-\lambda)} \geq \frac{1}{2}, \quad (9)$$

where the last inequality holds for all  $\lambda \geq 0$ ; this inequality is easily verified to hold (with equality) for  $\lambda \rightarrow 0^+$ , while the left-hand side of the inequality can be shown (with some effort) to be monotone increasing in  $\lambda \geq 0$ , and therefore this inequality holds for all  $\lambda \geq 0$ . We defer a formal proof of this last inequality, restated in the following claim, to Appendix D.

**Claim 4.3.** *For all  $x \in \mathbb{R}_{\geq 0}$ , we have that  $\frac{1 - \left(1 + \sum_{q=1}^2 \prod_{r=1}^q \frac{x}{r + \frac{x}{1 - \exp(-x)}}\right)^{-1}}{1 - \exp(-x)} \geq \frac{1}{2}$ .*

Combining Equations (7) and (9) yields  $s_j^2 \geq 1/2 \cdot x_j^*$  for all buyer types  $j \in \mathcal{B}$ . We conclude that, by linearity of expectation and Corollary 3.4, the expected average revenue of Algorithm 1 for  $\mathcal{I}$  is at least half that of the optimal offline policy in expectation.  $\square$



Note that in our proof of Lemma 4.2 we used a policy with capacity  $C = 2$ . Using the same proof strategy, the same policy with further restricted capacity of  $C = 1$  can be shown to yield a competitive ratio of  $\approx 0.435$  (see also Corollary B.3 from Appendix B). On the other hand, monotonicity of  $\mathbb{P}_C[A \geq 1]$  as a function of  $C$  implies that Equation (9) holds for any capacity  $C \geq 2$ . Intuitively, such higher capacity should allow for strictly higher competitive ratio. However, this turns out to not be the case; in Section 5 we show that no online policy, regardless of inventory capacity and even not restricted to posted prices, is better than  $1/2$ -competitive, and so this capacity-two posted-price policy is optimally competitive among all online policies.

#### 4.1.2 Proof of Theorem 1.2

In this section we prove our results for approximating  $\text{OPT}_{\text{on}}(\mathcal{I})$  for the single-good problem, restated below.

**Theorem 1.2.** *There exists a posted-price policy for the single-good stationary prophet inequality problem which is a 0.656-approximation of the optimal online policy in expectation.*

*Proof.* Fix an instance  $\mathcal{I}$  of the single-good stationary prophet inequality problem where the seller has inventory capacity  $C \in \mathbb{Z}_{>0}$ . Consider Algorithm 1 where  $\mathbf{x}^* = \arg \max_{\mathbf{x}} \text{LP}_{\text{on}}(\mathcal{I})$ ,  $\alpha = 1$ , and  $w \triangleq \min \left\{ 1 - \exp(-\lambda), \lambda - \sum_{j \in \mathcal{B}} x_j^* \right\}$ , which is a posted-price policy by Observation 4.1. By Equation (7) and the linearity of expectation, our policy's approximation ratio is at least  $\mathbb{P}_C[A \geq 1]/w$ , where, as before,  $A$  is the number of available items. We therefore turn to lower bounding  $\mathbb{P}_C[A \geq 1]/w$ .

For this, we will invoke our lower bound on  $\mathbb{P}_C[A \geq 1]$  of Corollary 2.4, which requires upper bounds on  $\gamma^*$ —the rate at which buyers arrive and pass the probabilistic check in Line 4. By definition of  $w$ , we have the following bound on  $\gamma^*$ :

$$1 + \gamma^* = \frac{w + \sum_{j \in \mathcal{B}} x_j^*}{w} \leq \frac{\lambda - \sum_{j \in \mathcal{B}} x_j^* + \sum_{j \in \mathcal{B}} x_j^*}{w} = \frac{\lambda}{w}. \quad (10)$$

**Warm-up  $1 - 1/e$  bound:** Using the above bound on  $1 + \gamma^*$  and appealing to the lower bound of Corollary 2.4, we find that our policy with  $C = \infty$  has approximation ratio at least  $1 - 1/e$ , since for any  $w \in [0, 1]$  (as is the case for our  $w \leq 1 - \exp(-\lambda) \leq 1$ ), we have that

$$\frac{\mathbb{P}_{\infty}[A \geq 1]}{w} \geq \frac{1 - \left( \sum_{q=0}^{\infty} \frac{1}{q!} \left( \frac{\lambda}{1 + \gamma^*} \right)^q \right)^{-1}}{w} \geq \frac{1 - \left( \sum_{q=0}^{\infty} \frac{w^q}{q!} \right)^{-1}}{w} = \frac{1 - \exp(-w)}{w} \geq 1 - 1/e.$$

In order to improve on the above natural bound, we will rely on the following two claims.

**Claim 4.4.** *If  $w \leq 1 - \exp(-12/5)$ , then, for  $g_1(C, x) \triangleq \frac{1 - \left( \sum_{q=0}^C \frac{x^q}{q!} \right)^{-1}}{x}$ , we have that*

$$\frac{\mathbb{P}_C[A \geq 1]}{w} \geq g_1(C, 1 - \exp(-12/5)).$$

*Proof.* By Corollary 2.4 and Equation (10), we have  $\frac{\mathbb{P}_C[A \geq 1]}{w} \geq \frac{1 - \left( \sum_{q=0}^C \frac{w^q}{q!} \right)^{-1}}{w} = g_1(C, w)$ . The claim then follows since  $g_1(C, w)$  is monotone decreasing as a function of  $w$  (see Fact D.1).  $\square$

The more intricate claim is the following bound on  $\mathbb{P}_C[A \geq 1]/w$  for  $w$  close to 1.

**Claim 4.5.** *If  $w > 1 - \exp(-12/5)$ , then, for  $g_2(C, x) \triangleq \frac{1 - \left(\frac{1}{6} + \frac{5}{6} \cdot \sum_{q=0}^C \frac{1}{q!} \left(\frac{6 \cdot x}{5}\right)^q\right)^{-1}}{x}$ , we have that*

$$\frac{\mathbb{P}_C[A \geq 1]}{w} \geq g_2(C, 1).$$

*Proof.* Observe that the lower bound on  $\mathbb{P}_C[A \geq 1]$  based on Corollary 2.4 that we used in Claim 4.4 follows by noting that  $r + \gamma^* \leq r \cdot (1 + \gamma^*)$  for all  $r \geq 1$ , which is lossy for large  $r$ . In particular, for  $r \geq 2$  we have

$$r - 1 \leq 2 \cdot r - \frac{12}{5} = \frac{12}{5} \cdot \left(\frac{5}{6} \cdot r - 1\right) \leq \frac{\lambda}{w} \cdot \left(\frac{5}{6} \cdot r - 1\right),$$

where the second inequality follows from the fact that  $w \geq 1 - \exp(-12/5)$  implies  $\lambda \geq 12/5$  and the fact that  $w \leq 1 - \exp(-\lambda) \leq 1$ . This, combined with Equation (10) yields

$$r + \gamma^* = r - 1 + 1 + \gamma^* \leq \frac{\lambda}{w} \cdot \left(\frac{5}{6} \cdot r - 1\right) + \frac{\lambda}{w} = \frac{5}{6} \cdot r \cdot \frac{\lambda}{w}.$$

Rearranging terms, we thus have that for all  $r \geq 2$ ,

$$\frac{\lambda}{r + \gamma^*} \geq \frac{6}{5} \cdot \frac{w}{r}, \quad (11)$$

This combined with Lemma 2.3 and Equation (8) yields

$$\frac{\mathbb{P}_C[A \geq 1]}{w} = \frac{1 - \left(1 + \sum_{q=1}^C \prod_{r=1}^q \frac{\lambda}{r + \gamma^*}\right)^{-1}}{1 - \exp(-\lambda)} \geq \frac{1 - \left(\frac{1}{6} + \frac{5}{6} \cdot \sum_{q=0}^C \frac{1}{q!} \left(\frac{6 \cdot w}{5}\right)^q\right)^{-1}}{w} = g_2(C, w).$$

The claim then follows since  $g_2(C, w)$  is monotone decreasing in  $w$  (see Fact D.2).  $\square$

Combining Equation (7) and Claims 4.4 and 4.5, we obtain the following:

$$s_j^C \geq \min\{g_1(C, 1 - \exp(-12/5)), g_2(C, 1)\} \cdot x_j^*. \quad (12)$$

The above bound is at least 0.656 for all  $C \geq 5$ . By linearity of expectation, we conclude that our policy's average gain is at least a 0.656 fraction of the average gain of the optimal online policy.  $\square$

**A small (but large enough) inventory suffices.** Our proof of Theorem 1.2 shows that a capacity of  $C = 5$  is sufficient to obtain a 0.656 approximation of  $\text{OPT}_{\text{on}}(\mathcal{I})$ . Opening up this proof and evaluating Equation (12) for different  $C$ , we obtain a number of bounds for different inventory capacity  $C$ ; e.g., a capacity of  $C = 3$  is sufficient to break the natural barrier of  $1 - 1/e$  for this problem, and a capacity of  $C = 1$  yields a  $1/2$  approximation. In contrast, we also prove in Lemma 4.6 (proof deferred to Appendix D) that sufficiently small inventory does harm the approximation ratio; so, for example, our bound for  $C = 1$  is optimal. See Table 1 for comparison of upper and lower bounds in terms of  $C$ .

**Lemma 4.6.** *For any  $C \in \mathbb{Z}_{>0}$ , no online policy with inventory capacity  $C$  is greater than a  $C/(C+1)$ -approximation of the optimal online (unbounded inventory) policy.*

	Inventory size $C$				
	1	2	3	4	5
Lower bound (algorithm)	1/2	0.615	0.647	0.655	0.656
Upper bound (impossibility)	1/2	2/3	3/4	4/5	5/6

Table 1: Lower and upper bounds on the approximation factor of the policy from Theorem 1.2 relative to the optimal online policy when the seller has inventory capacity  $C$ .

## 4.2 Multi-good problem

In this section we analyze Algorithm 1 applied to a multi-good instance.

We start by introducing the following terminology used in this section’s analysis of Algorithm 1. We say that a buyer of type  $j \in \mathcal{B}$  *reaches* good  $i \in \mathcal{G}$  if the buyer has not yet been sold an item when he considers good  $i$  in Line 2 of Algorithm 1, and we denote this event by  $R_{ij}$ . We denote the number of available items of good  $i$  by  $A_i$ . Lastly, we say that the seller *permits* the sale of an item of good  $i$  to a buyer of type  $j$  if the probabilistic check on Line 4 would pass, without regard for whether or not the buyer has already been sold an item of another good or the availability of the good. We denote this event by  $P_{ij} \sim \text{Ber}(p_{ij})$ . Importantly, the seller permits a sale independently of the availability of any good.

By definition of Algorithm 1, it is clear that an item of good  $i$  is sold to a buyer of type  $j$  if the buyer reaches good  $i$ , at least one item of good  $i$  is available, and the seller permits the sale of good  $i$  to the buyer. Suppose the seller has inventory capacity  $C \in \mathbb{Z}_{>0}$ , and let  $s_{ij}^C$  denote the expected rate at which items of good  $i$  are sold to buyers of type  $j \in \mathcal{B}$  under Algorithm 1. By the PASTA property (Lemma 2.5), we have the following:

$$s_{ij}^C = \gamma_j \cdot \mathbb{P}_C [R_{ij} \wedge A_i \geq 1 \wedge P_{ij}] = \alpha \cdot \frac{\mathbb{P}_C [R_{ij} \wedge A_i \geq 1]}{w_i} \cdot x_{ij}^*. \quad (13)$$

Inspecting Equation (13), it is clear that lower bounding  $s_{ij}^C$  is more challenging in the multi-good problem than in the single good, as the event that a buyer of type  $j$  reaches a good  $i \in \mathcal{G}$  and the availability of good  $i$  are correlated via the availability of other goods. Collina et al. [9] approached this problem by considering a number of Poisson processes which they showed to either stochastically dominate or be dominated by the processes of interest, and further, by arguing about the correlations between them. Although we similarly introduce a stochastic dominance relation, our analysis also relies heavily on the same queuing theory techniques employed in the single-good problem for analyzing stationary distributions of CTMCs.

To lower bound  $\mathbb{P}_C [R_{ij} \wedge A_i \geq 1]$ , we introduce some additional notation. We denote by  $\tilde{R}_{ij}$  the event that a buyer of type  $j$  is not permitted by the seller to buy a good  $i' \in \mathcal{G}$  that is present (even if unavailable) and precedes good  $i$  in the random ordering from Line 2. Although similar to  $R_{ij}$ , observe that  $\tilde{R}_{ij}$  does not depend on the availability of any good. We also let  $\tilde{A}_i$  denote the number of items of good  $i$  available when  $i$  comes first in the ordering from Line 2. We bound the probability that a buyer of type  $j$  reaches good  $i$  and the good is available in terms of  $\tilde{R}_{ij}$  and  $\tilde{A}_i$  in the following claim.

**Lemma 4.7.** *For any good  $i \in \mathcal{G}$  and buyer type  $j \in \mathcal{B}$ ,*

$$\mathbb{P}_C [R_{ij} \wedge A_i \geq 1] \geq \mathbb{P}_C [\tilde{R}_{ij}] \cdot \mathbb{P}_C [\tilde{A}_i \geq 1]. \quad (14)$$

See Appendix D.2 for the proof, which proceeds by introducing a new stochastic process which relaxes the constraints of the multi-good problem so that, in some sense, each good (simultaneously)

comes first in the random ordering. For each good  $i \in \mathcal{G}$ , this only increases the probability that an item of good  $i$  is sold to an arriving buyer, thus creating more “downwards pressure” on the number of items available. It is intuitive, therefore, that the dynamics under Algorithm 1 stochastically dominate those of this new process, and as a result, we can lower bound the probability that a buyer reaches a good and there is an item of that good available under Algorithm 1 by considering the probability of these same events under the dominated stochastic process. By construction, these events, which we relate to  $\tilde{R}_{ij}$  and  $\tilde{A}_i \geq 1$ , are independent under this new process, and we can therefore lower bound each separately.

In particular, lower bounding  $\mathbb{P}_C [\tilde{R}_{ij}]$ , as in the following lemma (see Appendix D.2 for proof), is straightforward as it depends only on the presence, not availability, of goods, which we can characterize exactly from the upper bound of Corollary 2.4.

**Lemma 4.8** ([9]). *For any good  $i \in \mathcal{G}$  and buyer type  $j \in \mathcal{B}$ , we have that*

$$\mathbb{P}_C [\tilde{R}_{ij}] \geq 1 - \frac{\alpha}{2}.$$

Combining Lemmas 4.7 and 4.8, we have the following theorem regarding the competitive ratio of Algorithm 1 in the multi-good problem.

**Theorem 1.3.** *There exists a  $15/56$ -competitive policy for the multi-good problem.*

*Proof.* Fix an instance  $\mathcal{I}$  of the multi-good stationary prophet inequality problem where the seller has inventory capacity  $C = 2$  and consider Algorithm 1 where  $\mathbf{x}^* = \arg \max_{\mathbf{x}} \text{LP}_{\text{off}}(\mathcal{I})$  and  $w_i \triangleq 1 - \exp(-\lambda_i/\mu_i)$  for all  $i \in \mathcal{G}$ . For any good  $i \in \mathcal{G}$  and buyer type  $j \in \mathcal{B}$ , it follows from Equation (13) and Lemma 4.7 that the expected rate at which items of good  $i$  are sold to buyers of type  $j$  under Algorithm 1 is at least

$$s_{ij}^2 \geq \alpha \cdot \left(1 - \frac{\alpha}{2}\right) \cdot \frac{\mathbb{P}_2 [\tilde{A}_i \geq 1]}{w_i} \cdot x_{ij}^* \geq \alpha \cdot \left(1 - \frac{\alpha}{2}\right) \cdot \frac{1 - \left(1 + \sum_{q=1}^2 \prod_{r=1}^q \frac{\lambda_i/\mu_i}{r + \alpha \cdot \frac{\lambda_i/\mu_i}{1 - \exp(-\lambda_i/\mu_i)}}\right)^{-1}}{1 - \exp(-\lambda_i/\mu_i)} \cdot x_{ij}^*,$$

where the second inequality holds due to Equation (9), as the probability that an item of good  $i$  is available when  $i$  comes first in the ordering (i.e.,  $\tilde{A}_i \geq 1$ ) is exactly the probability that an item of good  $i$  is available in the single-good problem. For  $\alpha = 3/4$ , we have that

$$s_{ij}^2 \geq \frac{3}{4} \cdot \left(1 - \frac{3}{8}\right) \cdot \frac{4}{7} \cdot x_{ij}^* = \frac{15}{56} \cdot x_{ij}^*.$$

(The proof of the  $4/7$  lower bound is analogous to that of Claim 4.3, and is omitted.) It follows from linearity of expectation and Corollary 3.4 that the expected average reward of Algorithm 1 for  $\mathcal{I}$  is at least  $15/56$  that of the optimal offline policy in expectation.  $\square$

**Tiny inventory results.** Similar to the single-good setting, a direct implication of the proof of Theorem 1.3 is that inventory capacity of 2 or greater is sufficient to guarantee that Algorithm 1 (with the specifications mentioned) is a  $15/56$ -competitive policy.

## 5 Proof of Theorem 1.1 (impossibility result)

In this section we prove that our single-good online policy’s competitive ratio of  $1/2$  is optimal among all online policies. Our proof of this hardness result of Theorem 1.1 follows from what can be viewed as the adaptation of the oft-cited example for the tightness of the  $1/2$  competitive ratio for the classic prophet inequality problem.<sup>3</sup>

*Proof of Theorem 1.1 (impossibility result).* We consider the following simple instance which we denote by  $\mathcal{I}_\varepsilon$ . The supply rate of the seller’s good is  $\lambda = \varepsilon$  and the perish rate is  $\mu = 1$ . There are two buyer types: the rare “big spender” who arrives with rate  $\gamma_1 = \varepsilon$  and bids  $v_1 = 1 + 1/\varepsilon$ , and the common “miser” who arrives with rate  $\gamma_2 = \infty$  and bids  $v_2 = 1$ . One can trivially achieve expected average revenue  $\varepsilon$  for instance  $\mathcal{I}_\varepsilon$  online: simply sell each item to the common buyer which arrives immediately after it. On the other hand, for small  $\varepsilon$ , an expected average revenue of roughly  $2 \cdot \varepsilon$  is achievable offline, as we now show.

**Claim 5.1.**  $\text{OPT}_{\text{off}}(\mathcal{I}_\varepsilon) \geq \varepsilon + 1 - \exp(-\varepsilon/(1+\varepsilon))$ .

*Proof.* We consider the offline policy which always permits a sale to a rare buyer (provided at least one item is available) and permits a sale to a common buyer only at the moment before an item perishes. Note that this is indeed a valid offline policy.

By the PASTA property (Lemma 2.5), letting  $A$  denote the number of items available upon arrival of a buyer, the rate at which items of the good are sold to the rare buyer is  $\varepsilon \cdot \mathbb{P}[A \geq 1]$ . Since each item that is not sold to a rare buyer is sold to a common buyer who arrives just before the item perishes, the rate of selling to common buyers is the residual rate:  $\varepsilon - \varepsilon \cdot \mathbb{P}[A \geq 1]$ . Therefore, the expected average revenue of this offline policy is

$$\left(1 + \frac{1}{\varepsilon}\right) \cdot \varepsilon \cdot \mathbb{P}[A \geq 1] + 1 \cdot (\varepsilon - \varepsilon \cdot \mathbb{P}[A \geq 1]) = \varepsilon + \mathbb{P}[A \geq 1].$$

Lower bounding  $\mathbb{P}[A \geq 1]$  as in Corollary 2.4, we have that the expected average revenue of this offline policy (and consequently of the optimal offline policy) is at least  $\varepsilon + 1 - \exp(-\varepsilon/(1+\varepsilon))$ .  $\square$

Next, we show that the aforementioned trivial online policy which sells each item immediately to a common buyer and has expected average reward  $\varepsilon$ , is optimal among all online policies.

**Claim 5.2.**  $\text{OPT}_{\text{on}}(\mathcal{I}_\varepsilon) \leq \varepsilon$ .

*Proof.* By Corollary 3.4,  $\text{OPT}_{\text{on}}(\mathcal{I}_\varepsilon) \leq \text{LP}_{\text{on}}(\mathcal{I}_\varepsilon)$ . For instance  $\mathcal{I}_\varepsilon$ ,  $\text{LP}_{\text{on}}$  is

$$\begin{aligned} \max \quad & (1 + 1/\varepsilon) \cdot x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq \varepsilon \\ & 1/\varepsilon \cdot x_1 \leq \min\{1 - \exp(-\varepsilon), \varepsilon - (x_1 + x_2)\} \\ & x_1, x_2 \geq 0. \end{aligned}$$

Fix some optimal solution  $\{x_1^*, x_2^*\}$ . The value of this solution is

$$\left(1 + \frac{1}{\varepsilon}\right) \cdot x_1^* + x_2^* \leq x_1^* + x_2^* + \varepsilon - (x_1^* + x_2^*) = \varepsilon.$$

Therefore, the expected average revenue of the optimal online policy is at most  $\varepsilon$ .  $\square$

Combining Claims 5.1 and 5.2 and taking  $\varepsilon$  to zero, the theorem follows.  $\square$

<sup>3</sup>The example consists of two buyers, the first with bid 1, and the second with bid  $1/\varepsilon$  with probability  $\varepsilon$  and 0 otherwise, for  $\varepsilon \rightarrow 0$ . While the expected maximum is  $2 - \varepsilon$ , no online strategy has expectation greater than 1.

# APPENDIX

## A Omitted proofs of Section 2

In this section, we provide proofs of lemmas deferred from Section 2, starting with results following from the queuing theory literature.

**Lemma 2.3.** *For any online policy with inventory capacity  $C \in \mathbb{Z}_{>0}$  and which sells any available item of good  $i \in \mathcal{G}$  to buyers which arrive at rate  $\gamma^*$ , the stationary probability of an item of good  $i \in \mathcal{G}$  being available satisfies*

$$\mathbb{P}_C [A_i \geq 1] = 1 - \left( 1 + \sum_{q=1}^C \prod_{r=1}^q \frac{\lambda_i}{r \cdot \mu_i + \gamma^*} \right)^{-1}.$$

*Proof.* The number of items of good  $i$  available under an online policy which sells any available item to a buyer is captured by the CTMC in Figure 1. This is a birth-death chain on state space  $\{0, 1, \dots, C\}$  with transition rates  $\alpha_q = \lambda_i$  from state  $q - 1$  to state  $q$  and  $\beta_q = q \cdot \mu_i + \gamma^*$  from state  $q$  to state  $q - 1$ , for all  $1 \leq q \leq C$  as in Figure 1.

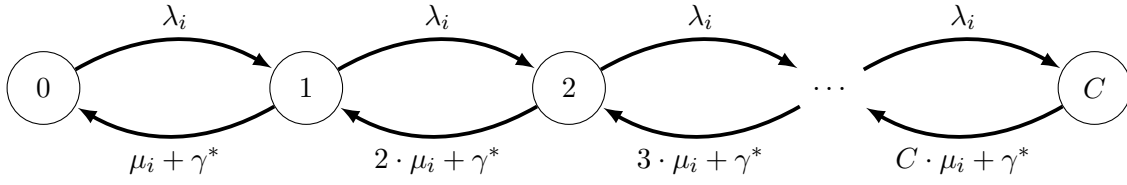


Figure 1: CTMC of the number of items of good  $i$  available under a policy selling available items to buyers which arrive at rate  $\gamma^*$  when the seller has inventory capacity  $C$ .

From [5, Section 3.1], the stationary probability of having no items available (i.e. being at state 0) is

$$\mathbb{P}_C [A_i = 0] = \left( 1 + \sum_{q=1}^C \prod_{r=1}^q \frac{\alpha_r}{\beta_r} \right)^{-1} = \left( 1 + \sum_{q=1}^C \prod_{r=1}^q \frac{\lambda_i}{r \cdot \mu_i + \gamma^*} \right)^{-1}. \quad \square$$

**Corollary 2.4.** *For any online policy with inventory capacity  $C \in \mathbb{Z}_{>0}$  and which sells any available item of good  $i \in \mathcal{G}$  to buyers which arrive at rate  $\gamma^*$ , the stationary probability of an item of good  $i \in \mathcal{G}$  being available satisfies*

$$\mathbb{P}_C [A_i \geq 1] \in \left[ 1 - \left( \sum_{q=0}^C \frac{1}{q!} \left( \frac{\lambda_i}{\mu_i + \gamma^*} \right)^q \right)^{-1}, 1 - \left( \sum_{q=0}^C \frac{1}{q!} \left( \frac{\lambda_i}{\mu_i} \right)^q \right)^{-1} \right].$$

*Proof.* The lower and upper bounds follow immediately from Lemma 2.3 and the fact that  $r \cdot \mu_i \leq r \cdot \mu_i + \gamma^* \leq r \cdot (\mu_i + \gamma^*)$  for all  $r \geq 1$ .  $\square$

## B Bounds implied by prior work

In [3], Aouad and Saritaç present an algorithm for their dynamic matching problem, which they prove yields a  $\frac{1}{4}(1 - 1/e)$ -approximation of the optimal online algorithm for their problem. Essentially,

their approach reduces their problem to the multi-good prophet inequality problem, for which they provide a  $(1 - 1/e)$ -approximation of the optimal online policy. We refer to [3] for more details. In Section 3 we show that this bound is inherent to their approach, since the LP benchmark they rely on cannot be used to prove a better than  $1 - 1/e$  approximation.

## B.1 Competitive policies

As stated before, the approach of Collina et al. [9], which gives a  $1/8$ -competitive algorithm for their problem, can be shown to have an improved bound for our problem, as the following lemma asserts.

**Lemma B.1.** *Via the proof strategy of Collina et al. [9], the algorithm from [9] for the single-good stationary prophet inequality problem can be shown to be a  $1/3$ -competitive posted-price policy.*

*Proof.* Fix an instance  $\mathcal{I}$  of the single-good stationary prophet inequality problem. Note that in the single-good problem, the algorithm from [9] reduces to Algorithm 1 with  $\alpha = 1$  and  $\mathbf{x}^* = \arg \max_{\mathbf{x}} \text{RB}_{\text{off}}(\mathcal{I})$  and  $w \triangleq \min\{1, \lambda\}$ . By a similar argument as in the proof of Observation 4.1, this definition of  $w$  guarantees that  $p_j$  is a valid probability for all  $j \in \mathcal{B}$ .

In order to lower bound the rate  $s_j$  at which buyers of type  $j$  are sold items of the good, the authors of [9] consider the following events:

- $E^1$ : An item of the good arrives.
- $E^2$ : A buyer of any type arrives and the seller permits a sale.
- $E^3$ : When there is exactly one item of the good available, this item perishes. If there are more/fewer than one item available,  $E_3$  follows an independent Poisson clock of rate 1.
- $E_j^4$ : A buyer of type  $j$  arrives and the seller permits a sale.

The authors of [9] note that a buyer of type  $j$  is sold an item of the good if when event  $E_j^4$  occurs, the most recent of events  $E^1, E^2, E^3$  to have occurred prior to  $E_j^4$  is  $E^1$ . If this is indeed the case, that means that when the buyer of type  $j$  arrives and the seller permits the sale, there is at least one item of the good available. Notice that these four events are independent Poisson processes and therefore the rate at which buyers of type  $j$  are sold items of the good is simply the rate at which event  $E_j^4$  occurs times the probability that  $E^1$  was the most recent of events  $E^1, E^2, E^3$ , which by the time reversibility of Poisson processes, is equal to the probability that  $E^1$  occurs before either  $E^2$  or  $E^3$ . Therefore, we have

$$s_j \geq \lambda_{E_j^4} \cdot \frac{\lambda_{E^1}}{\lambda_{E^1} + \lambda_{E^2} + \lambda_{E^3}},$$

where we let  $\lambda_E$  denote the rate at which event  $E$  occurs. Clearly,  $\lambda_{E^1} = \lambda$ ,  $\lambda_{E^3} = 1$ , and for any buyer type  $j \in \mathcal{B}$ ,  $\lambda_{E_j^4} = \gamma_j \cdot \alpha \cdot \frac{x_j^*}{\gamma_j \cdot w} = \frac{\alpha}{w} \cdot x_j^*$ . Also,

$$\lambda_{E^2} = \sum_{j \in \mathcal{B}} \gamma_j \cdot \alpha \cdot \frac{x_j^*}{\gamma_j \cdot w} = \alpha \sum_{j \in \mathcal{B}} \frac{x_j^*}{w} \leq \alpha \cdot \frac{\lambda}{w}.$$

As a result,

$$s_j \geq \frac{\alpha}{w} \cdot \frac{\lambda}{\lambda + \alpha \cdot \frac{\lambda}{w} + 1} \cdot x_j^* = \alpha \cdot \frac{\lambda}{\min\{1, \lambda\} \cdot (\lambda + 1) + \alpha \cdot \lambda} \cdot x_j^* \geq \frac{\alpha}{2 + \alpha} \cdot x_j^*, \quad (15)$$

where the last inequality follows from the following claim.

**Claim B.2.** For all  $x \in \mathbb{R}_{>0}$  and  $\alpha \in [0, 1]$ ,

$$\frac{x}{\min\{1, x\} \cdot (x + 1) + \alpha \cdot x} \geq \frac{1}{2 + \alpha}. \quad (16)$$

*Proof.* Let  $f(x)$  be the left hand side of Equation (16). First suppose  $\min\{1, x\} = 1$  (i.e.,  $x \geq 1$ ). Under this assumption,

$$f(x) = \frac{x}{x \cdot (1 + \alpha) + 1} = \frac{1}{1 + \alpha + \frac{1}{x}},$$

which is increasing in  $x$ . Therefore, the minimum is achieved at  $x = 1$ , in which case we have  $f(1) = 1/(2 + \alpha)$ . If instead  $\min\{1, x\} = x$  (i.e.,  $x \leq 1$ ),

$$f(x) = \frac{x}{x \cdot (x + 1) + \alpha \cdot x} = \frac{1}{x + 1 + \alpha}$$

which is decreasing in  $x$ . Therefore, the minimum is again achieved at  $x = 1$ , completing the claim.  $\square$

As  $\alpha/(2 + \alpha)$  is increasing in  $\alpha$ , this quantity is maximized for  $\alpha = 1$ , in which case we have  $s_j \geq 1/3 \cdot x_j^*$ . Since this holds for all buyer types  $j \in \mathcal{B}$ , by linearity of expectation and Corollary 2.2, the expected average reward of the algorithm from [9] for  $\mathcal{I}$  is at least a third that of the optimal offline policy in expectation. Furthermore, since  $\alpha = 1$ , an analogous proof of Observation 4.1 for  $\mathbf{x}^* = \arg \max_{\mathbf{x}} \text{RB}_{\text{off}}(\mathcal{I})$  and  $w \triangleq \min\{1, \lambda\}$  implies that this algorithm is a posted-price policy.  $\square$

A corollary of Lemma B.1 is that when using the solution to  $\text{LP}_{\text{off}}$  instead of  $\text{RB}_{\text{off}}$  (and the corresponding definition of  $w$ ), the techniques from [9] immediately yield a 0.435 competitive ratio, although as we show in Section 4, this is still not the best possible among all online policies.

**Corollary B.3.** Via the proof strategy of Collina et al. [9], the competitive ratio of the algorithm from [9] improves to 0.435 when  $\mathbf{x}^* = \text{LP}_{\text{off}}(\mathcal{I})$  and  $w \triangleq 1 - \exp(-\lambda)$ .

*Proof.* Fix an instance  $\mathcal{I}$  of the single-good stationary prophet inequality problem. Note that the algorithm from [9] with  $\mathbf{x}^* = \arg \max_{\mathbf{x}} \text{LP}_{\text{off}}(\mathcal{I})$  and  $w \triangleq 1 - \exp(-\lambda)$  is exactly Algorithm 1, and we can lower bound the rate at which items of the good are sold to buyers of type  $j \in \mathcal{B}$  as in Lemma B.1. From Equation (15) we have

$$s_j = \alpha \cdot \frac{\lambda}{(1 - \exp(-\lambda)) \cdot (\lambda + 1) + \alpha \cdot \lambda} \cdot x_j^*.$$

In order to show that the right-hand expression above is at least  $0.435 \cdot x_j^*$ , we hold  $\alpha$  fixed, let  $f(\alpha, \lambda) = \alpha \cdot \lambda / ((1 - \exp(-\lambda)) \cdot (\lambda + 1) + \alpha \cdot \lambda)$ , and consider the derivative of  $f$  with respect to  $\lambda$ :

$$\frac{\partial f(\alpha, \lambda)}{\partial \lambda} = \alpha \cdot \frac{e^\lambda (e^\lambda - 1 - \lambda - \lambda^2)}{(1 + \lambda - e^\lambda (1 + \lambda + \alpha \cdot \lambda))^2}.$$

The roots of  $\partial f(\alpha, \lambda)/\partial \lambda$  are simply the roots of  $e^\lambda - 1 - \lambda - \lambda^2$ , one of which is trivially  $\lambda_1^* = 0$ . In order to determine the other root(s), of which we show there is in fact only one, we simplify using the Taylor expansion of  $e^\lambda$ :

$$e^\lambda - 1 - \lambda - \lambda^2 = \sum_{k=2}^{\infty} \frac{\lambda^k}{k!} - \lambda^2 = \lambda^2 \cdot \left( \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{k!} - 1 \right) = \lambda^2 \cdot \left( \sum_{k=3}^{\infty} \frac{\lambda^{k-2}}{k!} - \frac{1}{2} \right) = \lambda^2 \cdot \left( \sum_{k=1}^{\infty} \frac{\lambda^k}{(k+2)!} - \frac{1}{2} \right).$$



Clearly any other root  $\lambda^* \neq 0$  satisfies

$$\sum_{k=1}^{\infty} \frac{(\lambda^*)^k}{(k+2)!} = \frac{1}{2},$$

and since the left-hand side is strictly increasing for  $\lambda^* \geq 0$ , there is exactly one value that satisfies this equality, which we denote by  $\lambda_2^*$ . Numerically, we find  $\lambda_2^* \approx 1.793$ .

Therefore for all  $\lambda > 0$ , we have  $f(\alpha, \lambda) \geq f(\alpha, \lambda_2^*)$ . The maximum of  $f(\alpha, \lambda_2^*)$  for  $\alpha \in [0, 1]$  is achieved at the right boundary, and we have  $f(1, \lambda_2^*) \geq 0.435$ . Consequently, for all buyer types  $j \in \mathcal{B}$ ,  $s_j \geq 0.435 \cdot x_j^*$ , and the competitive ratio follows.  $\square$

Although at first glance they may seem quite different, the techniques from [9] detailed in Lemma B.1 can actually be related to our proof of Theorem 1.1 by the stationary distribution of the CTMC that we analyze. Indeed, the lower bound on the probability that an item is available in the single-good problem implied by the analysis from [9] simply captures the dynamics of this CTMC when the seller's inventory capacity is constrained to 1. From Lemma 2.3, the probability that an item is available under the algorithm from [9] when  $C = 1$  is at least

$$1 - \left(1 + \frac{\lambda}{1 + \alpha \cdot \frac{\lambda}{w}}\right)^{-1} = 1 - \frac{1 + \alpha \cdot \frac{\lambda}{w}}{\lambda + 1 + \alpha \cdot \frac{\lambda}{w}} = \frac{\lambda}{\lambda + 1 + \alpha \cdot \frac{\lambda}{w}}.$$

This exactly matches the lower bound on the probability that event  $E^1$  occurred most recently of events  $E^1, E^2, E^3$  from Equation (15), which Collina et al. [9] use to bound the probability that one or more items are available. In contrast, by considering the dynamics of the CTMC with capacity of just 2 or greater, our analysis enables us to improve the competitive ratio of the algorithm from [9] from  $1/3$  to  $(1 - 1/(e-1)) \approx 0.418$ , as in the following lemma, and optimally to  $1/2$  in Theorem 1.1 when combined with our tighter benchmark.

**Lemma B.4.** *The algorithm from [9] for the single-good stationary prophet inequality problem is a  $(1 - 1/(e-1))$ -competitive posted-price policy.*

*Proof.* Fix an instance  $\mathcal{I}$  of the single-good stationary prophet inequality problem. Recall from Lemma B.1 that in the single-good stationary prophet inequality problem, the algorithm from [9] reduces to Algorithm 1 with  $\mathbf{x}^* = \arg \max_{\mathbf{x}} \text{RB}_{\text{off}}(\mathcal{I})$  and  $w \triangleq \min\{1, \lambda\}$ .

By Constraint 1, the arrival rate of buyers to whom the seller permits a sale is  $\gamma^* \leq \lambda/w = \lambda/\min\{1, \lambda\}$ , and therefore by Corollary 2.4 we have the following, for any buyer type  $j \in \mathcal{B}$ :

$$s_j = \frac{\mathbb{P}[A \geq 1]}{w} \cdot x_j^* \geq \frac{1 - \left(1 + \sum_{q=1}^{\infty} \prod_{r=1}^q \frac{\lambda}{r + \frac{\lambda}{\min\{1, \lambda\}}}\right)^{-1}}{\min\{1, \lambda\}} \cdot x_j^* \quad (17)$$

**Claim B.5.** *For all  $x \in \mathbb{R}_{>0}$ ,*

$$\frac{1 - \left(1 + \sum_{q=1}^{\infty} \prod_{r=1}^q \frac{x}{r + \frac{x}{\min\{1, x\}}}\right)^{-1}}{\min\{1, x\}} \geq 1 - \frac{1}{e-1}.$$

*Proof.* We consider two cases. First suppose  $\min\{1, x\} = 1$ . The claim reduces to proving that for all  $x \geq 1$ ,

$$1 + \sum_{q=1}^{\infty} \prod_{r=1}^q \frac{x}{r+x} \geq e-1.$$

By a simple inductive proof, the left-hand side of the above equation is increasing in  $x$  and therefore for  $x \geq 1$ ,

$$1 + \sum_{q=1}^{\infty} \prod_{r=1}^q \frac{x}{r+x} \geq 1 + \sum_{q=1}^{\infty} \prod_{r=1}^q \frac{1}{r+1} = \sum_{q=1}^{\infty} \frac{1}{q!} = e - 1.$$

Otherwise, if  $\min\{1, x\} = x$ , we must show that for all  $x \in [0, 1]$

$$\frac{1 - \left(1 + \sum_{q=1}^{\infty} \prod_{r=1}^q \frac{x}{r+1}\right)^{-1}}{x} \geq 1 - \frac{1}{e-1}.$$

Since  $1 + \sum_{q=1}^{\infty} \prod_{r=1}^q \frac{x}{r+1} = 1 + \frac{1}{x} \cdot (e^x - 1 - x) = \frac{1}{x} \cdot (e^x - 1)$ , this is equivalent to proving that

$$\frac{1}{x} - \frac{1}{e^x - 1} \geq 1 - \frac{1}{e-1}$$

for all  $x \in [0, 1]$ . We denote the left-hand side of the expression above by  $f(x)$  and take its derivative:

$$f'(x) = \frac{e^x}{(e^x - 1)^2} - \frac{1}{x^2}.$$

We show that  $f'$  is negative for all  $x \in [0, 1]$  and therefore the minimum of  $f$  on the interval  $[0, 1]$  is achieved at the left boundary. To this end, it suffices to show that  $x \cdot e^{x/2} \leq e^x - 1$  for all  $x \in [0, 1]$ . By the Taylor expansion of  $e^x$ ,

$$x \cdot e^{x/2} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{2^k \cdot k!} \leq \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} = \sum_{k=1}^{\infty} \frac{x^k}{k!} = e^x - 1,$$

where the inequality holds since  $2^k \geq k+1$  for all  $k \geq 1$ . Therefore,  $f(x) \geq f(1) = 1 - 1/(e-1)$  for all  $x \in [0, 1]$ .  $\square$

Equation (17) and Claim B.5 together yield  $s_j \geq (1 - 1/(e-1)) \cdot x_j^*$  for all buyer types  $j \in \mathcal{B}$ . The lemma follows from linearity of expectation and Corollary 2.2.  $\square$

## C Omitted proofs of Section 3

In this section we provide proofs on the gap between the reward benchmarks  $\text{RB}_{\text{off}}$  and  $\text{RB}_{\text{on}}$  compared to  $\text{OPT}_{\text{off}}$  and  $\text{OPT}_{\text{on}}$ . That is, we prove the following restated observations.

**Observation 3.1.** *There exist instances  $\mathcal{I}$  of the single-good stationary prophet inequality problem for which  $\text{OPT}_{\text{off}}(\mathcal{I}) \leq (1 - 1/(e-1)) \cdot \text{RB}_{\text{off}}(\mathcal{I})$ .*

**Observation 3.2.** *There exist instances  $\mathcal{I}$  of the single-good stationary prophet inequality problem for which  $\text{OPT}_{\text{on}}(\mathcal{I}) \leq (1 - 1/e) \cdot \text{RB}_{\text{on}}(\mathcal{I})$ .*

To prove the above observations, we consider instances of the stationary prophet inequality problem with a single good and a single buyer type. The convenience of such an instance  $\mathcal{I}$  is that the optimal policies are trivial: the seller sells an item to any buyer that arrives, provided at least one item is available, with the optimal offline policy selling to the earliest departing item.

*Proof of Observation 3.1.* Consider the following instance, which we denote by  $\mathcal{I}_1$ : the seller's good is supplied and perishes at rate  $\lambda = \mu = 1$ , and there is a single buyer type which arrives with rate  $\gamma = 1$  and bids  $v = 1$ . By the PASTA property [29], the rate at which items are sold is simply the rate at which buyers arrive and observe an available item. From the upper bound of Corollary 2.4, we have

$$\mathbb{P}[A \geq 1] \leq 1 - \left(1 + \sum_{q=1}^{\infty} \frac{1}{(q+1)!}\right)^{-1} = 1 - \frac{1}{e-1},$$

where  $A$  denotes the number of available items. We conclude that  $\text{OPT}_{\text{off}}(\mathcal{I}_1) \leq v \cdot \gamma \cdot \mathbb{P}[A \geq 1] = 1 - 1/(e-1)$ . On the other hand, we have that  $\text{RB}_{\text{off}}(\mathcal{I}_1) = \max\{x \mid x \leq 1, x \geq 0\} = 1$ .  $\square$

*Proof of Observation 3.2.* Consider the following instance, which we denote by  $\mathcal{I}_\lambda$ : the seller's good is supplied at rate  $\lambda \geq 2$  and perishes at rate  $\mu = \lambda - 1$ , and there is a single buyer type which arrives with rate  $\gamma = 1$  and bids  $v = 1$ . By Lemmas 2.3 and 2.5, the expected average revenue of this optimal online policy, which simply sells an item to a buyer whenever possible, is the rate at which buyers arrives, times the stationary probability that there is at least one item available:

$$\text{OPT}_{\text{on}}(\mathcal{I}) = v \cdot \gamma \cdot \mathbb{P}[A \geq 1] = \mathbb{P}[A \geq 1] = 1 - \left(1 + \sum_{q=1}^{\infty} \prod_{r=1}^q \frac{\lambda}{r \cdot (\lambda - 1) + 1}\right)^{-1}.$$

Therefore, we have that  $\lim_{\lambda \rightarrow \infty} \text{OPT}_{\text{on}}(\mathcal{I}_\lambda) = 1 - 1/e$ . On the other hand, we have that  $\text{RB}_{\text{on}}(\mathcal{I}_\lambda) = \max\{x \geq 0 \mid x \leq \min\{1, \lambda, \lambda/(\lambda - 1)\}\}$ , where we note that Equation (5) and Equation (3) both simplify to  $x \leq \lambda/(\lambda - 1)$  for the instance  $\mathcal{I}_\lambda$ . Therefore, for any value of  $\lambda \geq 1$ , we have that  $\text{RB}_{\text{on}}(\mathcal{I}_\lambda) \geq 1$ . Taking  $\lambda \rightarrow \infty$ , the observation follows.  $\square$

## D Omitted proofs of Section 4

In this section we provide proofs of claims and lemmas deferred from Section 4, starting with proofs concerning approximability of the single-good problem.

### D.1 Single-good problem

#### D.1.1 Approximating $\text{OPT}_{\text{off}}$

**Claim 4.3.** For all  $x \in \mathbb{R}_{\geq 0}$ , we have that  $\frac{1 - \left(1 + \sum_{q=1}^2 \prod_{r=1}^q \frac{x}{r + \frac{x}{1 - \exp(-x)}}\right)^{-1}}{1 - \exp(-x)} \geq \frac{1}{2}$ .

*Proof.* Denote the LHS of this inequality by  $f(x)$ . We note that  $\lim_{x \rightarrow 0^+} f(0) = \frac{1}{2}$ , and so we would like to prove that  $f(x)$  is monotone increasing in  $x$ . Taking the derivative of the LHS and simplifying it, we get the following (rather unwieldy) derivative.

$$f'(x) := \frac{\sum_{i=0}^3 g_i(x)}{e^{-x} \cdot (2 + 2x + x^2 + e^{2x} \cdot (2 + 5x + 3x^2) - e^x \cdot (4 + 7x + 3x^2))^2},$$

where, grouping by powers of  $e^x$ , these  $g_i(x)$  are defined as follows.

$$g_i(x) = \begin{cases} e^0 \cdot (-4 - 8x - 6x^2 - 4x^3 - x^4) & i = 0 \\ e^x \cdot (12 + 24x + 19x^2 + 12x^3 + 4x^4) & i = 1 \\ e^{2x} \cdot (-12 - 24x - 17x^2 - 9x^3 - 3x^4) & i = 2 \\ e^{3x} \cdot (4 + 8x + 4x^2) & i = 3. \end{cases}$$

Now, the denominator of the above form for  $f'(x)$  is easily seen to be positive for all  $x \geq 0$  (indeed, it is positive for all  $x \in \mathbb{R}$ , since it is the product of an exponential and a square). Therefore, to prove that  $f'(x) \geq 0$  for all  $x \geq 0$ , we need only prove that  $g(x) := \sum_{i=0}^3 g_i(x) \geq 0$  for all  $x \geq 0$ . Now,  $g(x)$  is the sum of products of analytic functions which are in particular equal to their Taylor expansions around zero. Therefore,  $g(x)$  is equal to its Taylor expansion around zero, and we can write it as

$$g(x) = \sum_{n=0}^{\infty} a_n \cdot x^n,$$

where, using the coefficients of the Taylor expansion of  $e^{k \cdot x} = \sum_{n=0}^{\infty} \frac{k^n}{n!} \cdot x^n$ , we have that

$$a_n = b_n - 4 \cdot \mathbf{1}[n = 0] - 8 \cdot \mathbf{1}[n = 1] - 6 \cdot \mathbf{1}[n = 2] - 4 \cdot \mathbf{1}[n = 3] - 1 \cdot \mathbf{1}[n = 4],$$

where

$$\begin{aligned} b_n &= \frac{12 - 12 \cdot 2^n + 4 \cdot 3^n}{n!} + \frac{24 - 24 \cdot 2^{n-1} + 8 \cdot 3^{n-1}}{(n-1)!} \\ &+ \frac{19 - 17 \cdot 2^{n-2} + 4 \cdot 3^{n-2}}{(n-2)!} + \frac{12 - 9 \cdot 2^{n-3}}{(n-3)!} + \frac{4 - 3 \cdot 2^{n-4}}{(n-4)!}. \end{aligned}$$

The above  $a_n$  are all non-negative. This can be proven by inspection for  $n \leq 45$ , while for  $n \geq 45$ , we have that  $a_n \geq \frac{4 \cdot 3^n}{n!} - \frac{(12+24+17+9+3) \cdot 2^n}{(n-4)!} \geq \frac{4 \cdot 3^n}{n!} - \frac{65 \cdot 2^n}{n!/n^4} \geq 0$ , where the last inequality is equivalent to  $(3/2)^n \geq \frac{65}{4} \cdot n^4$ , which holds for all  $n \geq 45$ . We conclude that, since all  $a_n$  are non-negative, we have that  $g(x) = \sum_{n=0}^{\infty} a_n \cdot x^n \geq 0$  for all  $x \geq 0$ . Recalling that this implies that  $f'(x) \geq 0$ , we find that, indeed,  $f(x) \geq \frac{1}{2}$  for all  $x \geq 0$ , as claimed.  $\square$

### D.1.2 Approximating $\text{OPT}_{\text{on}}$

In our proof of Theorem 1.2, we defined two auxiliary functions,  $g_1(C, x) \triangleq (1 - (\sum_{q=0}^C \frac{x^q}{q!}))^{-1}/x$  and  $g_2(C, x) \triangleq \frac{1 - (\frac{1}{6} + \frac{5}{6} \cdot \sum_{q=0}^C \frac{1}{q!} (\frac{6 \cdot w}{5})^q)^{-1}}{w}$ , which we claimed are monotone decreasing, as we now prove.

**Fact D.1.** *For any fixed  $C$ , the function  $g_1(C, w)$  is monotone decreasing in  $w$ .*

*Proof.* Taking the derivative of  $g_1(C, w)$  with respect to  $w$  yields

$$\frac{\partial g_1(C, w)}{\partial w} = \frac{1}{w} \cdot \left( \frac{\sum_{q=0}^{C-1} \frac{w^q}{q!}}{\left(\sum_{q=0}^C \frac{w^q}{q!}\right)^2} - g_1(C, w) \right),$$

and it therefore suffices to show that  $g_1(C, w) \geq \left(\sum_{q=0}^{C-1} \frac{w^q}{q!}\right) / \left(\sum_{q=0}^C \frac{w^q}{q!}\right)^2$ . Rearranging terms and using the definition of  $g_1(C, w)$ , we find that this is equivalent to the following inequality, which indeed holds for all  $w \geq 0$ ,

$$1 + w \cdot \frac{\sum_{q=0}^{C-1} \frac{w^q}{q!}}{\sum_{q=0}^C \frac{w^q}{q!}} \geq 1. \quad \square$$

**Fact D.2.** *For any fixed  $C$ , the function  $g_2(C, w)$  is monotone decreasing in  $w$ .*

*Proof.* Taking the derivative of  $g_2(C, w)$  with respect to  $w$  yields

$$\frac{\partial g_2(C, w)}{\partial w} = \frac{1}{w} \cdot \left( \frac{\sum_{q=0}^{C-1} \frac{1}{q!} \left(\frac{6 \cdot w}{5}\right)^q}{\left(\frac{1}{6} + \frac{5}{6} \cdot \sum_{q=0}^C \frac{1}{q!} \left(\frac{6 \cdot w}{5}\right)^q\right)^2} - g_2(C, w) \right),$$

and it therefore suffices to show that  $g_2(C, w) \geq \left(\sum_{q=0}^{C-1} \frac{1}{q!} \left(\frac{6 \cdot w}{5}\right)^q\right) / \left(\frac{1}{6} + \frac{5}{6} \cdot \sum_{q=0}^C \frac{1}{q!} \left(\frac{6 \cdot w}{5}\right)^q\right)^2$ . Rearranging terms using the definition of  $g_2(C, w)$ , we find that this is equivalent to showing the following inequality, which indeed holds for all  $w \geq 0$ .

$$1 + w \cdot \frac{\sum_{q=0}^{C-1} \frac{1}{q!} \left(\frac{6 \cdot w}{5}\right)^q}{\frac{1}{6} + \frac{5}{6} \cdot \sum_{q=0}^C \frac{1}{q!} \left(\frac{6 \cdot w}{5}\right)^q} \geq 1. \quad \square$$

Finally, we prove that limiting the inventory sufficiently results in a worse approximation of the optimal (unbounded-inventory) online algorithm.

**Lemma 4.6.** *For any  $C \in \mathbb{Z}_{>0}$ , no online policy with inventory capacity  $C$  is greater than a  $C/(C+1)$ -approximation of the optimal online (unbounded inventory) policy.*

*Proof.* We consider the following instance of the single-good stationary prophet inequality problem: the seller's good is supplied at rate  $\lambda > 0$  and perishes at rate  $\mu = 1$ , and there is a single buyer type which arrives at rate  $\gamma = \lambda$  and bids  $v = 1$ . As there is only one buyer type, the optimal online policy, regardless of the seller's inventory capacity, is to always sell an available item to any buyer that arrives. Therefore, by the PASTA property [29] and Lemma 2.3, the expected average revenue of this policy when the seller has inventory capacity  $C \in \mathbb{Z}_{>0}$  is

$$v \cdot \gamma \cdot \mathbb{P}_C[A \geq 1] = \lambda \cdot \left( 1 - \left( 1 + \sum_{q=1}^C \prod_{r=1}^q \frac{\lambda}{r + \lambda} \right)^{-1} \right).$$

As  $\lambda$  goes to infinity, the expected average revenue of the optimal online policy when the seller has inventory capacity  $C$  relative to that of the optimal online policy when the seller has unbounded inventory capacity goes to

$$\lim_{\lambda \rightarrow \infty} \frac{\lambda \cdot \mathbb{P}_C[A \geq 1]}{\lambda \cdot \mathbb{P}[A \geq 1]} = \lim_{\lambda \rightarrow \infty} \frac{1 - \left( 1 + \sum_{q=1}^C \prod_{r=1}^q \frac{\lambda}{r + \lambda} \right)^{-1}}{1 - \left( 1 + \sum_{q=1}^{\infty} \prod_{r=1}^q \frac{\lambda}{r + \lambda} \right)^{-1}} = \frac{1 - \left( 1 + \sum_{q=1}^C 1 \right)^{-1}}{1 - \left( 1 + \sum_{q=1}^{\infty} 1 \right)^{-1}} = \frac{C}{C + 1}. \quad \square$$

## D.2 Multi-good problem

**Lemma 4.7.** *For any good  $i \in \mathcal{G}$  and buyer type  $j \in \mathcal{B}$ ,*

$$\mathbb{P}_C[R_{ij} \wedge A_i \geq 1] \geq \mathbb{P}_C[\tilde{R}_{ij}] \cdot \mathbb{P}_C[\tilde{A}_i \geq 1]. \quad (14)$$

*Proof.* Fix a buyer type  $j \in \mathcal{B}$  and an ordering  $\sigma$  of the goods. Note that the execution of Algorithm 1 when a buyer of type  $j$  arrives is equivalent to the following: Before the buyer considers any of the goods, the seller determines which goods to permit a sale of, meaning the seller samples a set of permissible goods from the product distribution  $Ber(p_{1j}) \times \cdots \times Ber(p_{nj})$ . Then, as the buyer iterates through the goods according to order  $\sigma$ , the seller sells this buyer an item of the first

good that he reaches that is both available and permissible for this buyer. Therefore, in order for the buyer to reach good  $i$ , there must be no items of good  $i'$  available for each good  $i'$  that precedes  $i$  in the order  $\sigma$  (i.e,  $\sigma(i') < \sigma(i)$ ) and which is permissible for the buyer.

Fix the set  $H_j \subseteq \mathcal{G}$  of permissible goods for this buyer, and let  $D_{i,\sigma,H_j} = \{i' \in H_j : \sigma(i') < \sigma(i)\}$ . Since every available item is also present, having no items of good  $i'$  present for each  $i' \in D_{i,\sigma,H_j}$  is a sufficient condition to guarantee that the buyer reaches good  $i$ . Therefore, letting  $Y \in \mathbb{R}^{2n}$  be the vector whose elements, which we refer to as  $Y_{A_i}$  and  $Y_{P_i}$ , represent the number of items of good  $i$  available and the negative of the number of items of good  $i$  present, respectively, under Algorithm 1, we have

$$\mathbb{P}_C [R_{ij} \wedge A_i \geq 1 \mid \sigma, H_j] \geq \mathbb{P}_C \left[ Y \geq e_{A_i} - \infty \cdot \sum_{i' \in \bar{D}_{i,\sigma,H_j}} e_{P_{i'}} \mid \sigma, H_j \right]. \quad (18)$$

We use  $e_{A_i}$  and  $e_{P_i}$  to denote the vectors with all zeros except at the elements corresponding to  $A_i$  and  $P_i$  which are 1. We can think of  $Y$  as simply the (augmented) state of the marketplace under Algorithm 1, where the set  $\mathcal{Y}_C \subseteq \mathbb{R}^{2n}$  of valid states is such that for any  $y \in \mathcal{Y}_C$ , we have  $0 \leq y_{A_i} \leq C$ ,  $y_{P_i} \leq 0$ , and  $y_{A_i} \leq |y_{P_i}|$ . Under Algorithm 1, the stochastic process governing  $Y$  is described by intensity matrix  $Q_C$ , where for any  $y, y' \in \mathcal{Y}_C$ ,

$$Q_C(y, y') = \begin{cases} \lambda_i & y' = y + e_{A_i} - e_{P_i} \text{ and } y_{A_i} < C \\ y_{A_i} \cdot \mu_i & y' = y - e_{A_i} + e_{P_i} \\ (|y_{P_i}| - y_{A_i}) \cdot \mu_i & y' = y + e_{P_i} \\ \alpha \cdot \sum_{j \in \mathcal{B}} \mathbb{P}_C [R_{ij} \mid y] \cdot \frac{x_{ij}^*}{w_i} & y' = y - e_{A_i} \text{ and } y_{A_i} > 0 \\ 0 & \text{o.w.} \end{cases}$$

and  $Q_C(y, y) = -\sum_{y' \in \mathcal{Y}_C: y' \neq y} Q_C(y, y')$ .

Although the availability of good  $i$  and the presence of other goods  $i' \neq i$  are correlated under  $Q_C$ , we show that  $Y$  stochastically dominates a stochastic process  $\tilde{Y}$  under which they are in fact independent. Informally, we define the dynamics governing  $\tilde{Y}$  to correspond to, in some sense, every good (simultaneously) coming first in the ordering such that an arriving buyer reaches each good with probability 1. Put another way,  $\tilde{Y}$  can be thought of as a collection of  $n$  independent single-good instances, where each instance consists of a different good  $i \in \mathcal{G}$  and the full set of buyers  $\mathcal{B}$ . Of course, this does not reflect the dynamics of Algorithm 1 (or any other of feasible policy for the multi-good problem) but is still a useful tool, as we will see. More specifically, we let  $\tilde{Y}$  represent the state of a stochastic process on the same space  $\mathcal{Y}_C$  governed by intensity matrix  $\tilde{Q}_C$ , where for any  $y, y' \in \mathcal{Y}_C$ ,

$$\tilde{Q}_C(y, y') = \begin{cases} \lambda_i & y' = y + e_{A_i} - e_{P_i} \text{ and } y_{A_i} < C \\ y_{A_i} \cdot \mu_i & y' = y - e_{A_i} + e_{P_i} \\ (|y_{P_i}| - y_{A_i}) \cdot \mu_i & y' = y + e_{P_i} \\ \alpha \cdot \frac{\lambda_i}{w_i} & y' = y - e_{A_i} \text{ and } y_{A_i} > 0 \\ 0 & \text{o.w.} \end{cases}$$

and  $\tilde{Q}_C(y, y) = -\sum_{y' \in \mathcal{Y}_C: y' \neq y} \tilde{Q}_C(y, y')$ . Observe that  $Q_C$  and  $\tilde{Q}_C$  are identical except for the rate at which they transition to states with exactly one fewer available item, a rate which is higher for  $\tilde{Y}$ . Intuitively, this creates more “downwards pressure” for variable  $\tilde{Y}$  than for  $Y$ . This intuition is borne out by the following lemma, which we prove following this proof.

**Claim D.3.** For any permutation  $\sigma$  over  $\mathcal{G}$  and any  $H_j \subseteq \mathcal{G}$ , we have that

$$[Y \mid \sigma, H_j] \succeq [\tilde{Y} \mid \sigma, H_j].$$

Claim D.3 implies that for any  $y \in \mathcal{Y}$ ,  $\mathbb{P}_C [Y \geq y] \geq \mathbb{P}_C [\tilde{Y} \geq y]$  and therefore

$$\mathbb{P}_C \left[ Y \geq e_{A_i} - \infty \cdot \sum_{i' \in \bar{D}_{i,\sigma,H_j}} e_{P_{i'}} \mid \sigma, H_j \right] \geq \mathbb{P}_C \left[ \tilde{Y} \geq e_{A_i} - \infty \cdot \sum_{i' \in \bar{D}_{i,\sigma,H_j}} e_{P_{i'}} \mid \sigma, H_j \right]. \quad (19)$$

Due to the independence of the availability of good  $i$  and the presence of any other good  $i' \neq i$  under  $\tilde{Q}_C$ , we have

$$\mathbb{P}_C \left[ \tilde{Y} \geq e_{A_i} - \infty \cdot \sum_{i' \in \bar{D}_{i,\sigma,H_j}} e_{P_{i'}} \mid \sigma, H_j \right] = \mathbb{P}_C [\tilde{Y}_{A_i} \geq 1 \mid \sigma, H_j] \cdot \mathbb{P}_C [\tilde{Y}_{P_{i'}} = 0 \forall i' \in D_{i,\sigma,H_j} \mid \sigma, H_j]. \quad (20)$$

Observe that the event  $\tilde{Y}_{A_i} \geq 1$  is exactly the event  $\tilde{A}_i \geq 1$  and furthermore, that the availability of good  $i$  under  $\tilde{Q}$  does not depend on either the ordering  $\sigma$  or the set of permissible goods  $i' \in H_j$ . Also, conditional on  $\sigma$  and  $H_j$ , the event  $\tilde{Y}_{P_{i'}} = 0$  for all  $i' \in D_{i,\sigma,H_j}$  corresponds to the event that no good  $i'$  which is permissible for the buyer precedes  $i$  in the ordering and is present, which is exactly  $\tilde{R}_{ij}$ . Combining Equations (18) to (20), Equation (14) follows.  $\square$

*Proof of Claim D.3.* By Lemma 2.6, it suffices to show that for all  $\tilde{y}, y \in \mathcal{Y}_C$  and every upward closed set  $S \subseteq \mathcal{Y}_C$ ,

$$\sum_{\Delta: \tilde{y} + \Delta \in S} \tilde{Q}_C(\tilde{y}, z) \leq \sum_{\Delta: \tilde{y} + \Delta \in S} Q_C(y, z)$$

if  $\tilde{y} \leq y$  and either  $\tilde{y}, y \notin S$  or  $\tilde{y}, y \in S$ . We fix  $\tilde{y}, y$  and consider both the two cases individually.

**Case 1:** Suppose  $\tilde{y}, y \notin S$ . For every  $\Delta$  such that  $\tilde{y} + \Delta \in S$ , and  $y + \Delta \in \mathcal{Y}_C$ , we have  $y + \Delta \in S$  since  $\tilde{y} \leq y$  and  $S$  is upward closed. The upward closedness of  $S$  also implies that since  $\tilde{y} \notin S$ , there must exist some coordinate  $k$  such that  $\tilde{\Delta}_k > 0$ . Inspecting the intensity matrix  $\tilde{Q}$ , it is clear that there are exactly three forms that  $\Delta$  can take such that  $\tilde{Q}_C(\tilde{y}, \tilde{y} + \Delta) > 0$ : (1)  $\Delta_i^{(1)} = e_{A_i} - e_{P_i}$ , (2)  $\Delta_i^{(2)} = e_{P_i} - e_{A_i}$ , or (3)  $\Delta_i^{(3)} = e_{P_i}$ , for some  $i \in \mathcal{G}$ . Note that for (1),  $\tilde{y} \leq y$ ,  $\tilde{y} + \Delta \in S$ , and  $S \subseteq \mathcal{Y}_C$  together imply that  $\tilde{y}_{A_i} = y_{A_i} < C$ ; otherwise,  $\tilde{y} + e_{A_i} \leq y$  and therefore  $\tilde{y} + \Delta \leq y$ , meaning  $y \in S$  by the upward closedness of  $S$ , contradicting  $y \notin S$ . We conclude that if  $\tilde{y} + \Delta \in S$ , then  $y + \Delta \in \mathcal{Y}_C$ , and hence  $y_\Delta \in S$ . Similarly for (2) and (3),  $\tilde{y} \leq y$  and  $\tilde{y} + \Delta \in S$  imply that  $\tilde{y}_{P_i} = y_{P_i}$ ; otherwise, if  $\tilde{y}_{P_i} < y_{P_i}$ , then  $\tilde{y} + e_{P_i} \leq y$  and therefore  $\tilde{y} + \Delta \leq \tilde{y} + e_{P_i} \leq y$ , which, by the upward closedness of  $S$ , contradicts  $y \notin S$ . We now relate the intensities under  $\tilde{Q}_C$  and  $Q_C$ .

Fix  $\Delta$  such that  $y + \Delta \in S$  and first suppose that  $\Delta = \Delta_i^{(1)}$  for some  $i \in \mathcal{G}$ . Comparing the intensity matrices, we have that

$$\tilde{Q}_C(\tilde{y}, \tilde{y} + \Delta_i^{(1)}) = \lambda_i = Q_C(y, y + \Delta_i^{(1)}).$$

Next suppose that  $\Delta = \Delta_i^{(2)}$  for some  $i \in \mathcal{G}$ . Since  $\tilde{y} + \Delta_i^{(2)} \in S$ , by the upward closedness of  $S$ ,  $\tilde{y} + \Delta_i^{(3)} \in S$  and also  $y + \Delta_i^{(2)}, y + \Delta_i^{(3)} \in S$ . Furthermore, we have that

$$\tilde{Q}_C(\tilde{y}, \tilde{y} + \Delta_i^{(2)}) + \tilde{Q}_C(\tilde{y}, \tilde{y} + \Delta_i^{(3)}) = |\tilde{y}_{P_i}| \cdot \mu_i = |y_{P_i}| \cdot \mu_i = Q_C(y, y + \Delta_i^{(2)}) + Q_C(y, y + \Delta_i^{(3)}),$$

where the equality is due to the fact that  $\tilde{y}_{P_i} = y_{P_i}$ , as argued in the preceding paragraph. Lastly, suppose that  $\Delta = \Delta_i^{(3)}$  for some  $i \in \mathcal{G}$  such that  $\tilde{y} + \Delta_i^{(2)} \notin S$ . If  $\tilde{y}_{A_i} = y_{A_i}$ , then

$$\tilde{Q}_C\left(\tilde{y}, \tilde{y} + \Delta_i^{(3)}\right) = (|\tilde{y}_{P_i}| - \tilde{y}_{A_i}) \cdot \mu_i = (|y_{P_i}| - y_{A_i}) \cdot \mu_i = Q_C\left(y, y + \Delta_i^{(3)}\right).$$

If  $\tilde{y}_{A_i} < y_{A_i}$ , then  $\tilde{y} \leq y - e_{A_i}$ , which implies  $\tilde{y} + \Delta_i^{(3)} \leq y + \Delta_i^{(2)}$ , from which it follows that  $y + \Delta_i^{(2)} \in S$  by the upward closedness of  $S$ . Therefore,

$$\tilde{Q}_C\left(\tilde{y}, \tilde{y} + \Delta_i^{(3)}\right) = (|\tilde{y}_{P_i}| - \tilde{y}_{A_i}) \cdot \mu_i \leq |\tilde{y}_{P_i}| \cdot \mu_i = |y_{P_i}| \cdot \mu_i = Q_C\left(y, y + \Delta_i^{(2)}\right) + Q_C\left(y, y + \Delta_i^{(3)}\right).$$

Putting this all together, it follows that

$$\sum_{\Delta: \tilde{y} + \Delta \in S} \tilde{Q}_C(\tilde{y}, \tilde{y} + \Delta) \leq \sum_{\Delta: y + \Delta \in S} Q_C(y, y + \Delta).$$

**Case 2:** Suppose  $y, \tilde{y} \in S$ . In this case it suffices to show that

$$\sum_{\Delta: \tilde{y} + \Delta \in \mathcal{V}_C \setminus S} Q_C(y, \tilde{y} + \Delta) \leq \sum_{\Delta: y + \Delta \in \mathcal{V}_C \setminus S} Q_C(y, y + \Delta).$$

For every  $\Delta$  such that  $y + \Delta \notin S$ , we have that  $\tilde{y} + \Delta \notin S$  since  $\tilde{y} \leq y$  and  $S$  is upward closed. The upward closedness of  $S$  also implies that since  $y \in S$ , there must exist some coordinate  $k$  such that  $\Delta_k < 0$ . From the intensity matrix  $Q_C$ , there are three forms that  $\Delta$  can take such that  $Q_C(y, y + \Delta) > 0$ : (1)  $\Delta_i^{(1)} = e_{A_i} - e_{P_i}$ , (2)  $\Delta_i^{(2)} = e_{P_i} - e_{A_i}$ , or (4)  $\Delta_i^{(4)} = -e_{A_i}$ , for some  $i \in \mathcal{G}$ . Note that for (2) and (4),  $\tilde{y} \leq y$  and  $\tilde{y} \in S$  immediately imply that  $\tilde{y}_{A_i} = y_{A_i}$ ; otherwise, if  $\tilde{y}_{A_i} < y_{A_i}$ , then  $\tilde{y} \leq y - e_{A_i} \leq y + \Delta$ , which, by the upward closedness of  $S$ , contradicts  $y + \Delta \notin S$ .

Fix  $\Delta$  such that  $\tilde{y} + \Delta \notin S$  and first suppose  $\Delta = \Delta_i^{(1)}$  for some  $i \in \mathcal{G}$ . In this case, we have

$$Q_C\left(y, y + \Delta_i^{(1)}\right) = \lambda_i = \tilde{Q}_C\left(\tilde{y}, \tilde{y} + \Delta_i^{(1)}\right).$$

Next suppose  $\Delta = \Delta_i^{(2)}$  for some  $i \in \mathcal{G}$ . Since  $y_{A_i} = \tilde{y}_{A_i}$ ,

$$Q_C\left(y, y + \Delta_i^{(2)}\right) = y_{A_i} \cdot \mu_i = \tilde{y}_{A_i} \cdot \mu_i = \tilde{Q}_C\left(\tilde{y}, \tilde{y} + \Delta_i^{(2)}\right).$$

Lastly, suppose  $\Delta = \Delta_i^{(3)}$ . We have

$$Q_C\left(y, y + \Delta_i^{(3)}\right) = \alpha \cdot \sum_{j \in \mathcal{B}} \mathbb{P}_C[R_{ij}|y] \cdot \frac{x_{ij}^*}{w_i} \leq \alpha \cdot \frac{\lambda_i}{w_i} = \tilde{Q}_C\left(\tilde{y}, \tilde{y} + \Delta_i^{(3)}\right),$$

where the inequality follows from the fact that  $\mathbb{P}_C[R_{ij}|y] \leq 1$  trivially and Constraint 1 from LP<sub>off</sub>.

Therefore,

$$\sum_{\Delta: y + \Delta \in \mathcal{V}_C \setminus S} Q_C(y, y + \Delta) \leq \sum_{\Delta: \tilde{y} + \Delta \in \mathcal{V}_C \setminus S} \tilde{Q}_C(\tilde{y}, \tilde{y} + \Delta). \quad \square$$

**Lemma 4.8** ([9]). *For any good  $i \in \mathcal{G}$  and buyer type  $j \in \mathcal{B}$ , we have that*

$$\mathbb{P}_C\left[\tilde{R}_{ij}\right] \geq 1 - \frac{\alpha}{2}.$$



*Proof.* Fix some good  $i \in \mathcal{G}$  and some buyer  $j \in \mathcal{B}$  and consider Algorithm 1 when a buyer of type  $j$  arrives. By the upper bound of Corollary 2.4, the probability that the seller permits the sale of some good  $i' \neq i$  that is present and precedes good  $i$  in the randomly chosen ordering  $\sigma$  is

$$\sum_{i' \in \mathcal{G}} \mathbb{P}[\sigma(i') < \sigma(i)] \cdot \alpha \cdot \frac{x_{i'j}^*}{\gamma_j \cdot w_{i'}} \cdot \left(1 - \left(1 + \sum_{q=1}^C \left(\frac{\lambda_{i'}}{\mu_{i'}}\right)^q\right)^{-1}\right) \leq \frac{\alpha}{2} \cdot \sum_{i' \in \mathcal{G}} \frac{x_{i'j}^*}{\gamma_j} \leq \frac{\alpha}{2}.$$

The first inequality holds since  $w_{i'} = 1 - \exp(-\lambda_{i'}/\mu_{i'}) \geq 1 - (1 + \sum_{q=1}^C \frac{\lambda_{i'}}{\mu_{i'}})^{-1}$  for all  $C$ , and the second inequality follows from Constraint (2) from  $\text{LP}_{\text{off}}$ .  $\square$

## E Mechanism design implications

By standard results in mechanism design, our posted-price mechanisms imply truthful mechanisms for approximating the optimal social welfare and revenue of any mechanism for this problem. For completeness, we discuss implications of our policies to mechanism design for the single-good problem in our model.

In the mechanism design setting, buyers arrive as before with their real value  $v_j$  for the good again drawn from the known distribution  $\mathcal{D}$ , though here they may misreport their true value for the good. Our randomized posted-price mechanism sets a price-probability pair  $(\bar{v}, \bar{p})$ , and upon arrival of a buyer who observes an available item and bids  $b$ , the mechanism sells the item at price  $\bar{v}$  if the reported bid is strictly higher than  $\bar{v}$ , and sells the item at the same price with probability  $\bar{p}$  if the bid is equal to  $\bar{v}$  (else, an item is not sold). The utility of the buyer with true valuation  $v_j$  for the good is  $v_j - \bar{v}$ . It is easy to see that this policy is weakly DSIC (dominant-strategy incentive compatible) and individually rational.<sup>4</sup> Since the social welfare (buyer utility + seller revenue) is precisely the gain,  $v_j = (v_j - \bar{v}) + \bar{v}$ , it is immediate that our posted-price policies yield weakly DSIC and individually rational mechanisms which match our policies' approximation ratios.

**Corollary E.1.** *There exists a weakly DSIC and individually rational mechanism for the single-good stationary prophet inequality problem whose expected social welfare is a 1/2- and 0.656-approximation of the social-welfare-maximizing offline and online mechanisms, respectively.*

It is quite standard in the literature to go through Myerson's lemma [24] to leverage mechanisms which approximately maximize social welfare to obtain mechanisms which approximately maximize the seller's revenue (going via so-called "virtual valuations"). Unfortunately, this approach does not work for randomized policies under discrete distributions over buyer valuations, as in our setting. However, it is not hard to leverage our randomized posted-price policies to obtain deterministic posted-price policies, for which the above equivalence of Myerson does hold.

**Claim E.2.** *For any randomized posted-price policy  $\mathcal{P}$  for the single-good stationary prophet inequality problem, there exists a deterministic posted-price mechanism with at least half the expected average gain of  $\mathcal{P}$ .*

*Proof.* Fix a randomized posted-price mechanism  $\mathcal{P}$  for the stationary prophet inequality problem characterized by price-probability pair  $(\bar{v}, \bar{p})$ . If  $\bar{p} \in \{0, 1\}$ , then  $\mathcal{P}$  is in fact a deterministic posted-price mechanism and the claim is trivially true. Define two deterministic posted-price mechanisms which we refer to as  $\mathcal{P}^0$  and  $\mathcal{P}^1$ . In particular, mechanism  $\mathcal{P}^0$  accepts all bids strictly greater than

<sup>4</sup>For weakly DSIC mechanisms, a weakly dominant strategy for a buyer is to report his true value. An individually rational mechanism is one in which a buyer loses nothing from entering the market.

$\bar{v}$  but rejects any bid of  $\bar{v}$  or less. On the other hand, mechanism  $\mathcal{P}^1$  accepts all bids greater than or equal to  $\bar{v}$  but rejects any bid strictly less than  $\bar{v}$ . We denote the expected average gains of  $\mathcal{P}^0$  and  $\mathcal{P}^1$  as  $r^{\mathcal{P}^0}$  and  $r^{\mathcal{P}^1}$ , respectively.

Let  $r_{>\bar{v}}^{\mathcal{P}}$  denote the expected average gain under  $\mathcal{P}$  from selling the good to buyers with value strictly greater than  $\bar{v}$  and let  $r_{=\bar{v}}^{\mathcal{P}}$  denote the expected average gain under  $\mathcal{P}$  from selling the good to buyers with value exactly  $\bar{v}$ . Note that we trivially have  $r_{>\bar{v}}^{\mathcal{P}} + r_{=\bar{v}}^{\mathcal{P}} = r^{\mathcal{P}}$ . It is clear that  $r^{\mathcal{P}^0} \geq r_{>\bar{v}}^{\mathcal{P}}$  since the stationary probability under  $\mathcal{P}^0$  that an item is available when a buyer with value strictly greater than  $\bar{v}$  arrives is at least the same probability under  $\mathcal{P}$ . We also have that  $r^{\mathcal{P}^1} \geq r_{=\bar{v}}^{\mathcal{P}}$ . This is true since  $r^{\mathcal{P}^1}$  is at least the expected average gain of the policy that only sells items to buyers with value exactly  $\bar{v}$ , which is trivially an upper bound on  $r_{=\bar{v}}^{\mathcal{P}}$ . It follows then that  $r^{\mathcal{P}^0} + r^{\mathcal{P}^1} \geq r_{>\bar{v}}^{\mathcal{P}} + r_{=\bar{v}}^{\mathcal{P}} = r^{\mathcal{P}}$  and therefore,

$$\max \left\{ r^{\mathcal{P}^0}, r^{\mathcal{P}^1} \right\} \geq 1/2 \cdot r^{\mathcal{P}}. \quad \square$$

Combining the above with our algorithms of Theorems 1.1 and 1.2 and with Myerson’s lemma then yields the following.

**Corollary E.3.** *There exists a DSIC mechanism for the single-good stationary prophet inequality problem whose expected revenue is a 1/4- and 0.328-approximation of the revenue-maximizing offline and online DSIC mechanisms, respectively.*

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