Equilibrium Simulation

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Why Analyze Equilibrium?

An investor needs to have a *view* of the ways in which asset prices are determined.

Risk and return forecasts should reflect these views.

Asset prices are set by *investors* operating in *markets*.

Equilibrium prices are those at which no investors are willing to make further trades.

Asset prices will tend towards equilibrium until conditions change.

Good *asset pricing theory* is a key ingredient for good *investment practice* .

Asset Pricing Theories

Mean/Variance

State/Preference

Mean/Variance Analysis

All returns are jointly normally distributed, or

Investors care only about mean and variance of portfolio return

Markowitz Portfolio Analysis
- Normative theory
- Maximize portfolio expected return for given risk
- Portfolio Optimization

The Capital Asset Pricing Model
- Positive theory
- Expected Returns related to Beta Values
- Index Funds

Key Implications of the CAPM

The Market Risk/Reward Theorem

Only market risk is rewarded with higher expected return

The Market Risk/Reward Corollary

Don’t take non-market risk

State/Preference Analysis

Investors care about the entire distribution of portfolio return and returns need not be jointly normally distributed

Arrow/Debreu economies

Financial Engineering
    Normative theory

Pricing Kernels
    Positive theory

State/Preference Asset Pricing Theory for Dummies

States of the World

There are alternative future *states of the world*

One and only one will occur

For each state there is a *probability*

It is possible to buy and sell *state claims*

a claim for state *s* pays $1 if and only if state s occurs (similar to an insurance policy)

The markets are *complete* (every state claim can be traded)

When equilibrium is established, there will be a set of *state prices*, one for each state of the world.

Price Per Chance

PPC:

$$\frac{\text{State Price}}{\text{Probability of State}}$$

With any insurance policy, one should compare the price with the likelihood of cashing in on a claim

The higher the PPC, the less attractive is an investment

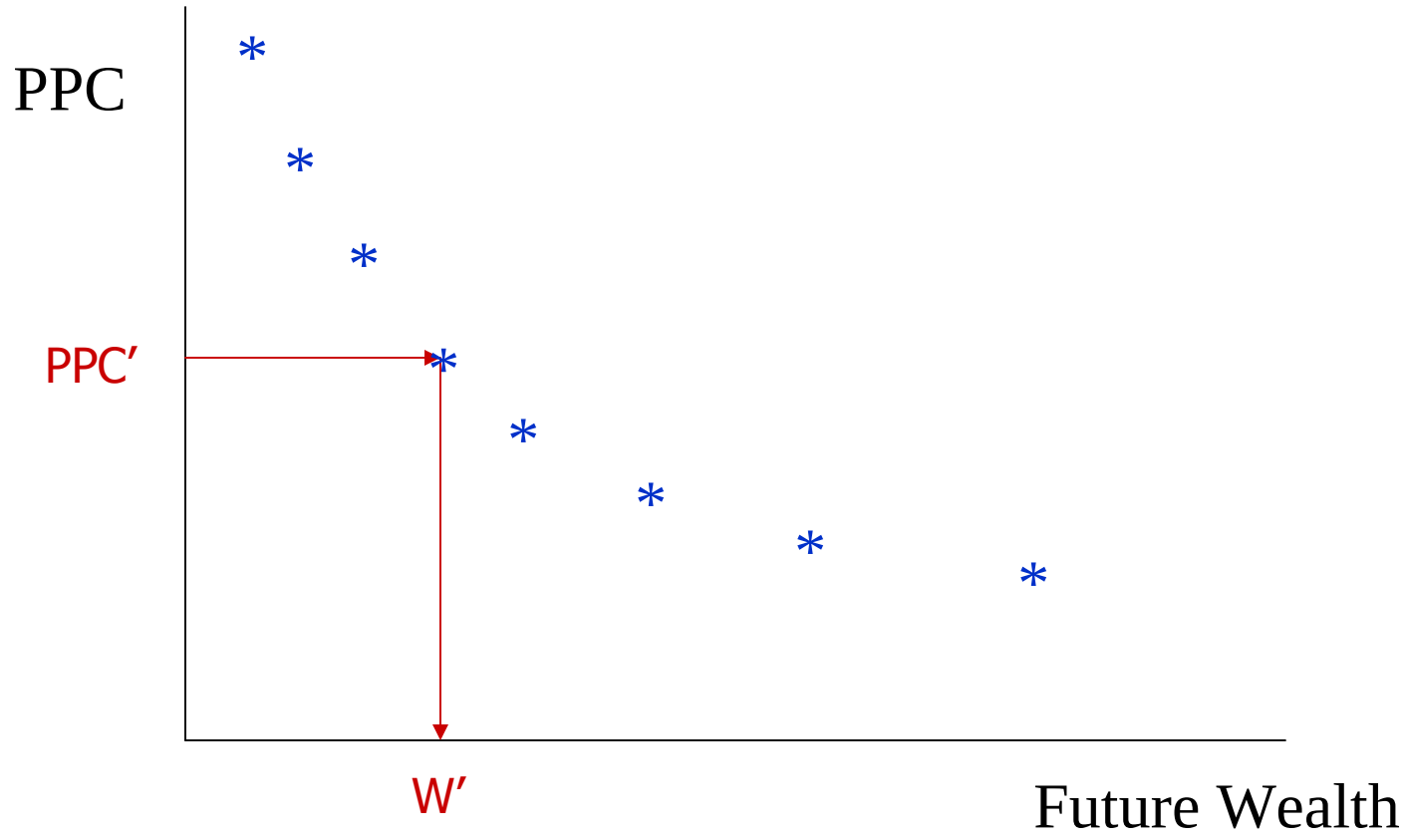
Rational Investment Allocations

Take more of something when it costs less

PPC is a measure of cost

Allocate current wealth to obtain more future wealth in states with lower PPCs

An Individual's Optimal Allocation

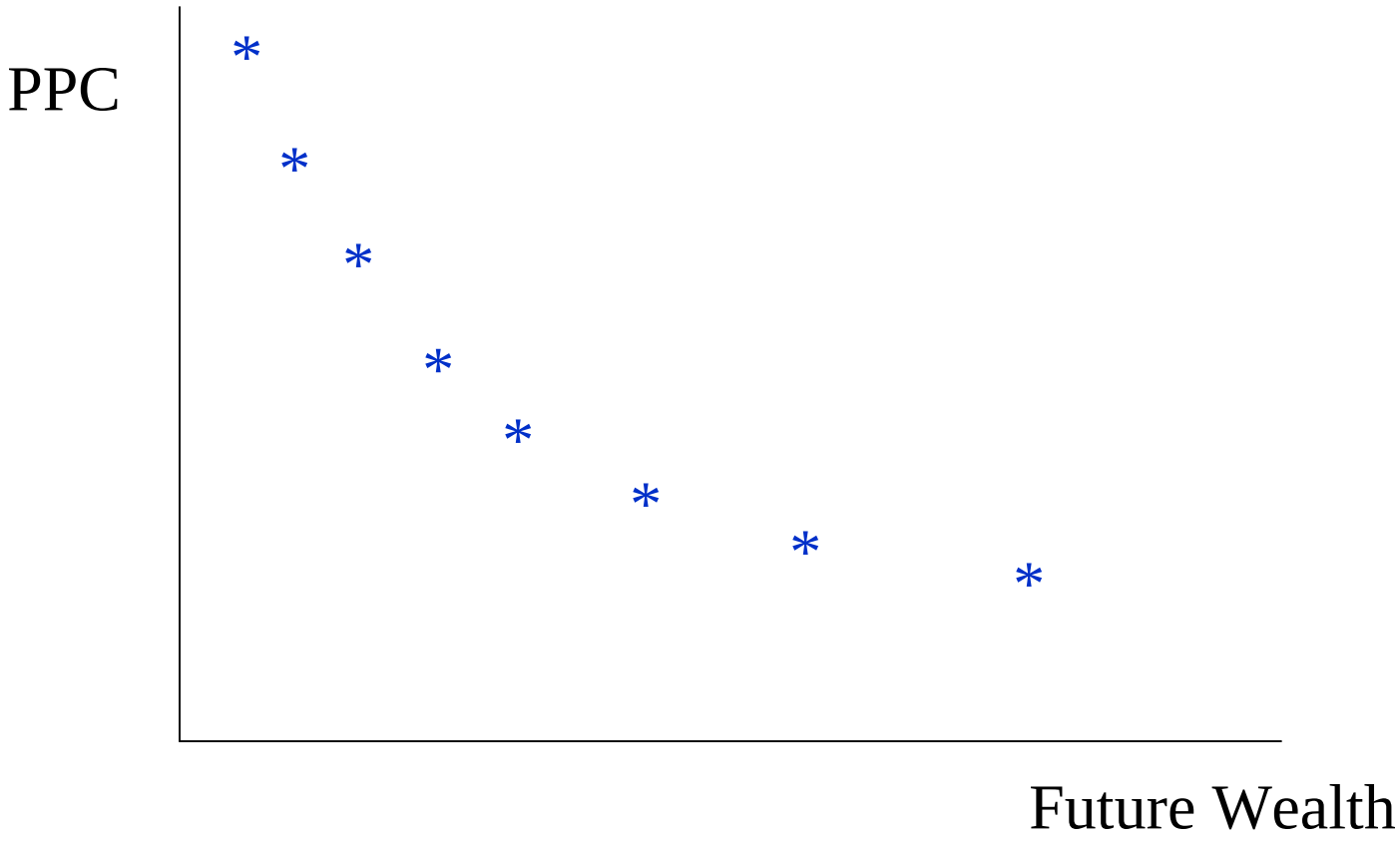


The Market Allocation

The market wealth in a state is the sum of the individuals' levels of wealth in that state

If each individual wants more wealth in state A than state B, the total desired market wealth in state A will be greater than in state B

The Market Portfolio



Equilibrium Conditions

Given production, the amount of market wealth in each state is given.

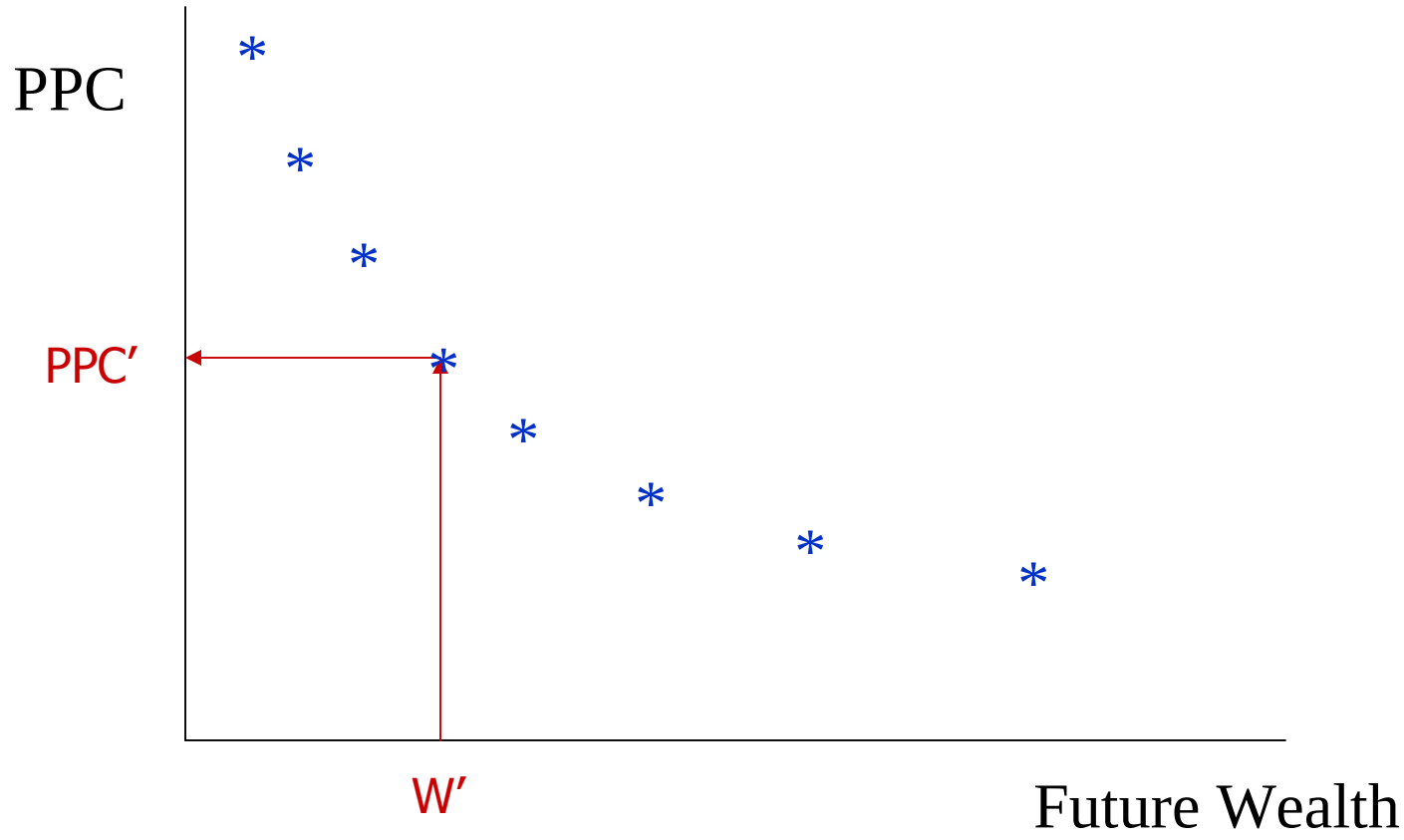
Thus prices must adjust until the individuals' collective demand for wealth in a state equals that available.

This implies that

States with the same wealth will have the same PPC, and

States with more wealth will have lower PPCs

Equilibrium Prices: the Pricing Kernel



Individual and Market Allocations

For each level of market wealth there is a PPC

Higher levels of market wealth ➔ lower PPCs

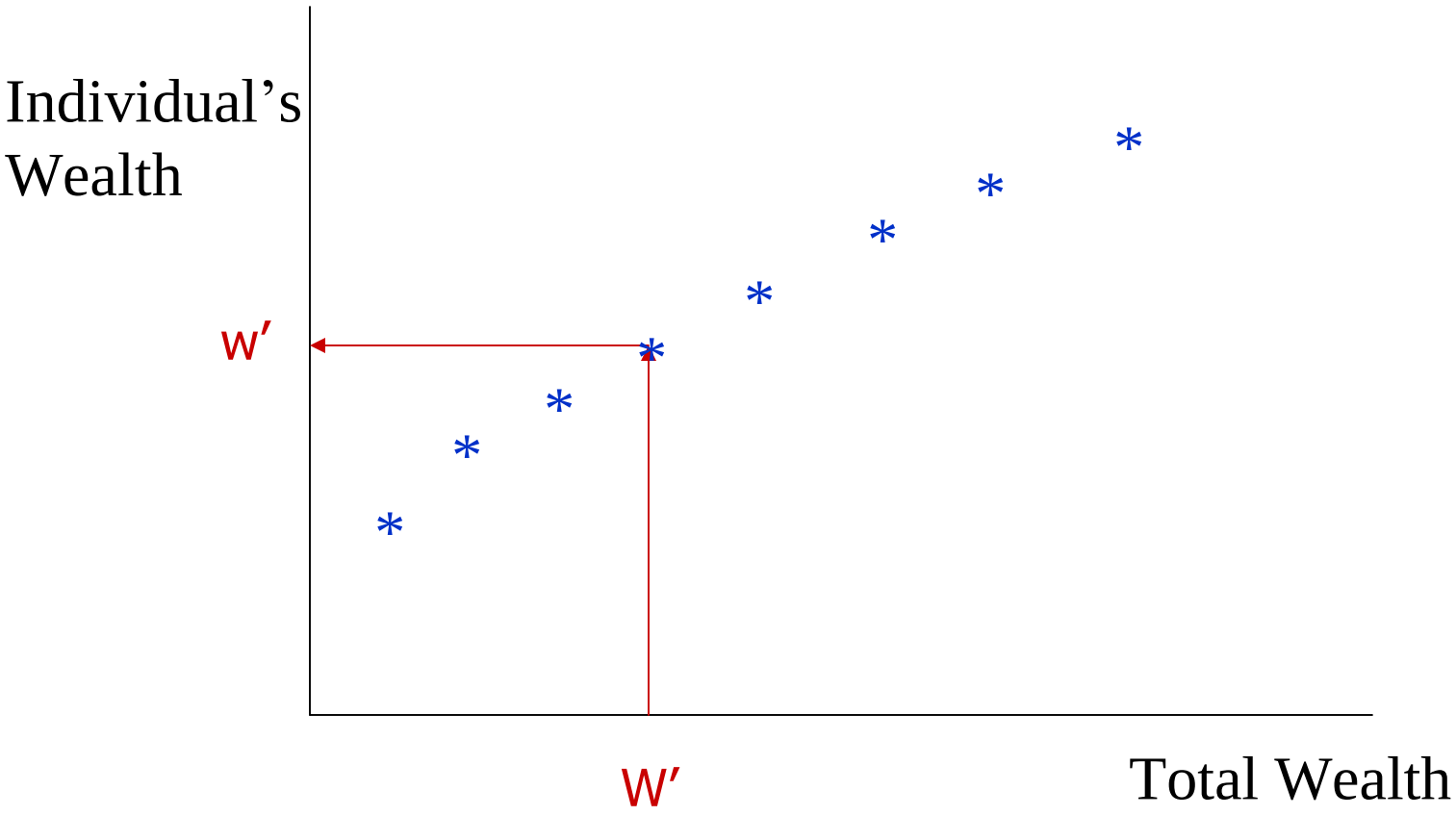
For each PPC there is a level of individual wealth

Lower PPCs ➔ higher levels of individual wealth

Thus each individual should arrange to have wealth that is related directly to market wealth

Higher levels of market wealth ➔ higher levels of individual wealth

Individual and Market Wealth



Risk and Expected Return

Risk and Expected Return (1)

The pricing kernel

$$m_s \equiv p_s/\pi_s$$

A kernel beta

$$\beta_i^k \equiv \frac{\text{cov}(R_i,m)}{\text{cov}(R_M,m)}$$

The kernel beta equation

$$(KBE) \qquad E(R_i) = r + \beta_i^k(E(R_M) - r)$$

Risk and Expected Return (2)

If m is a decreasing function of R_M

$$m = f(R_M)$$

Then

$$\frac{\mathrm{cov}(R_i, f(R_M))}{\mathrm{cov}(R_M, f(R_M))} \equiv \beta_i^{f(R_M)}$$

And

$$(MRRT) \qquad E(R_i) = r + \beta_i^{f(R_M)}(E(R_M) - r)$$

Risk and Expected Return (3)

If m is a linear function of R_M

$$m = a - bR_M$$

Conventionally, let

$$\beta_i \equiv \frac{\text{cov}(R_i, R_M)}{\text{var}(R_M)}$$

Then

$$(SML) \qquad E(R_i) = r + \beta_i \ (E(R_M) - r)$$

Equilibrium Simulation

The Key Question to be Addressed

To what extent do the implications of the CAPM and/or State/Preference Asset Pricing theory hold when markets are incomplete and investors:

do not have mean/variance preferences,

have sources of income outside the capital market,

make different predictions,

act in accordance with findings of behavioral research,

etc..

The Vehicle: APSIM,

Asset Price and **P**ortfolio Choice **Sim**ulator

Formulation:

- Discrete outcomes
- Discrete time

Process:

- Simulate trading to reach market equilibrium
- Analyze characteristics of the resulting equilibrium

The approach:

- Proceeds from first principles
- Uses simple mathematics
- Allows for complex economies and preferences
- Can analyze mean/variance as a special case

References

William F. Sharpe,

Investors and Markets: Portfolio Choices, Asset Prices and Investment Advice,

Princeton University Press, 2007

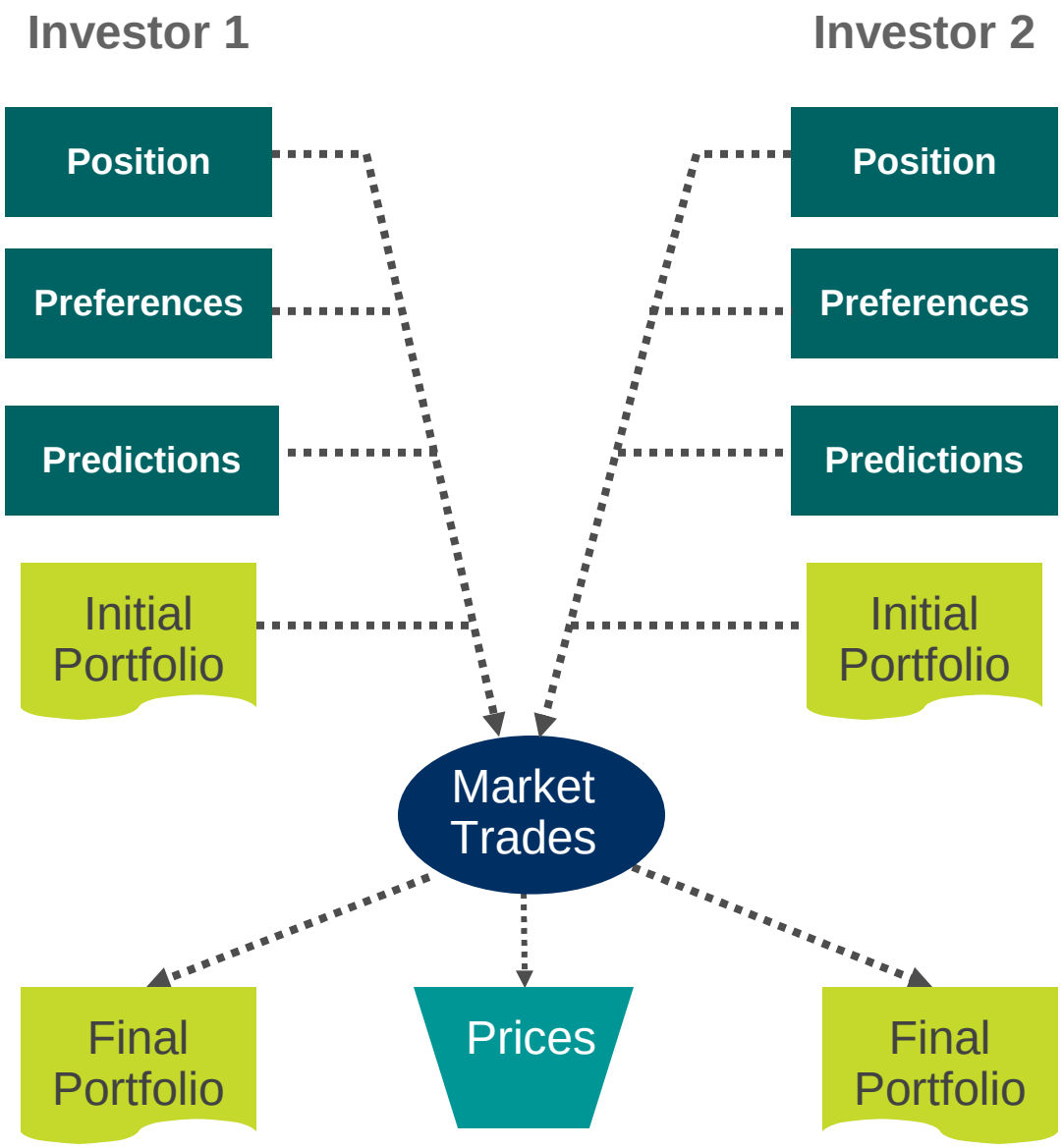
APSIM program, cases and manual
www.wsharpe.com

Equilibrium Simulation: Key Steps

Specify investors' initial conditions

Operate markets until no further trades are possible

Examine equilibrium properties


Investor 1
Investor 2
Position
Preferences
Predictions
Initial Portfolio
Position
Preferences
Predictions
Initial Portfolio
Market Trades
Final Portfolio
Prices
Final Portfolio

Case 1
Non-Mean/Variance Preferences

Case 1

Currency: fish

2 traders (Mario and Hue)

Preferences not mean/variance
- Constant relative risk aversion

4 future states of the world
- Total number of fish
- Favored locations

Agreement
- All predictions = actual probabilities

Incomplete market
- Only assets traded

Inputs

Case 1

Securities:	Consume	Bond	MFC	HFC
Now	1	0	0	0
BadS	0	1	5	3
BadN	0	1	3	5
GoodS	0	1	8	4
GoodN	0	1	4	8

Portfolios:	Consume	Bond	MFC	HFC
Mario	49	0	10	0
Hue	49	0	0	10

Probabilities:	Now	BadS	BadN	GoodS	GoodN
Probability	1	0.15	0.25	0.25	0.35

Preferences:	Time	RiskAversion
Mario	0.96	1.5
Hue	0.96	2.5

Trading

Do a *round* of trades:

For each security from 2 through n

Find investors' reservation prices

Select a trade price

Obtain bid and offered quantities

Make trades for the smaller of bids and offers

If any trades were made in the round, repeat

Equilibrium Portfolios and Consumptions

Case 1

Portfolios:	Consume	Bond	MFC	HFC
MARKET	98.00	0.00	10.00	10.00
Mario	48.77	-12.16	6.24	6.24
Hue	49.23	12.16	3.76	3.76

Consumptions:	Now	BadS	BadN	GoodS	GoodN
TOTAL	98.0	80.0	80.0	120.0	120.0
Mario	48.8	37.8	37.8	62.7	62.7
Hue	49.2	42.2	42.2	57.3	57.3

Equilibrium Prices

Case 1

Security Prices:	Consume	Bond	MFC	HFC
MARKET	1.00	0.96	4.35	4.89
Mario	1.00	0.96	4.35	4.89
Hue	1.00	0.96	4.35	4.89

State Prices:	Now	BadS	BadN	GoodS	GoodN
MARKET	1.00	0.21	0.35	0.16	0.23
Mario	1.00	0.21	0.35	0.16	0.23
Hue	1.00	0.21	0.35	0.16	0.23

Price Per Chance

Case 1

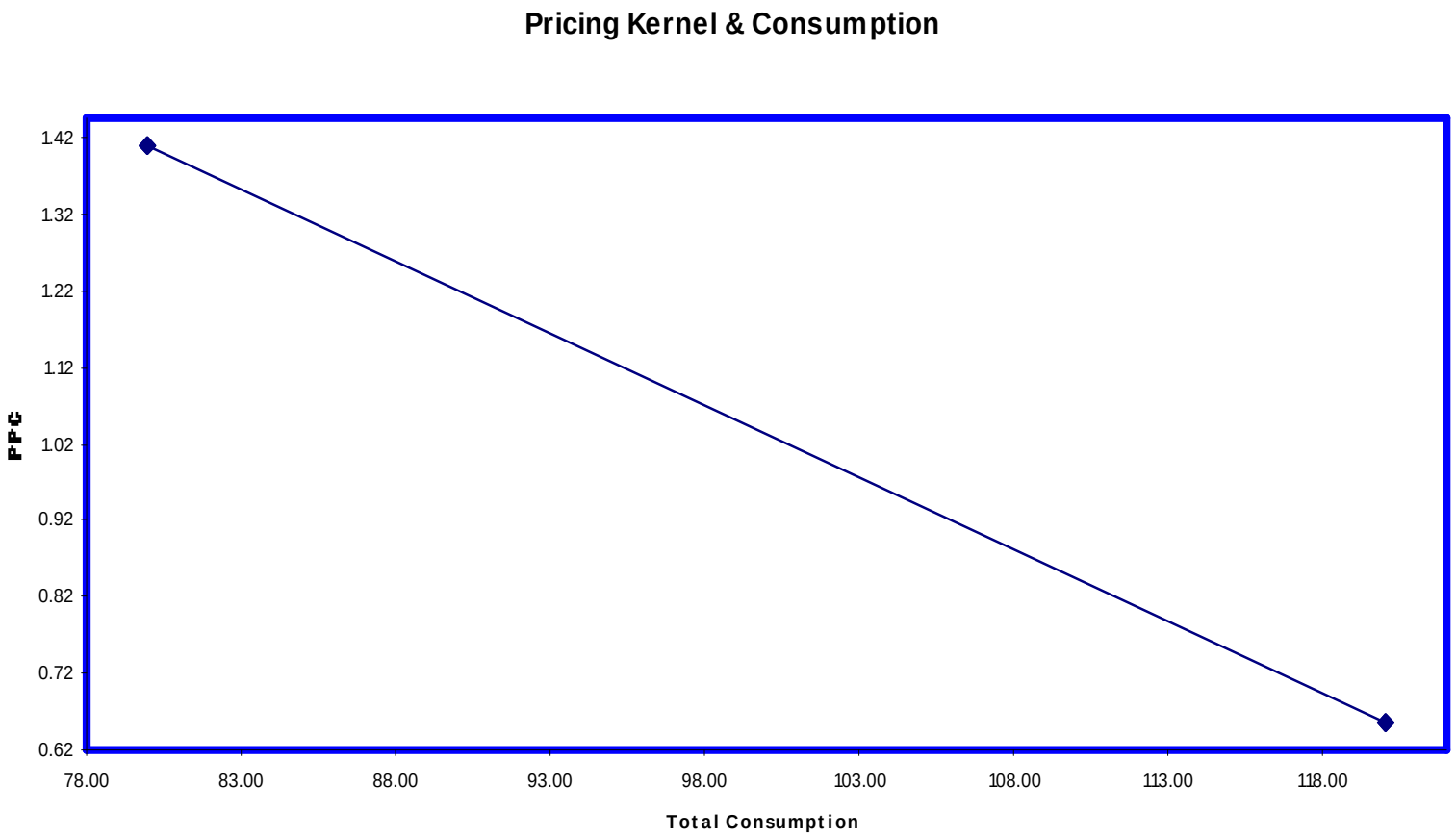
State Prices:	Now	BadS	BadN	GoodS	GoodN
MARKET	1.00	0.21	0.35	0.16	0.23

Probabilities:	Now	BadS	BadN	GoodS	GoodN
Probability	1	0.15	0.25	0.25	0.35

PPCs:	Now	BadS	BadN	GoodS	GoodN
PPC	1.00	1.41	1.41	0.66	0.66

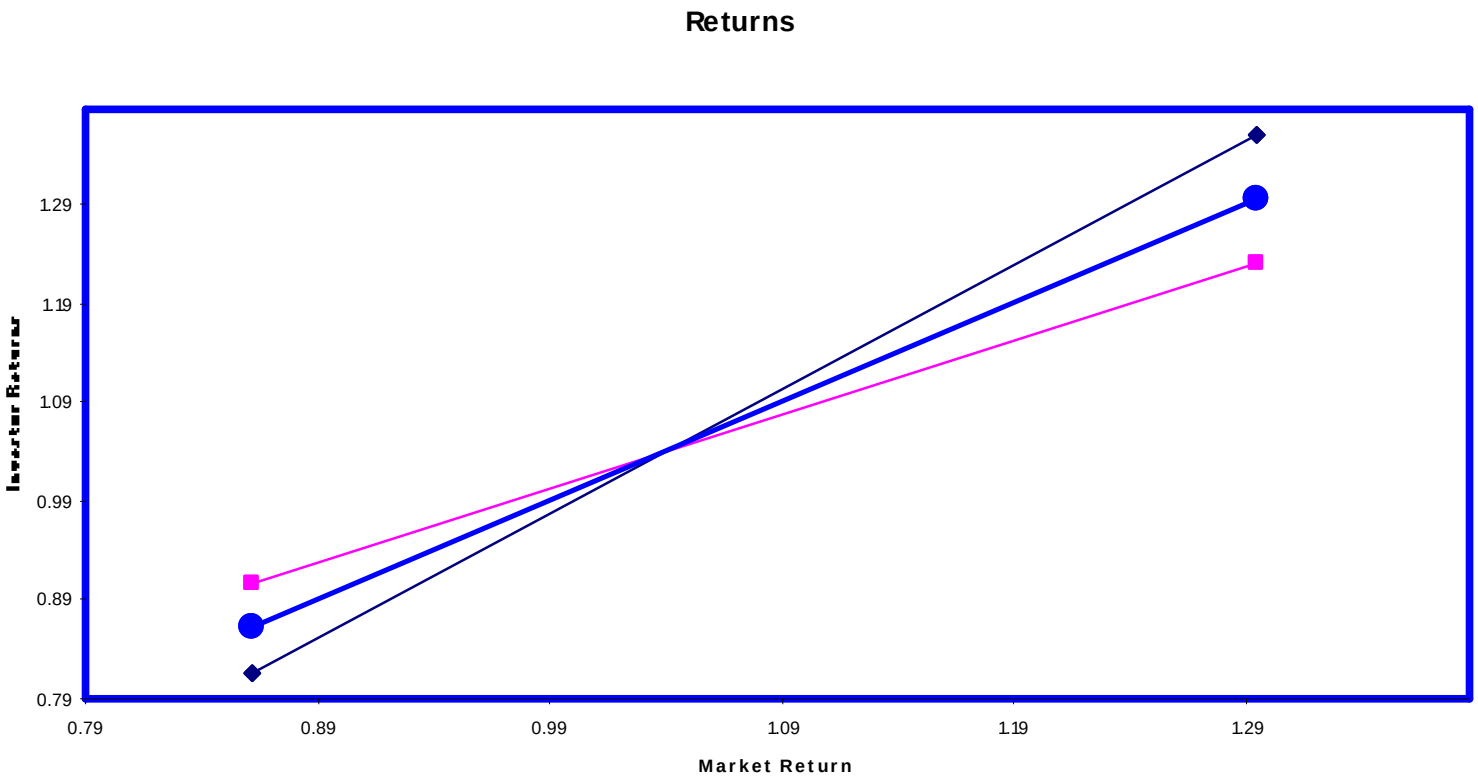
The Pricing Kernel

Case 1



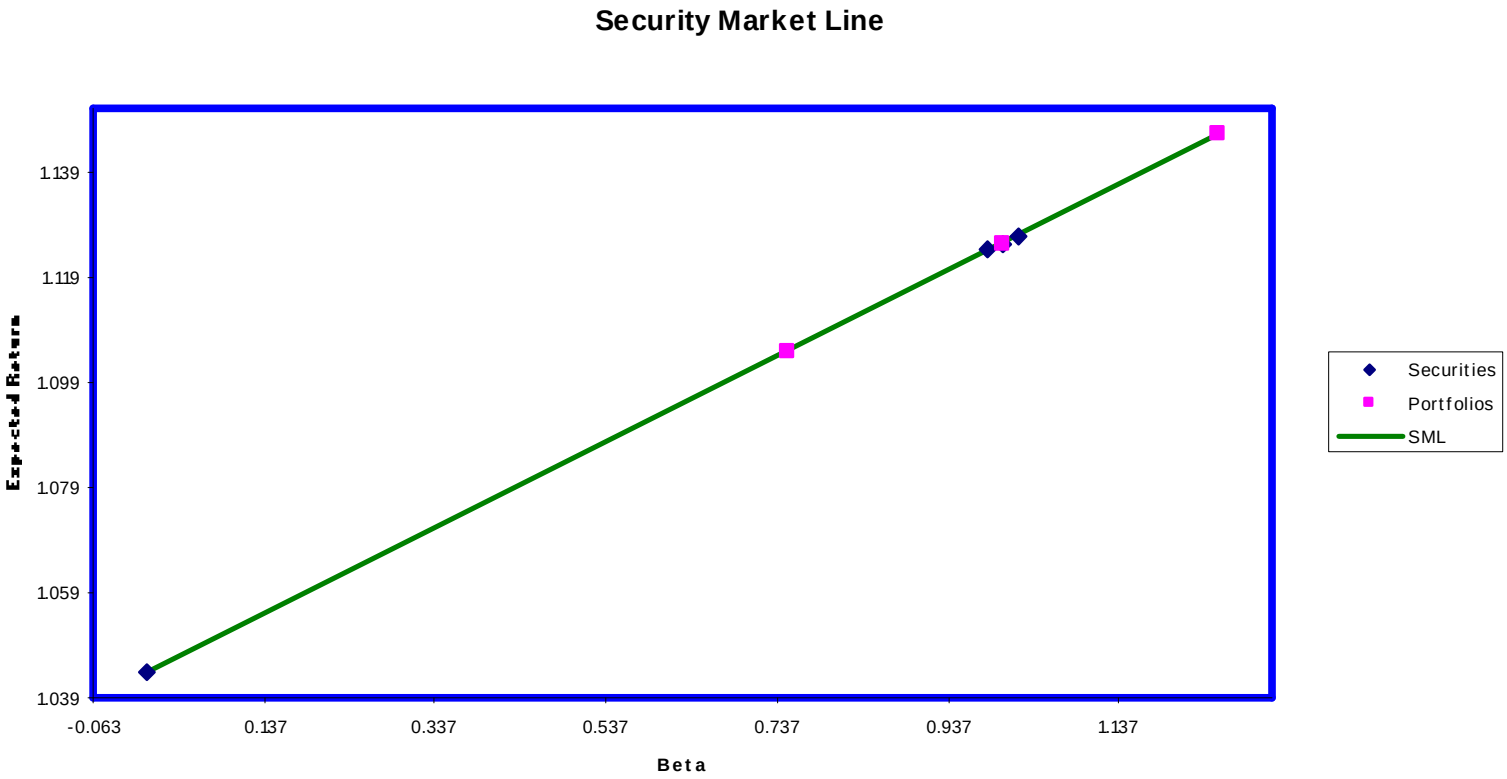
Investor and Market Returns

Case 1



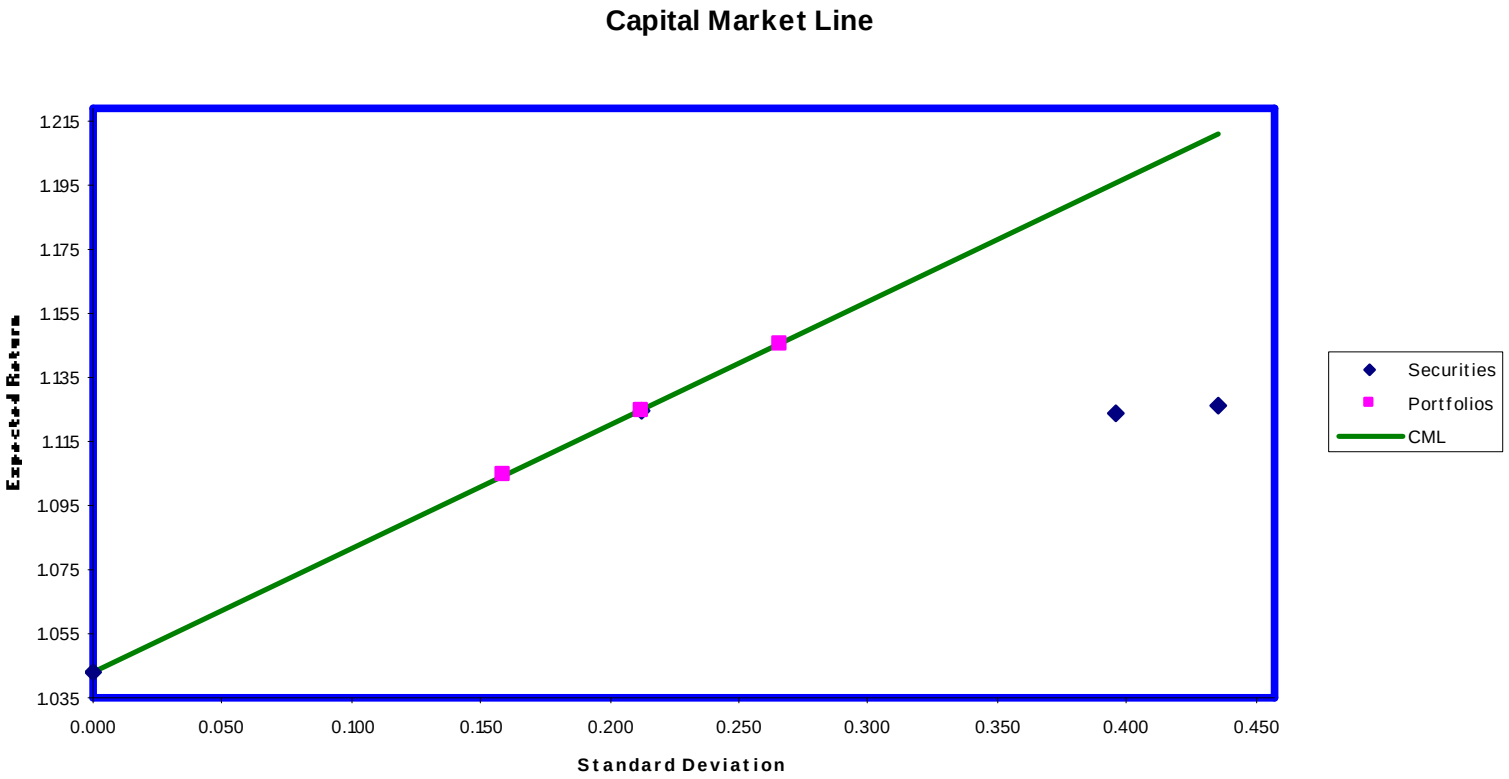
The Security Market Line

Case 1



The Capital Market Line

Case 1



Case 10
Outside Positions

Case 10

Same investors, states of the world and securities as in Case 1

Agreement

Investors have outside positions (salary income)

Total consumption is different in each state

Incomplete market

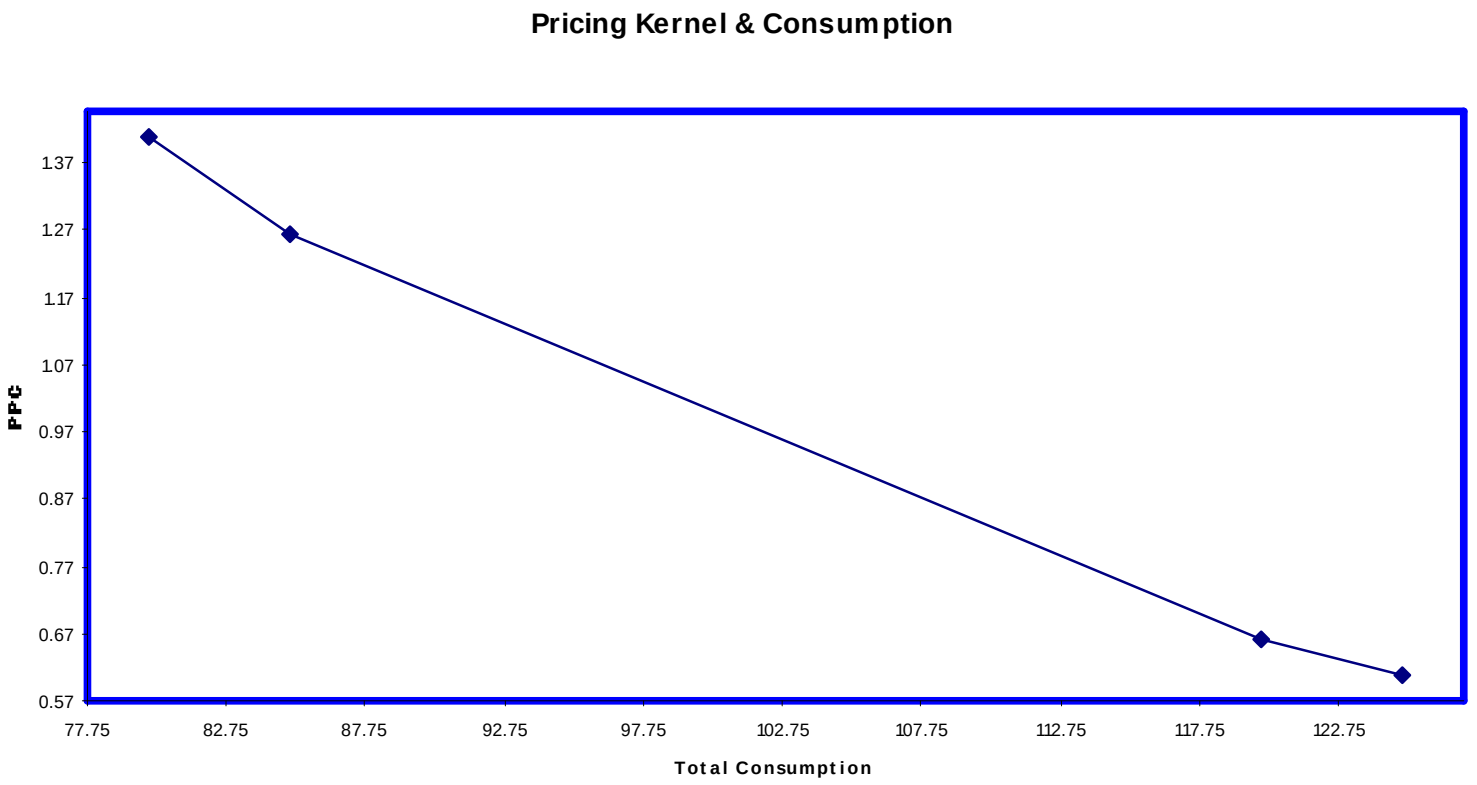
Salaries

Case 10

Salaries:	Now	BadS	BadN	GoodS	GoodN
Mario	0	30	15	45	20
Hue	0	15	25	20	40

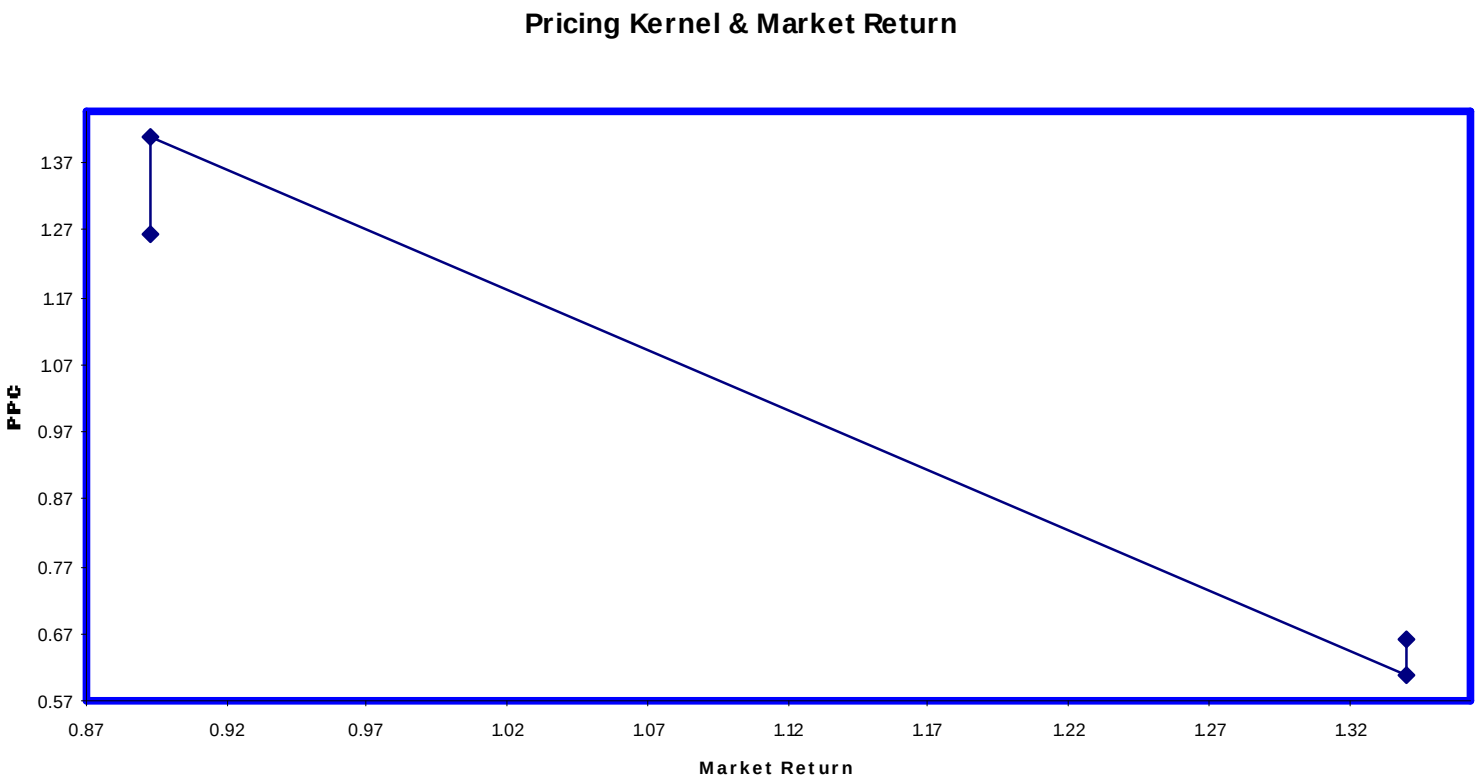
The Pricing Kernel & Consumption

Case 10



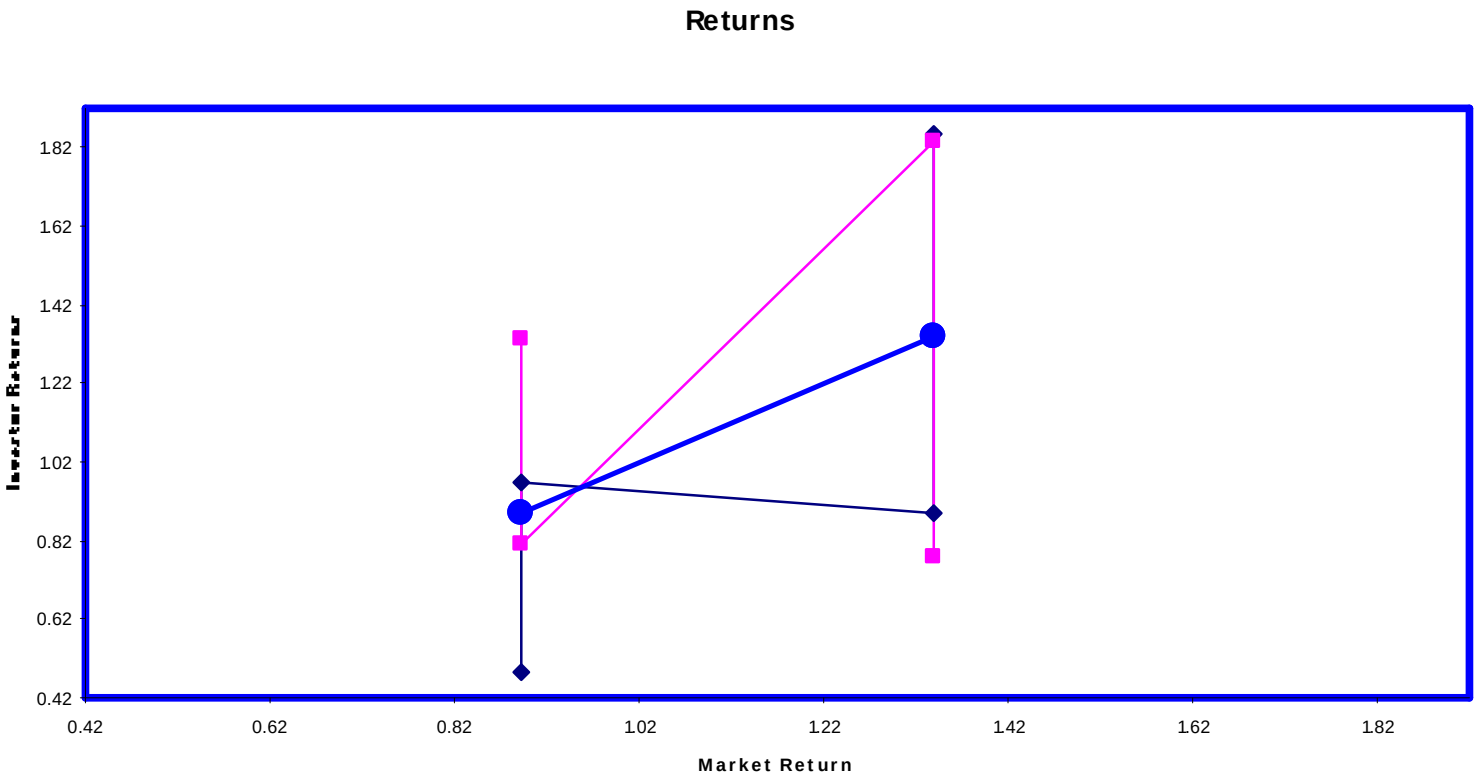
The Pricing Kernel & Market Return

Case 10



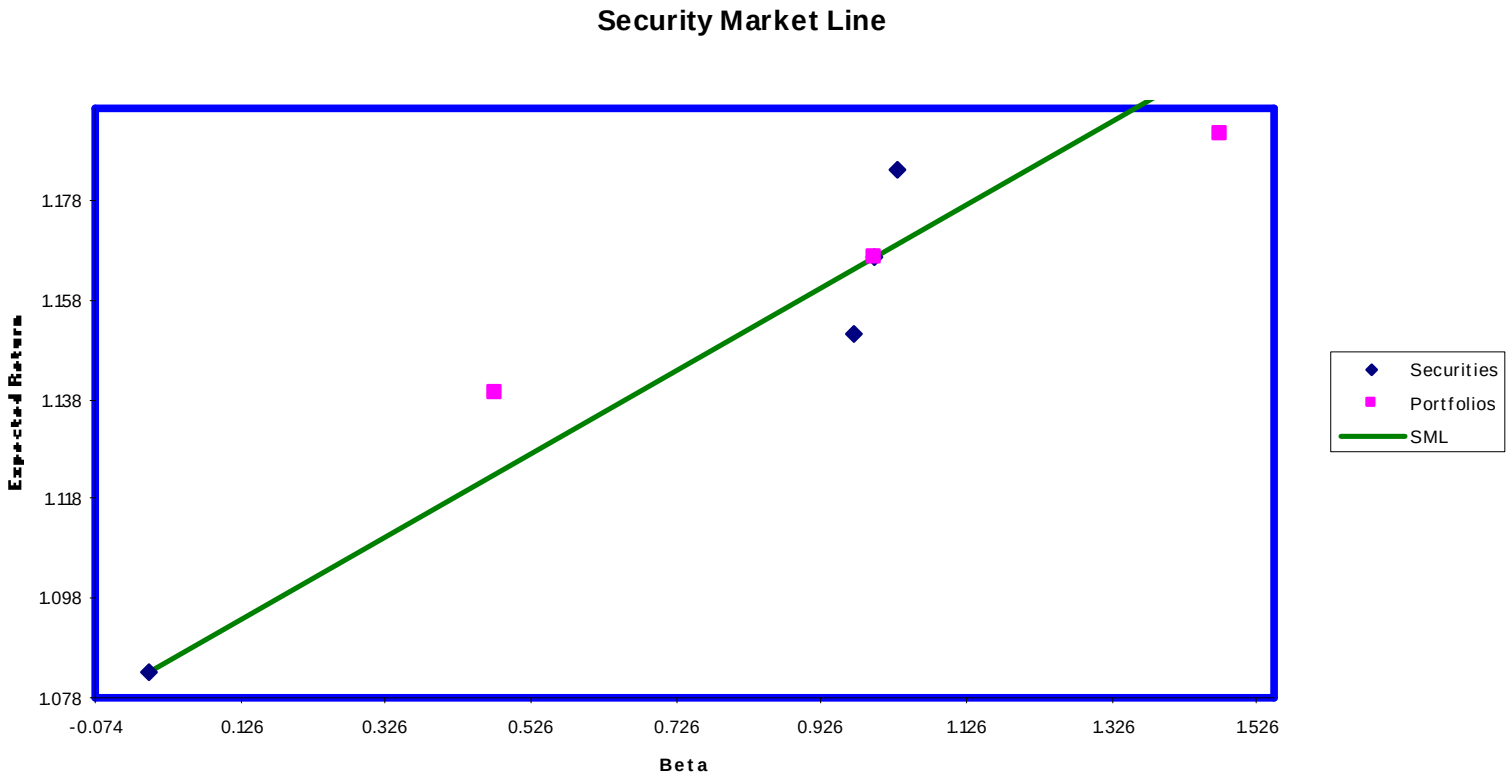
Investor and Market Returns

Case 10



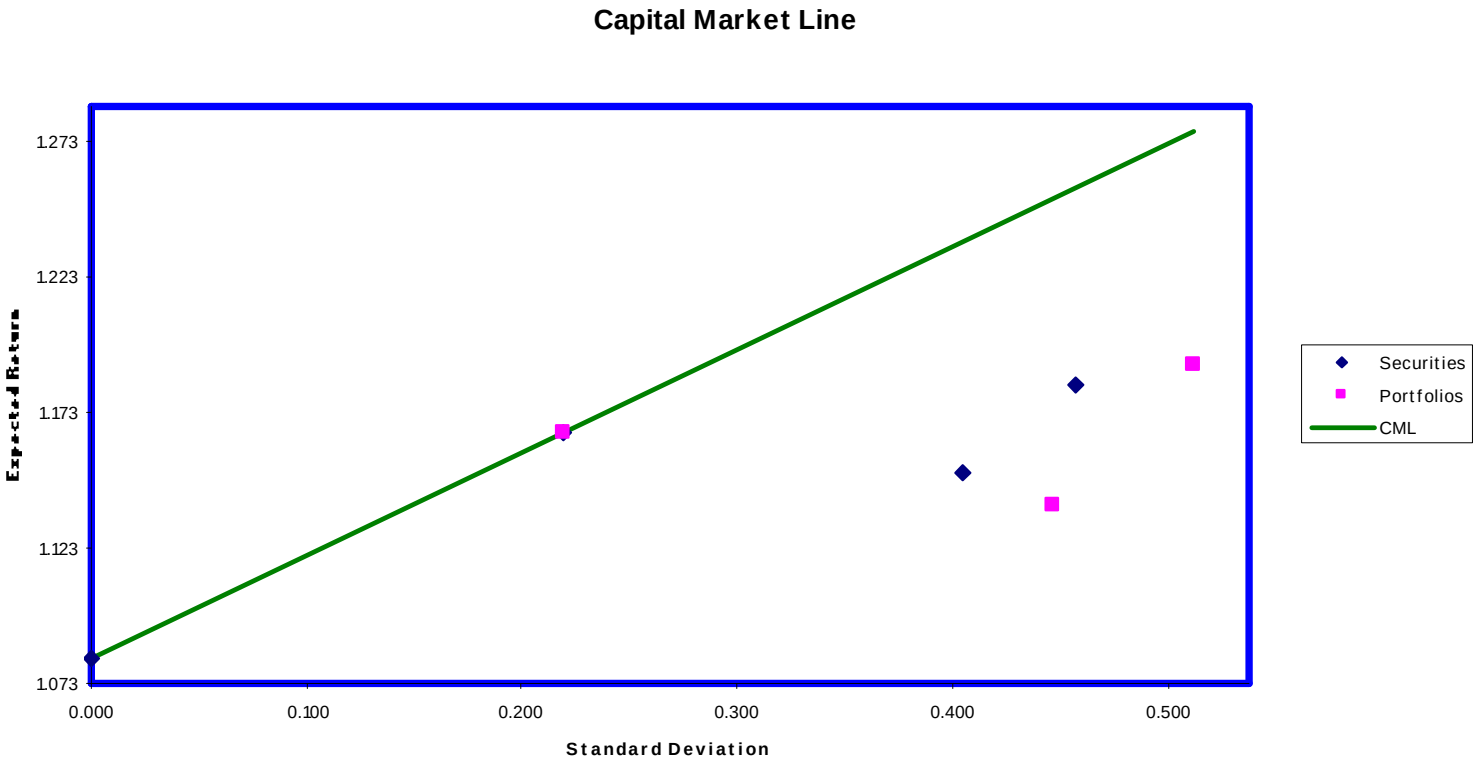
The Security Market Line

Case 10



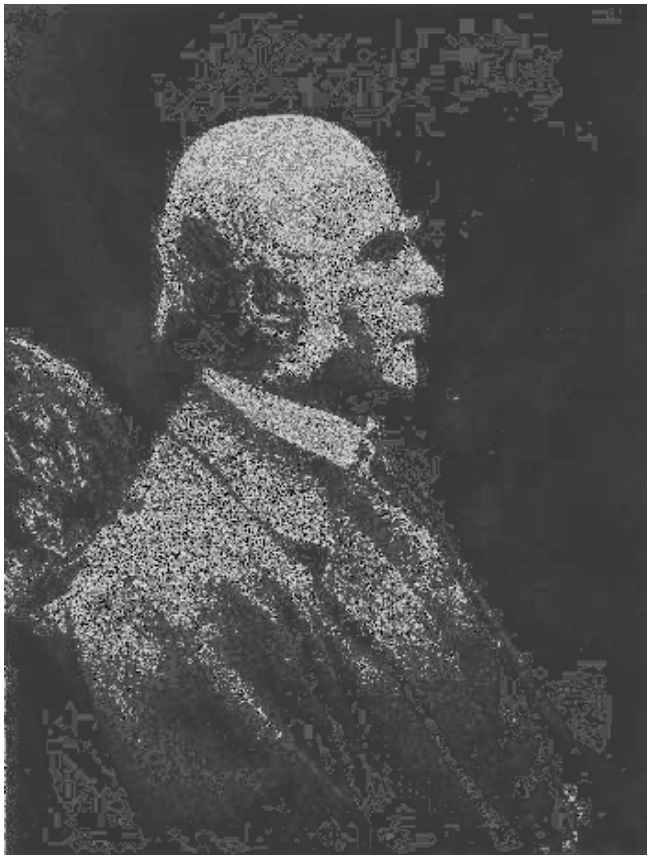
The Capital Market Line

Case 10

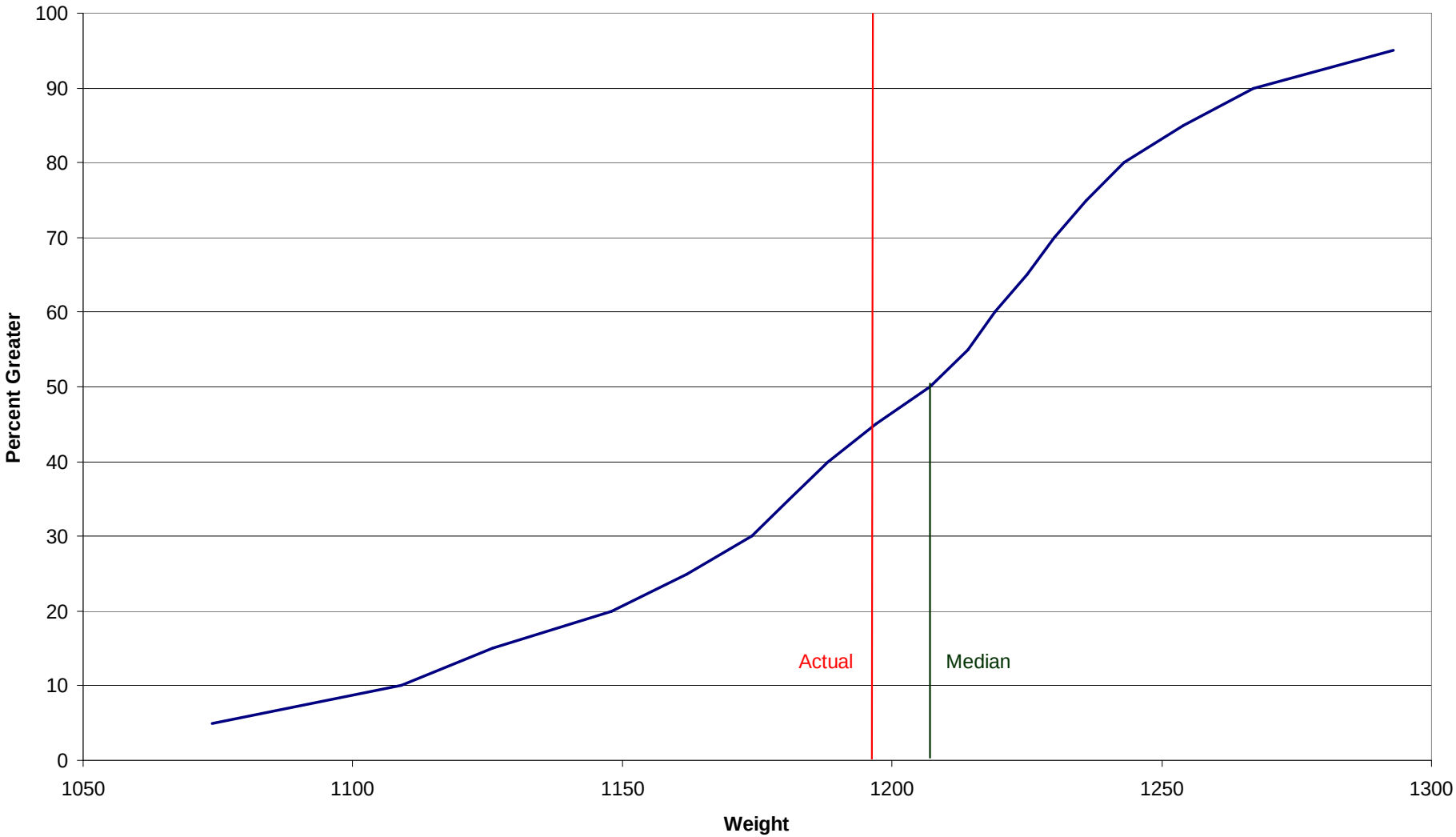


Case 15
Diverse Predictions

“Vox Populi”
Francis Galton, 1907



Estimates of Weight of Ox



The Index Fund Premise

None of us is as smart as all of us

Variation 1

Few of us are as smart as all of us

Variation 2

Few of us are as smart as all of us,
and it is hard to identify such people in advance

Variation 3

Few of us are as smart as all of us,
it is hard to identify them in advance,
and they may charge more than they are worth

Case 15

10 investors
- 5 like Mario
- 5 like Hue

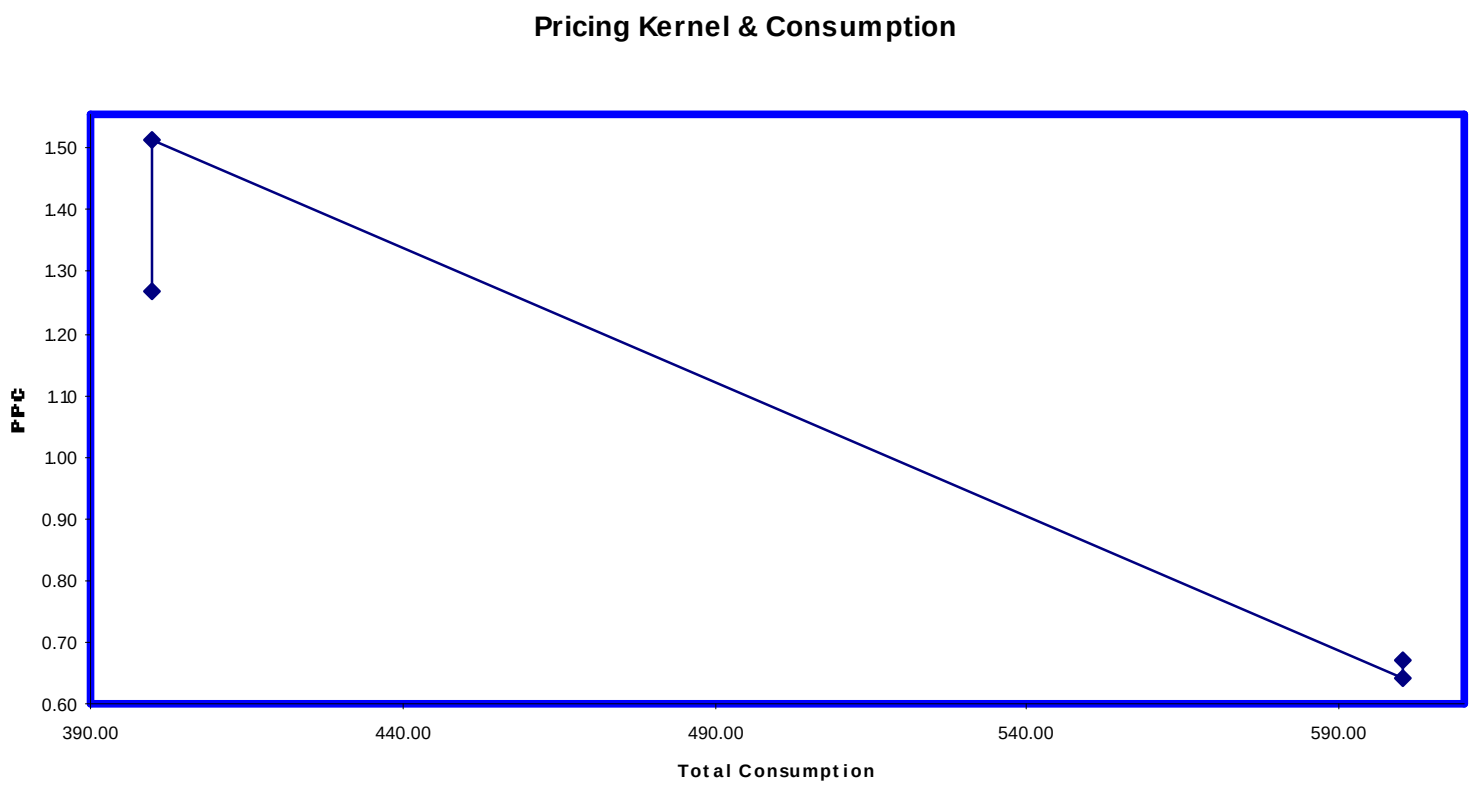
Disagreement

Predictions unbiased but subject to error
- Based on independent samples from true probability distribution

Incomplete market

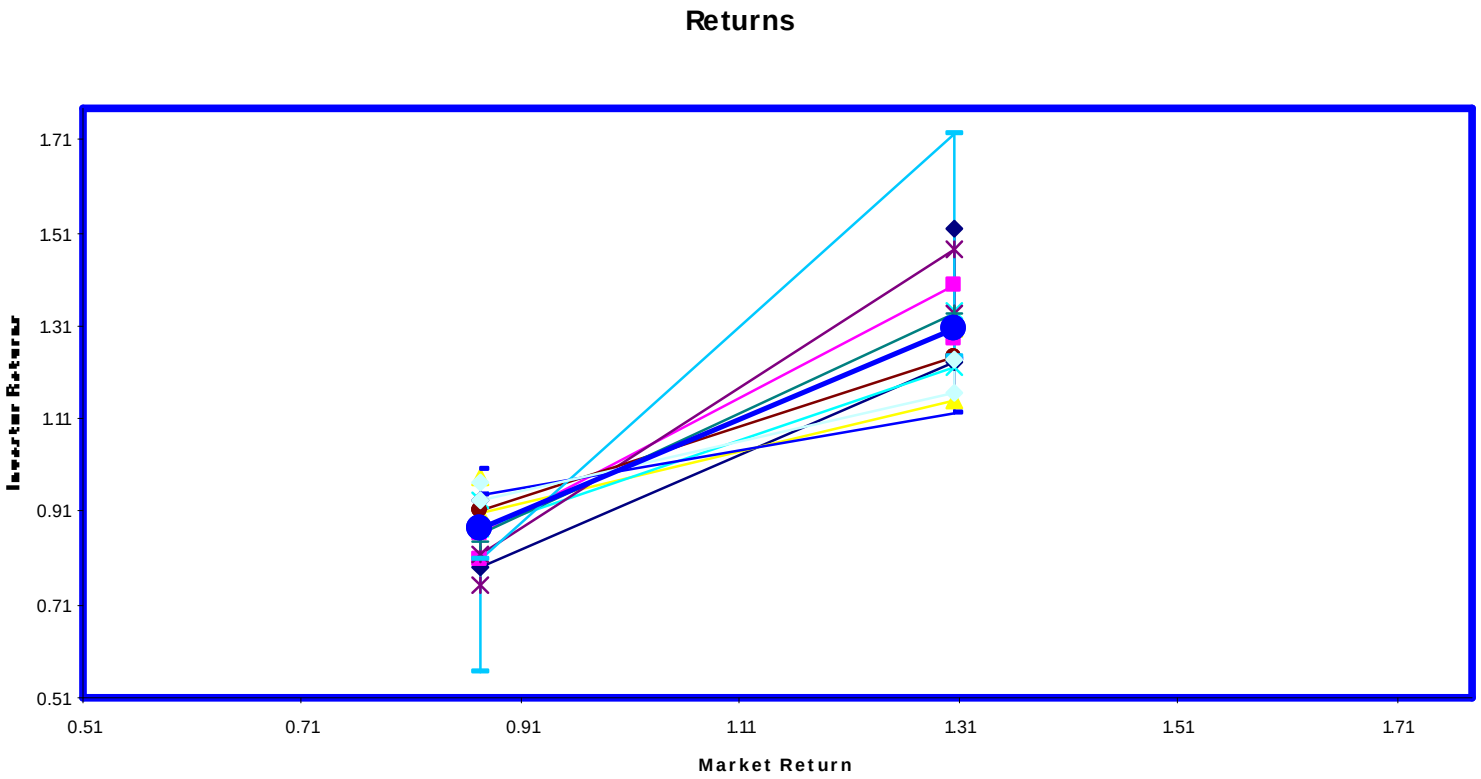
The Pricing Kernel

Case 15



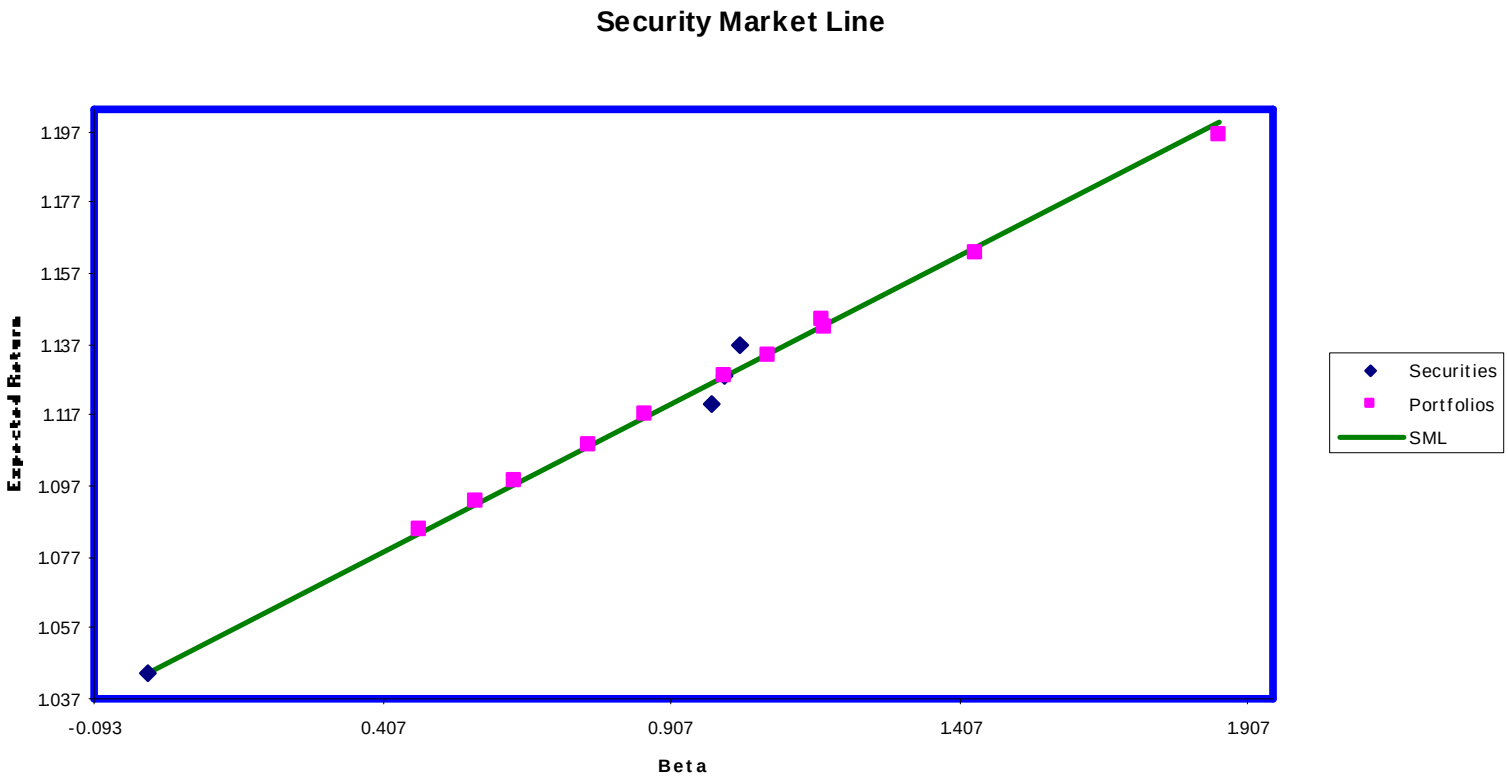
Investor and Market Returns

Case 15



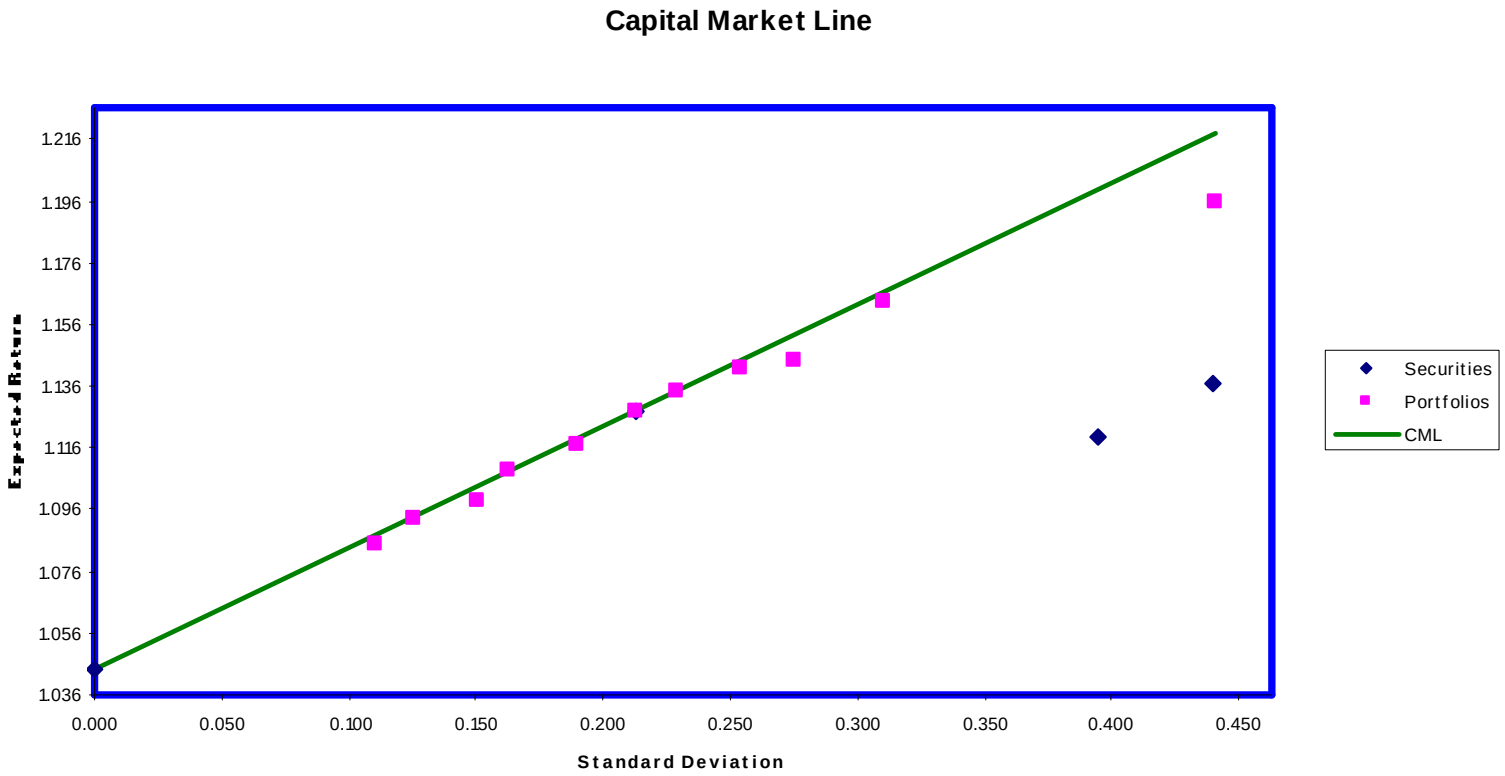
The Security Market Line

Case 15



The Capital Market Line

Case 15



Cases 21 and 23
Behavioral Preferences

Case 21

17 investors:

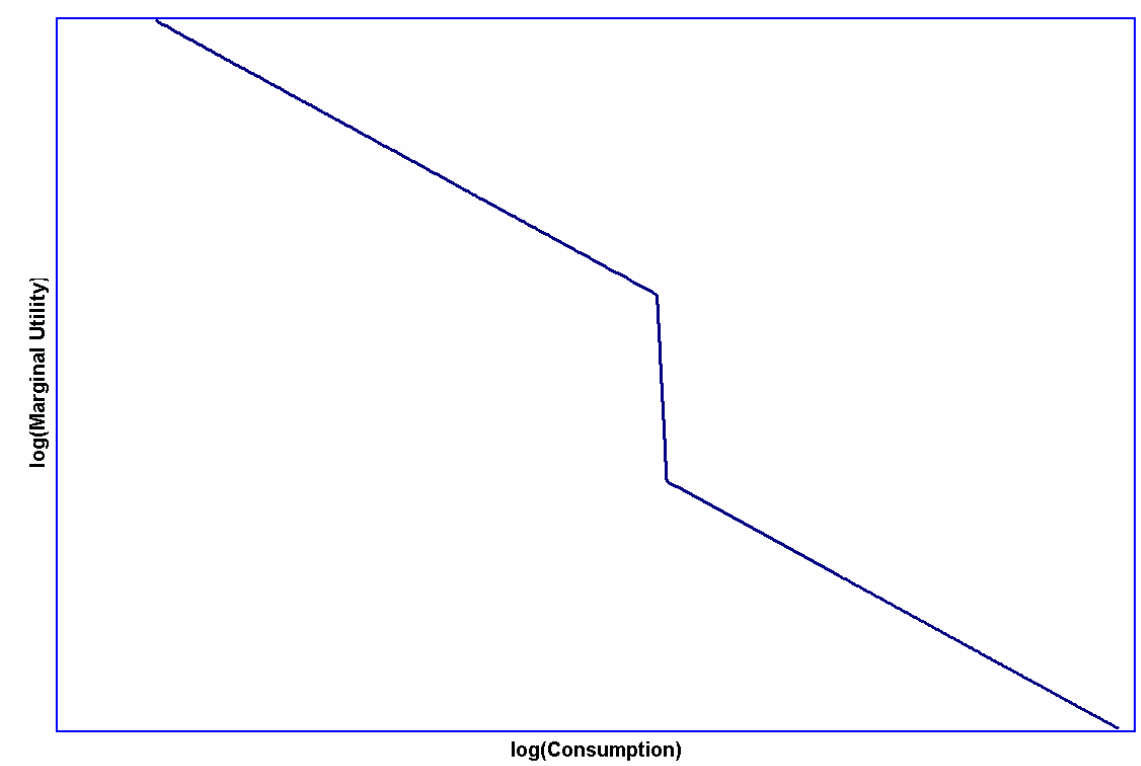
16 “standard” (constant relative risk aversion)

1 (Kevin) “behavioral” with a reference range (kinked marginal utility function)

Agreement

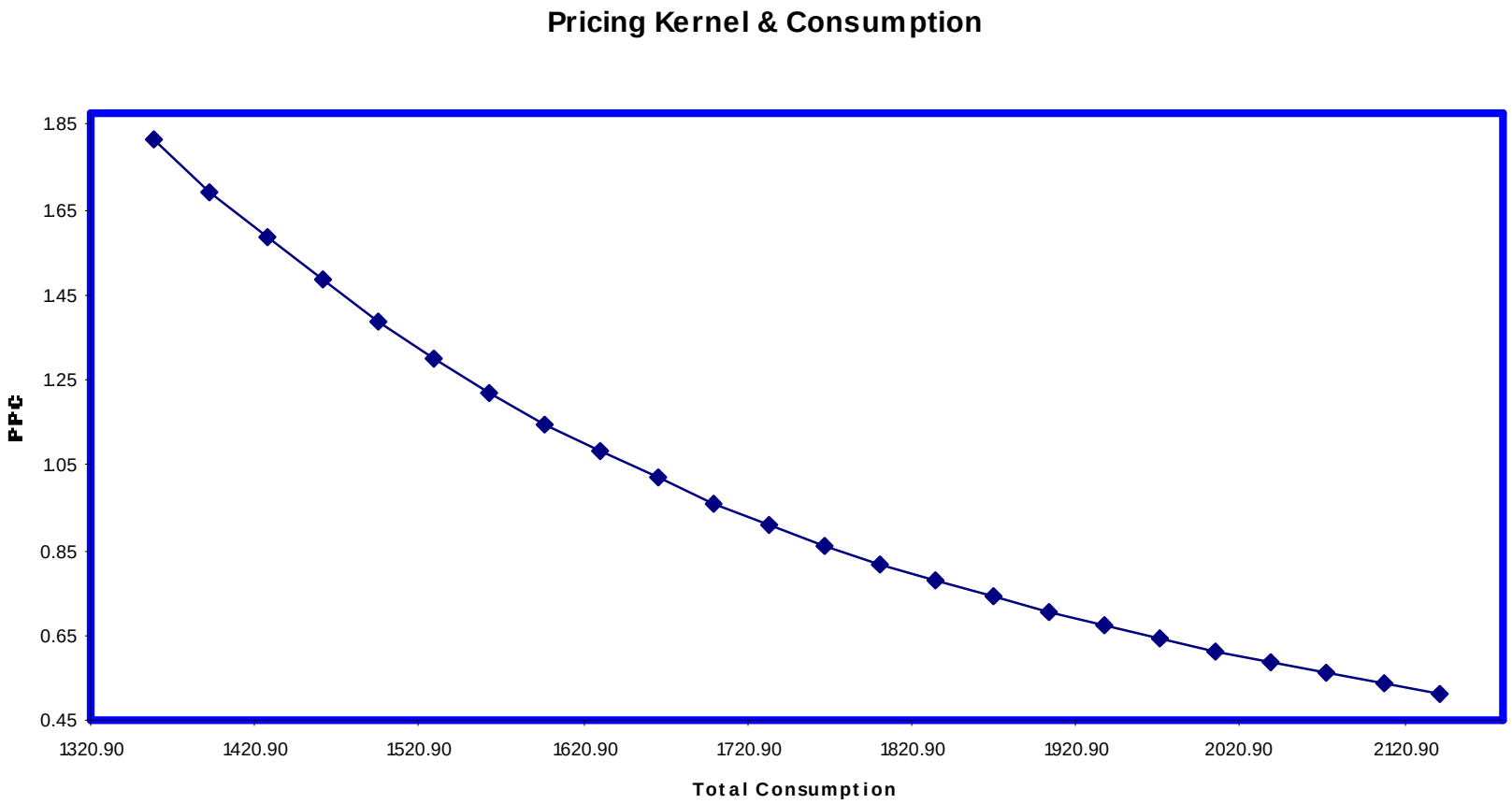
Complete market

A Kinked Marginal Utility Function



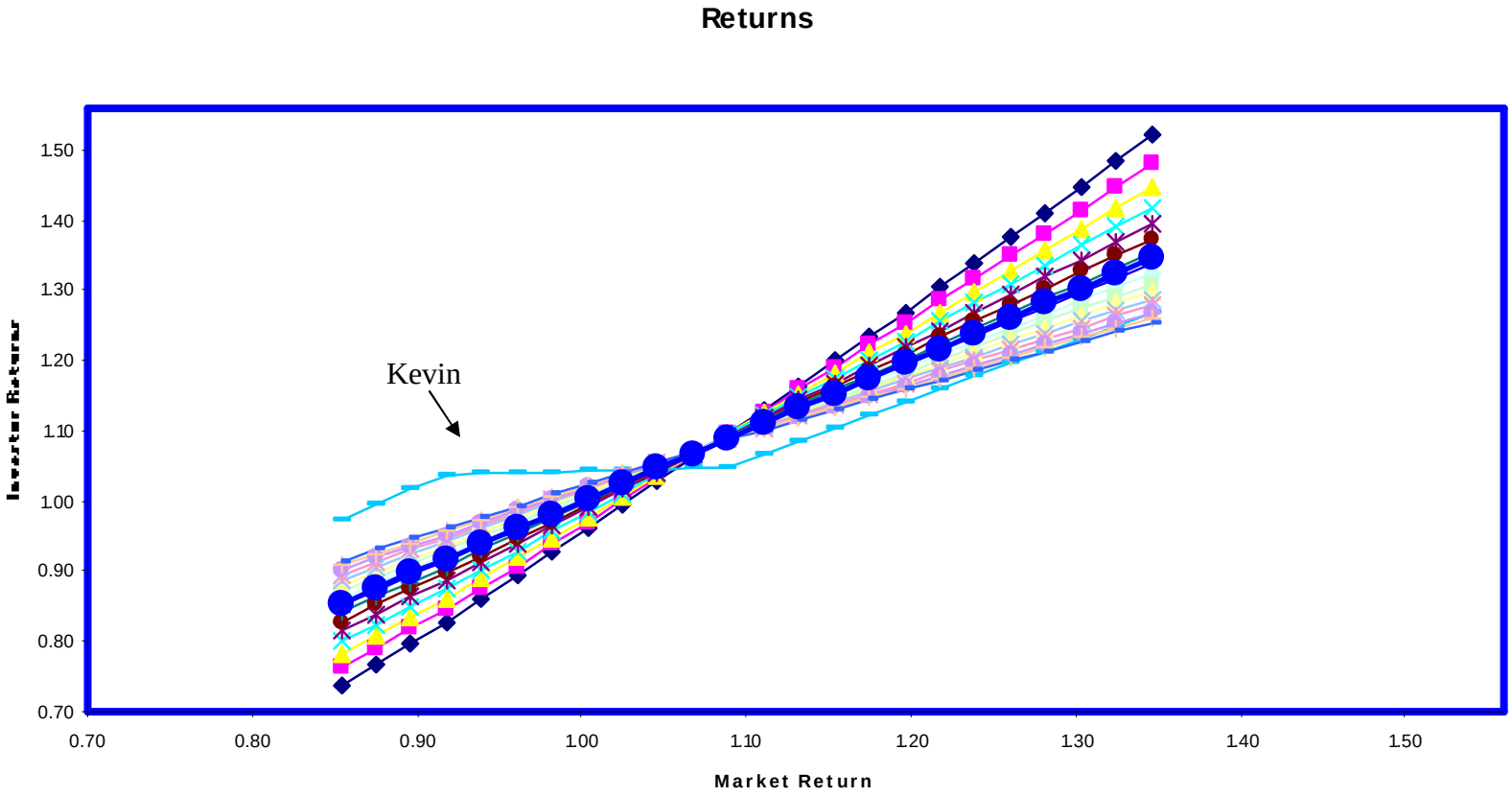
The Pricing Kernel

Case 21



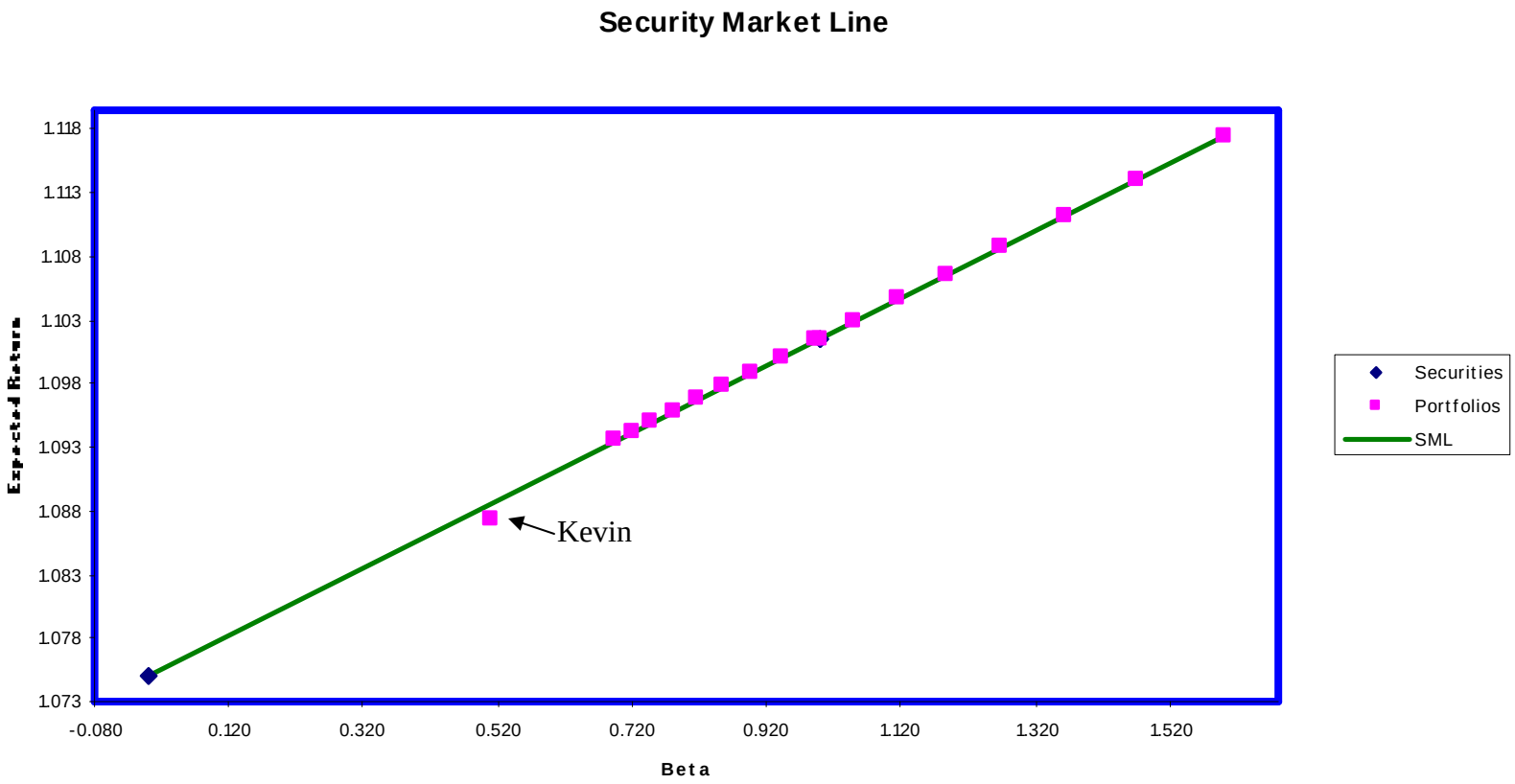
Investor and Market Returns

Case 21



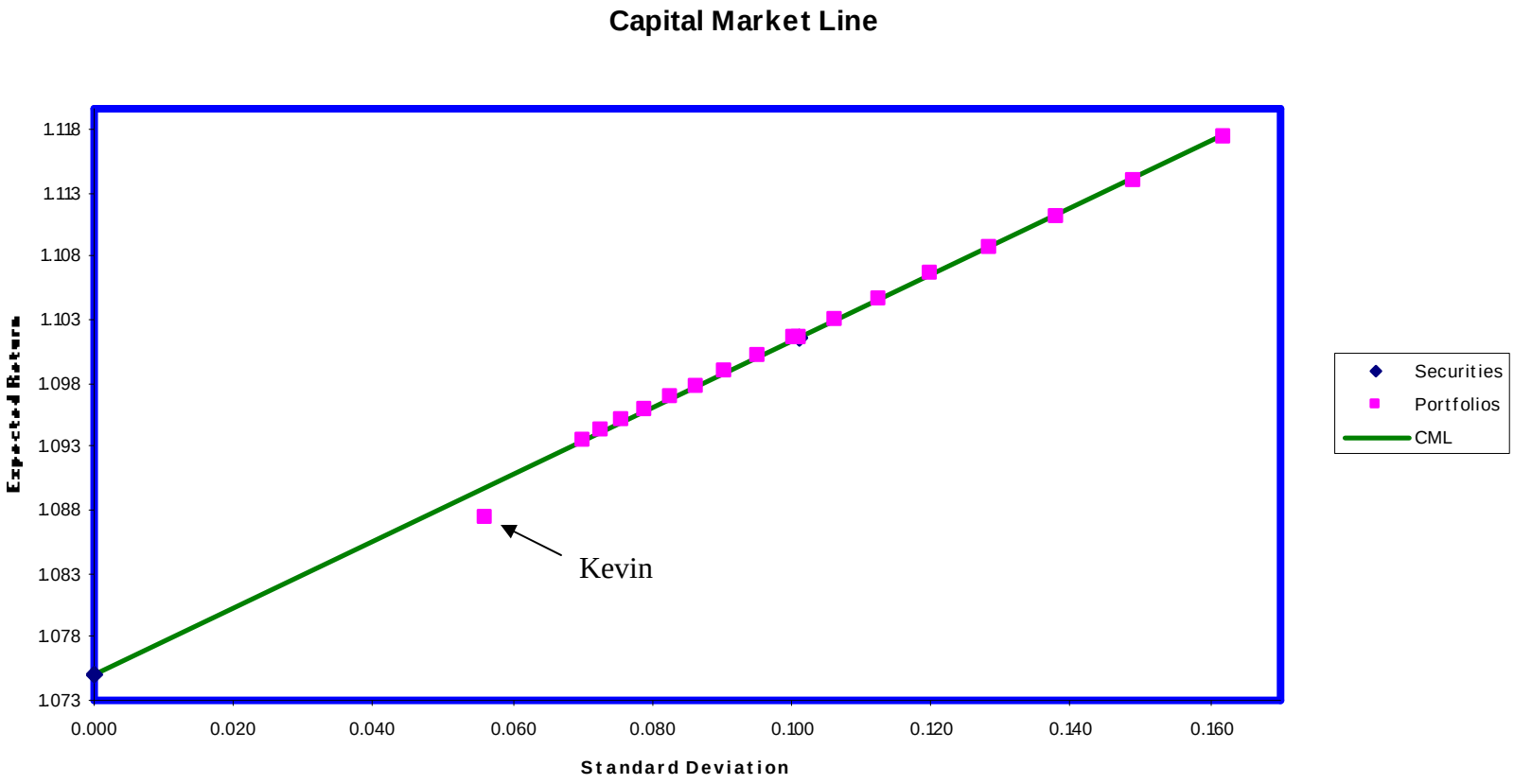
The Security Market Line

Case 21



The Capital Market Line

Case 21



Case 23

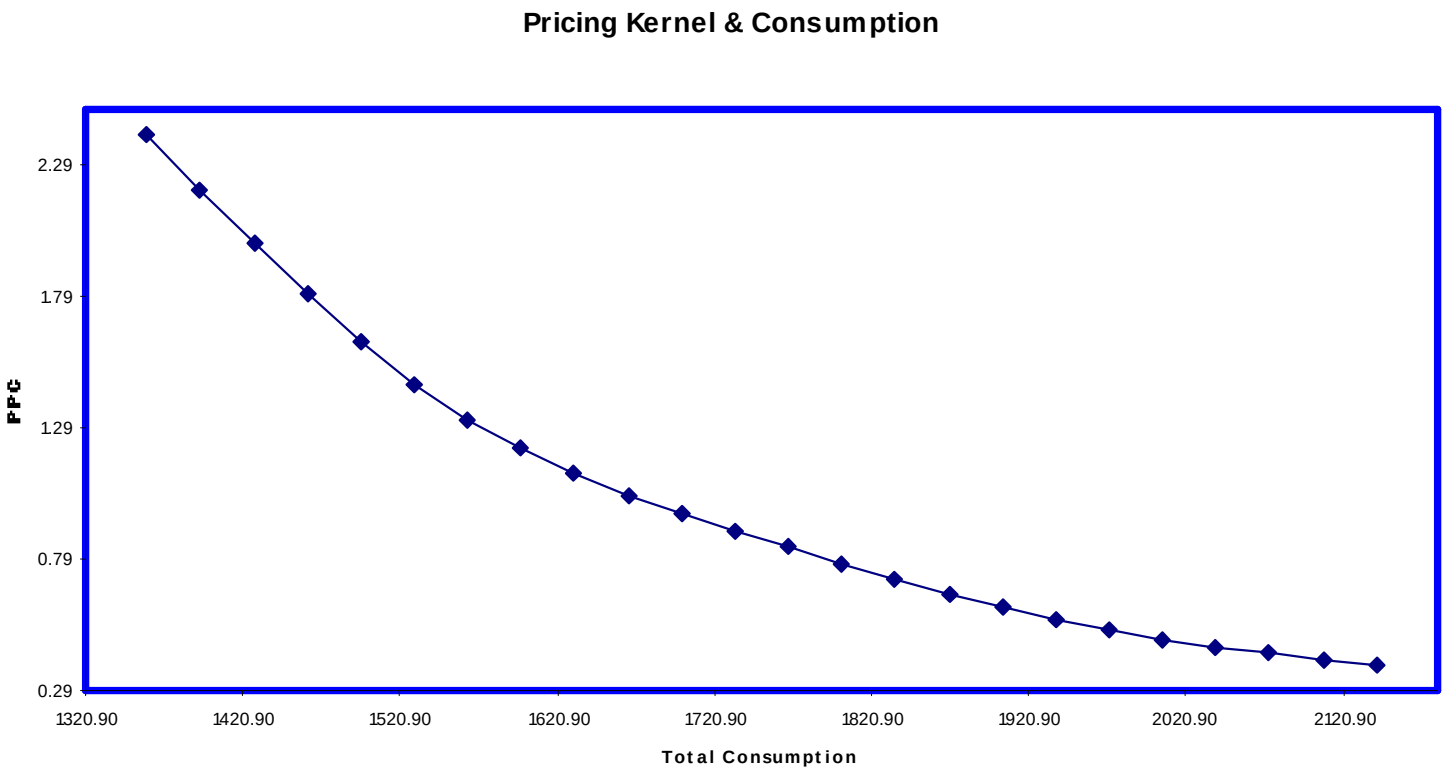
17 investors
- Each “behavioral”
- Reference ranges differ

Agreement

Complete market

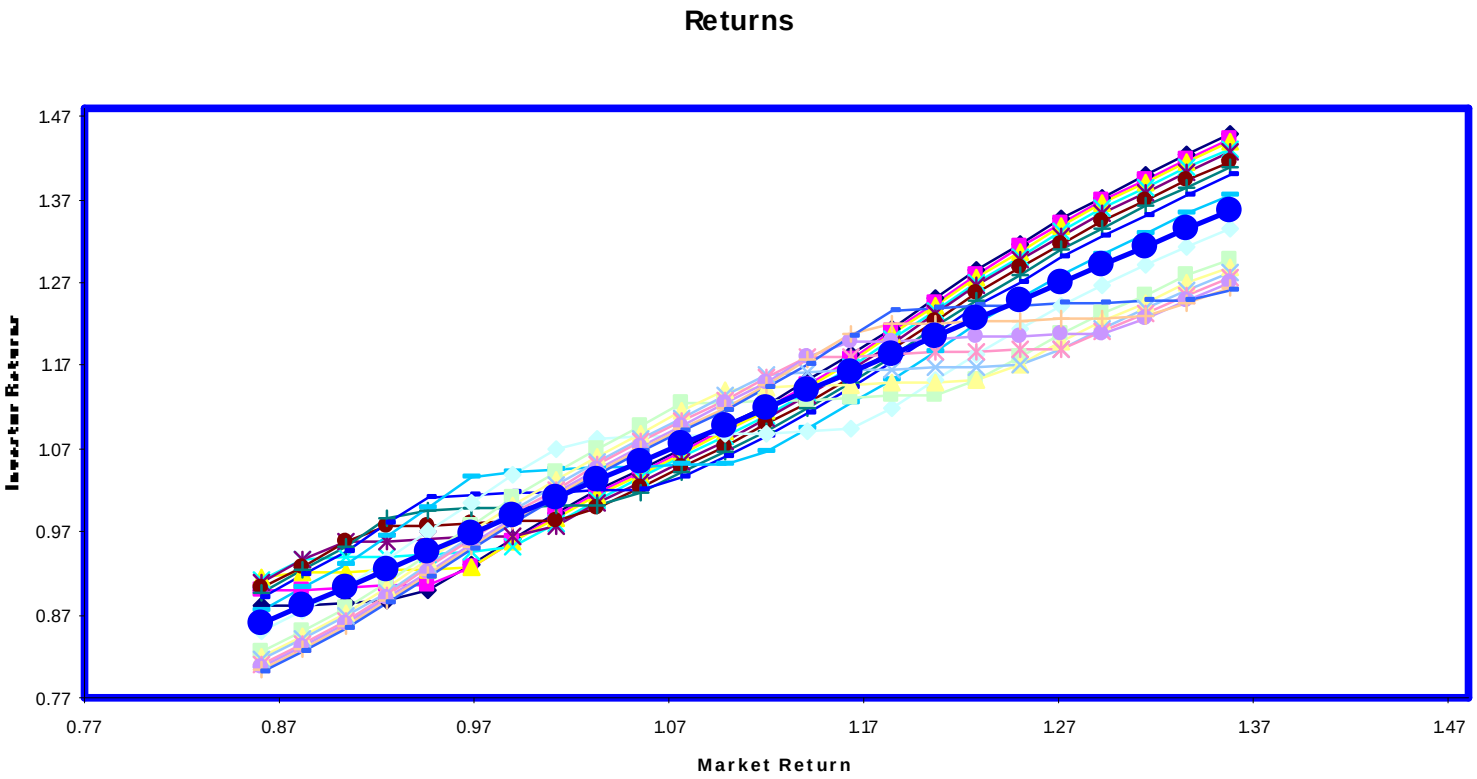
The Pricing Kernel

Case 23



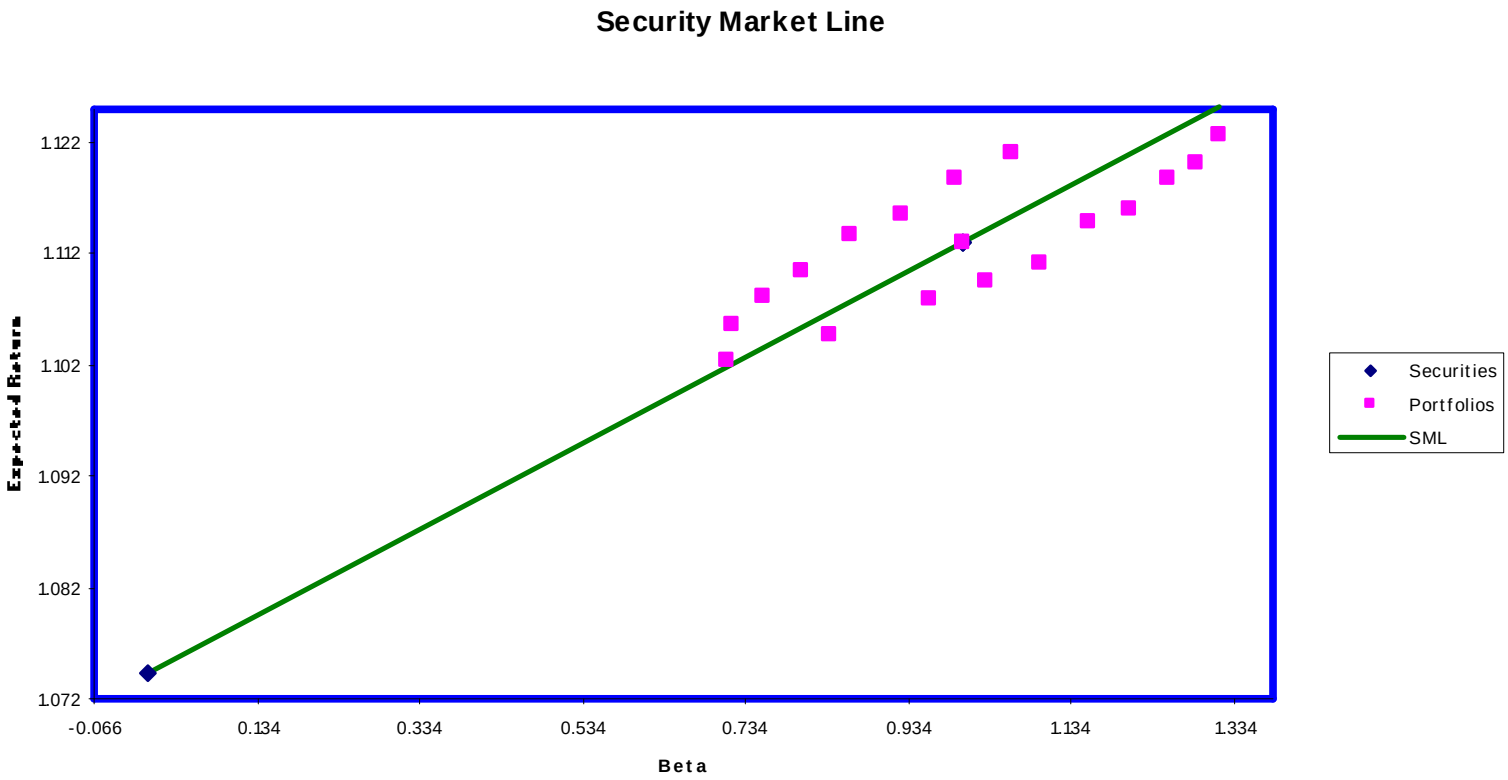
Investor and Market Returns

Case 23



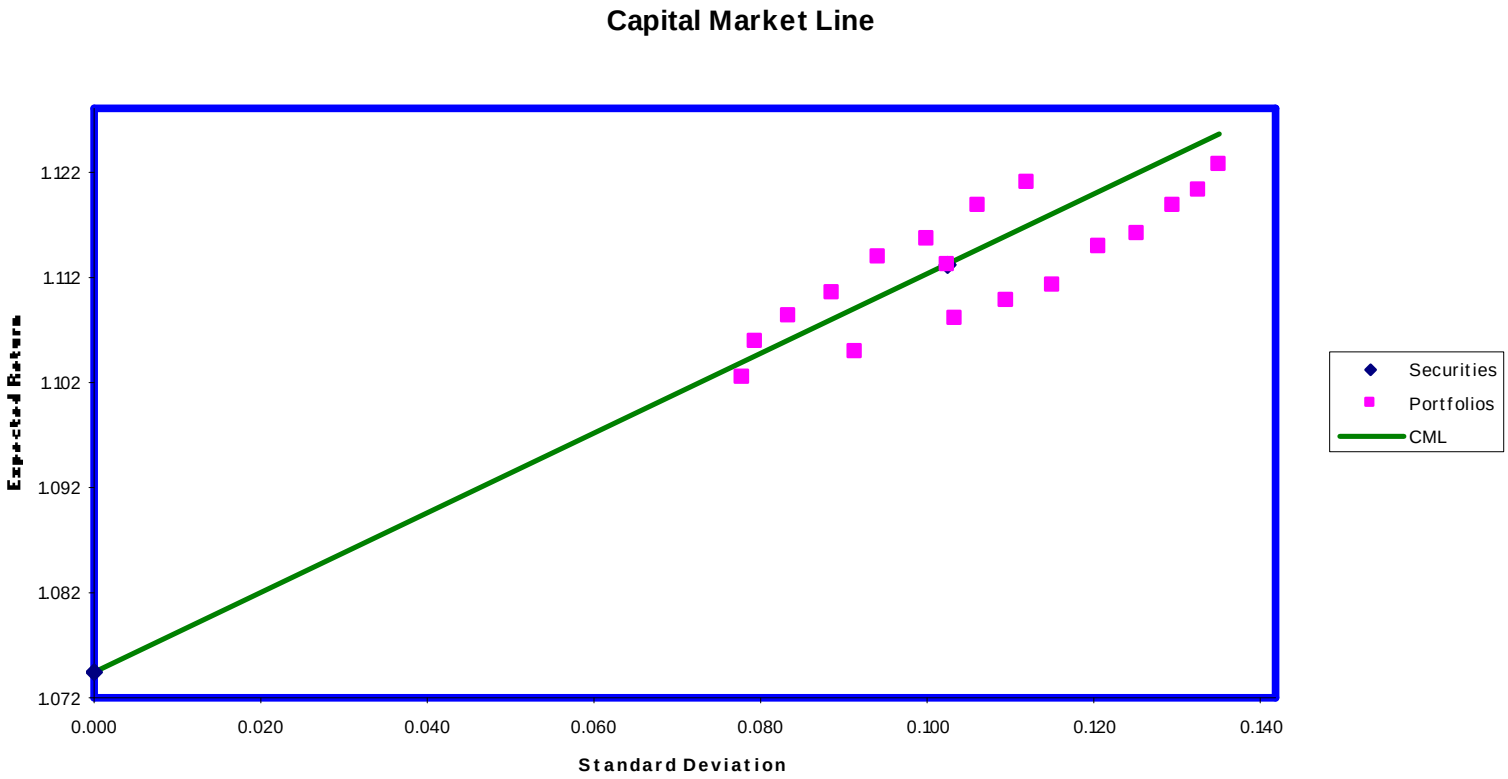
The Security Market Line

Case 23



The Capital Market Line

Case 23



Conclusions

General Observations

The MRRT version of the Market Risk/Reward Theorem holds relatively well in most cases

> Equivalently, asset prices are consistent with a pricing kernel that is a decreasing function of market return

The Market Risk/Reward Corollary fails in many cases

> Investors *do* hold portfolios with non-market risk and in at least some cases they *should* do so

Sound Personal Investment Advice

Diversify

to avoid unrewarded risk

Economize

to avoid unnecessary costs

Personalize

to take into account one's situation

Contextualize

to take into account the determinants of asset prices

Requirements for Good Investment Practice

A well thought-out view of the ways in which asset prices are determined:

an *equilibrium model and/*or *simulation*

A procedure for making forecasts of possible future returns that take into account the *current market values of assets*

such values reflect the opinions of investors worldwide concerning assets' future prospects

Without both ingredients it will be difficult or impossible to even know whether you are betting against the market and if so, in what manner.