An Introduction to (Irrotational) Shocks

Warren Li

July 30, 2024

Abstract

Expository lecture notes for lectures given during the SLMath "Mathematics of General Relativity and Fluids" Summer Graduate School 2024, July 22 – August 2.

1 Shock development and propagation

Following the lectures of Serban, our goal will be to understand in further detail the mathematical treatment of shocks in the equations of relativistic fluids. These equations describe the thermodynamical behaviour of gases in Minkowski space $(\mathbb{R}^{3+1}, \eta^{\mu\nu})$.

Recall that the main variables are a unit length future-directed timelike vector field u^μ (the fluid *velocity*), and two scalar fields ρ and s representing the *density* and *entropy* of the gas. With respect to these variables, the appropriate equations of motion are:

$$\nabla_u \rho + (\rho + p) \operatorname{div} u = 0, \tag{1.1}$$

$$\nabla_u s = 0, \tag{1.2}$$

$$(\rho + p) \nabla_u u^\mu + \Pi^{\mu\nu} \nabla_\nu p = 0, \tag{1.3}$$

where $\Pi^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu$ is a projection orthogonal to u^μ, and p is another scalar function in $\mathbb{R}^{1+3}$ representing the *pressure*. The system (1.1)–(1.3) becomes a closed first-order hyperbolic system of conservation laws after imposing an *equation of state* (coming from thermodynamics):

$$p = p(\rho, s). \tag{1.4}$$

This is not the only formulation of the equations of fluid dynamics. Another (see Serban's lecture) defines also the *particle density* $n = n(\rho, s)$, appropriately related to (1.4) via thermodynamics, as well as the particular current $I^\mu = nu^\mu$ and energy-momentum tensor $T^{\mu\nu} = (\rho+p)u^\mu u^\nu + p\eta^{\mu\nu}$. Then the system (1.1)–(1.3) is equivalent to:

$$\nabla_\mu I^\mu = 0, \quad \nabla_\mu T^{\mu\nu} = 0. \tag{1.5}$$

We consider equations (1.1)–(1.3) in the context of the *initial value problem*; in particular with initial data posed on $\Sigma_0 = \{t = 0\}$ in Minkowski space and which moreover is a constant state outside some compact set (i.e. with $\rho = \rho_0$, $s = s_0$ and $u^\mu = \left(\frac{\partial}{\partial t}\right)^\mu$ outside a compact set). For such data, local well-posedness in high-regularity Sobolev spaces follows from classical work of Kato [?].

1.1 Isentropic and irrotational fluids

For the remainder of this lecture we consider *isentropic* initial data with $s = s_0$ everywhere on Σ_0. In light of the adiabatic equation (1.2) this means that $s = s_0$ everywhere, or at least wherever the solution remains regular. For isentropic solutions, the equation of state becomes $p = p(\rho)$, and we define the particle density n and enthalpy h by:

$$n = \exp\left(\int_{\rho_0}^{\rho} \frac{1}{\tilde\rho + p(\tilde\rho)} \frac{dp}{d\rho}(\tilde\rho) d\tilde\rho\right), \quad h = \frac{p+\rho}{n}.$$

With these definitions, the main observation is that $dp = ndh$.

Introduce the 1-form β_μ as $\beta_\mu := -hu_\mu$. Using the equations (1.1) and (1.3), one computes the Lie derivative of β_μ with respect to u^μ as

$$\mathcal{L}_u \beta = dh. \tag{1.6}$$

We next define the *vorticity* 2-form by $\omega = -d\beta$. Using (1.6) and Cartan's magic formula $\mathcal{L}_X\alpha = d\iota_X\alpha + \iota_X d\alpha$, one thus has $\iota_u\omega = 0$ and thus also $\mathcal{L}_u\omega = 0$. Thereby ω is transported via the timelike vector field u – in particular if $\omega \equiv 0$ initially at Σ_0 then $\omega = 0$ wherever the solution remains smooth and perfectly described by the initial data.

We call such a solution an *irrotational* solution to (1.1)–(1.3). Ignoring topological issues, $\omega = 0$ implies that β is exact and therefore there exists a scalar *potential* ϕ such that

$$\beta_\mu = -hu_\mu =: \nabla_\mu\phi.$$

Furthermore, in the isentropic and irrotational case the variables ρ, p may be recovered from ϕ since $h = \sqrt{-\eta(\nabla\phi, \nabla\phi)}$ and via the equation of state, p and ρ can be considered functions of h.

Thereby in the irrotational and isentropic case, the dynamics reduces to one equation for the scalar quantity ϕ. This comes from the first equation in (1.5), and is equivalent to:

$$\nabla_\mu \left(\frac{p+\rho}{h^2}(d\phi)\, \nabla^\mu \phi \right) = 0.$$

Therefore, upon defining $G(d\phi) = \frac{\rho+p}{h^2}$ and $H^\mu(d\phi) = G \cdot \partial^\mu \phi$, we have the divergence form equation

$$\nabla_\mu \left(H^\mu(d\phi)\right) = 0. \tag{1.7}$$

Moreover, by applying the chain rule we can write this as a *quasilinear wave equation*

$$h^{\alpha\beta}(d\phi)\partial_\alpha\partial_\beta\phi = 0, \qquad h^{\mu\nu} = \frac{1}{2}\left(\frac{dH^\mu}{d\xi_\nu} + \frac{dH^\nu}{d\xi_\mu}\right), \tag{1.8}$$

where the hyperbolicity of the equation arises from the fact that $h^{\mu\nu}$ is a Lorentzian metric, namely (G times) the *inverse acoustical metric*, where $c^2 = \frac{dp}{d\rho}$, assumed to have $0 < c^2 \le 1$:

$$(h^{-1})^{\mu\nu} = G\left(\eta^{\mu\nu} - (c^{-2} - 1)u^\mu u^\nu\right) = G(-c^{-2}u^\mu u^\nu + \Pi^{\mu\nu}). \tag{1.9}$$

Hence u^μ is timelike and propagates at the speed of sound c, while directions normal to u^μ in Minkowski space remain spacelike with unchanged length. Note

$$h_{\mu\nu} = G^{-1}\left(\eta_{\mu\nu} - (c^2 - 1)u_\mu u_\nu\right) = G^{-1}(-c^2 u_\mu u_\nu + \Pi_{\mu\nu}). \tag{1.10}$$

It remains to study the quasilinear wave equation (1.8), with initial data $(\phi_0, \phi_1) = (\phi, \partial_t\phi)|_{\Sigma_0}$. Local well-posedness for $\phi_0 \in H^{s+1}, \phi_1 \in H^s$ follows for $s \ge \frac{d}{2} + 1$ by Hughes–Kato–Marsden [?]. So there exists a regular local solution to (1.8). However:

1. It is long been known that for general equations of state the solution to (1.8) does not exist globally in $\mathbb{R}^{1+3}$. (See Alinhac, Sideris, John.)

2. Even for "small data", meaning perturbations of the constant density state $\rho \equiv \rho_0, s \equiv s_0, u_\mu = (dt)_\mu$, one may expect breakdown of global regularity since upon Taylor expanding $h(d\phi)$ in (1.8) it can be checked that the nonlinearity does not obey the *null condition* with respect to the background acoustical metric $(h_0)_{\mu\nu} = -c_0^2 dt^2 + dx_1^2 + dx_2^2 + dx_3^2$, where $c_0^2 = \frac{dp}{d\rho}(\rho_0)$.

3. In Christodoulou's 2008 monograph "The Formation of Shocks in 3-Dimensional Fluids" a mechanism for this breakdown of regularity is understood, namely that while $\partial\phi$ remains

bounded the $\partial^2\phi$ derivatives will blow up on a codimension one singular hypersurface $\mathcal{B}$, whose past (2-dimensional) boundary $\partial_{-}\mathcal{B}$ is known as the pre-shock or crease.

See the later problem set for a explanation of this mechanism in spherical symmetry.

However, it turns out that although classical solutions eventually break down one expects to be able to continue them as *singular weak solutions*, in particular weak solutions containing hypersurfaces of discontinuity called *shocks*. This is because:

1. This is how it works in 1D hyperbolic systems of conservation laws. See work of Riemann, Lax, Hopf concerning the related 1D theory, e.g. for the Burgers equation.

2. It is known that if one has initial data "containing a shock", but regular away from the shock, then one can locally continue the solution with the shock. For the equation (1.8), or more precisely (1.7), this is Majda–Thomann 1987 [?], which we present in the following section. In fact shock propagation can be done for the full system (1.5), see Majda 1983 [?,?].

1.2 Propagation of Shocks

Here we present the main ideas behind the proof of local shock propagation as given in Majda–Thomann. Their result concerns general equations of the form:

$$\partial_\mu \left(H^\mu(\partial\phi) \right) = 0, \tag{1.11}$$

in $\mathbb{R}^{1+D}$, where the nonlinear function $H^\mu(\xi)$ is such that the bilinear form

$$h^{\mu\nu} = \frac{1}{2}\left(\frac{\partial H^\mu}{d\xi_\nu} + \frac{\partial H^\nu}{d\xi_\mu} \right), \tag{1.12}$$

is of Lorentzian signature. We define a *shock-containing* solution to be:

Definition 1.1. We say that ϕ is a shock-containing solution to (1.11) in a domain $\Omega \subset \mathbb{R}^{1+D}$ if there exists a regular hypersurface $\mathcal{S}$ dividing Ω into two regions $\Omega = \Omega^+ \cap \Omega^-$ and two regular (say H^s for sufficiently large s) functions ϕ^+ and ϕ^- defined in $\Omega^+ \cup \mathcal{S}$ and $\Omega^- \cup \mathcal{S}$ respectively, such that

$$\phi^\pm \text{ solves (1.11) classically in } \Omega^\pm$$

and moreover ϕ, which is equal to $\phi^\pm$ in $\Omega^\pm$ is a *weak solution* to the divergence form equation (1.11) in the sense that the following *jump conditions* hold on $\mathcal{S}$, where $[f] = f^+|_{\mathcal{S}} - f^-|_{\mathcal{S}}$:

$$[\phi] = 0, \tag{1.13}$$

$$\zeta_\mu \cdot [H^\mu(\partial\phi)] = 0, \tag{1.14}$$

where ζ_μ is a one-form conormal to the hypersurface $\mathcal{S}$.

Remark. It is clear that the condition (1.14) exactly corresponds to the function ϕ being a weak solution to the divergence form equation (1.11). The other jump condition (1.13) can be thought of as a condition needed for the irrotationality condition $\omega = 0$ to hold weakly.

As a result, these jump conditions are not the physical jump conditions required for our physical variables (u^μ, ρ, s) to solve the conservation laws (1.5) weakly. Indeed, one expects that across a shock there will be a positive increase in entropy and also a jump in vorticity. See Serban's talk.

Nevertheless, for the remainder of these lectures we restrict ourselves to shocks as in Definition 1.1. This is often known as the *restricted problem* where the modified jump condition (1.13) means that we force the solution to remain irrotational (and isentropic) either side of the jump. We now state the shock propagation theorem of Majda–Thomann.

Theorem 1.1 (Majda–Thomann 1987)**.** *Let $s > \frac{d}{2} + 1$. Consider shock front initial data on Σ_0 as follows: let $\mathcal{S}_0$ be a codimension one hypersurface in $\mathbb{R}^d$ dividing $\mathbb{R}^d$ into two regions Σ^+ and Σ^-, and let $(\phi_0^\pm, \phi_1^\pm)$ be functions on $\Sigma^\pm$ with $\phi_0^\pm \in H^{s+1}(\Sigma^\pm)$ and $\phi_1^\pm \in H^s(\Sigma^\pm)$. To ensure that Σ_0 is spacelike, we impose that $h^{\mu\nu}(\partial\phi_0^\pm, \phi_1^\pm)\partial_\mu t \partial_\nu t < 0$.*

Suppose moreover that the initial data $(\mathcal{S}_0, \phi_0^\pm, \phi_1^\pm)$ satisfies further conditions:

- *The determinism condition, and*
- *the linearized stability condition,*

as well as several higher order compatibility conditions. Then there exists $T > 0$ such that there exists a unique shock containing solution to (1.8) in $[0, T] \times \mathbb{R}^d$ with $\mathcal{S} \cap \Sigma_0 = \mathcal{S}_0$ and such that $\phi_0^\pm$, $\phi_1^\pm$ are equal to $\phi^\pm$ and $\partial_t\phi^\pm$ at Σ_0 respectively.

Remark. The theorem allows for shocks of any topology (e.g. spherical and planar shocks are both allowed) with $\mathcal{S}$ diffeomorphic to $[0, T] \times \mathcal{S}_0$. Also due to finite speed of propagation, Theorem 1.1 can be spatially localized and may be applied to multiple shocks, so long as the initial data for each shock satisfies the required conditions.

We spend the remainder of this section explaining the meaning of the determinism condition and the linearized stability condition above, justifying their necessity in Theorem 1.1.

The *determinism* condition was known already to Lax in the theory of 1D conservation laws and is the statement that the eventual shock $\mathcal{S}$ will be *spacelike* with respect to $h^{\mu\nu}(\partial\phi)$ from one side of the shock but *timelike* from the other. Note this can already be detected from initial data since we can determine ζ_μ on the $\mathcal{S}_0$; the spatial components ζ_{x^i} are found through the shape of $\mathcal{S}_0$ while ζ_t is determined using the jump condition (1.14). The determism condition is thus that:

$$h^{\mu\nu}(\partial\phi_0^+, \phi_1^+)\,\zeta_\mu\zeta_\nu < 0, \qquad h^{\mu\nu}(\partial\phi_0^-, \phi_1^-)\,\zeta_\mu\zeta_\nu > 0$$

or vice versa. In the sequel, we assume the above without loss of generality. So the eventual shock $\mathcal{S}$ will be spacelike on the $+$ side and timelike on the $-$ side.

Even in 1D, the determinism condition is necessary for both existence and uniqueness. It also explains how one locally solves the problem from the PDE viewpoint. Firstly, using the standard theory of quasilinear hyperbolic PDE (e.g. Hughes–Kato–Marsden) one solves (1.7) on the $+$ side; since $\mathcal{S}$ is spacelike on the $+$ side one can actually solve for ϕ^+ *beyond* the eventual shock.

That is, for $T > 0$ small there is a unique local solution $\tilde{\phi}^+$ in a domain $\tilde{\Omega}^+ \subset [0,T] \times \mathbb{R}^d$ diffeomorphic to Σ^+ which is *maximal* in the sense that $(\tilde{\Omega}^+, \tilde{\phi}^+)$ "contains" all other solutions, and that we eventually have $\mathcal{S} \subset \tilde{\Omega}^+$. The portion of $\tilde{\phi}^+$ beyond $\mathcal{S}$ will be unphysical, but is still helpful.

It remains to find both ϕ^-, as well as the location of the shock hypersurface $\mathcal{S}$. Since $\mathcal{S}$ is *timelike* on the $-$ side, using simply the initial data (ϕ_0^-, ϕ_1^-) is insufficient, and one instead solves (1.11) with the boundary conditions (1.13)–(1.14) as a hyperbolic free boundary problem with timelike boundary conditions. We consider (1.13) as an evolution equation for the free boundary $\mathcal{S}$ while (1.14) is considered to be a boundary condition for (1.11) on the timelike boundary $\mathcal{S}$.

This brings us to the second condition required in Theorem 1.1, the *linearized stability condition*. We motivate this second condition through a much simpler problem, a linear wave equation on a Lorentzian background with a fixed timelike boundary. Let $h_{\mu\nu}$ be a fixed Lorentzian metric on $\mathbb{R}^{1+d}$ with t timelike, and let $\mathcal{S}$ be a timelike hypersurface, such that Σ_0 and $\mathcal{S}$ bound a domain Ω lying to the future of Σ_0. Then consider $\psi : \overline{\Omega} \to \mathbb{R}$ solving

$$\Box_h \psi := (h^{-1})^{\mu\nu}\nabla_\mu\nabla_\nu\psi = F \quad \text{in } \Omega, \tag{1.15}$$

$$B\psi := B^\mu\nabla_\mu\psi = G \quad \text{on } \mathcal{S}. \tag{1.16}$$

Here B is a vector field in $\mathbb{R}^{1+d}$, while F and G are given inhomogeneities. For initial data given in $\Sigma \subset \Sigma_0$, the past boundary of Ω, it turns out the solvability and well-posedness properties of (1.15)–(1.16) depend on the geometric properties of B. Indeed, we have

Theorem 1.2 (Majda–Thomann)**.** *Let (ψ_0, ψ_1) be initial data for (1.15)–(1.16) with $\psi_0 \in H^{s+1}(\Sigma)$ and $\psi_1 \in H^s(\Sigma)$ where s is a nonnegative integer. Then assuming that:*

(i) there exists $\delta > 0$ such that $h(B, N_{\mathcal{S}}) \geq \delta$ everywhere on $\mathcal{S}$, where $N_{\mathcal{S}}$ is the outward pointing unit normal of $\mathcal{S}$ (non-transversality condition), and

(ii) the vector field $\hat{B} = B - h(B, N_{\mathcal{S}})N_{\mathcal{S}}$ representing the projection of B onto $\mathcal{S}$, is everywhere future-directed timelike on $\mathcal{S}$,

as well as several higher order compatibility conditions.

Then there exists a unique solution ψ to (1.15)–(1.16) in $\overline{\Omega}$ such that $\psi_0 = \psi|_\Sigma$ and $\psi_1 = \partial_t \psi|_\Sigma$. Furthermore, between the two spacelike surfaces Σ_0 and Σ_t the solution obeys the estimate:

$$\int_{\Sigma_t \cap \Omega} |\partial \partial^{\leq s} \psi|^2 + \int_{\mathcal{S}_{0,t}} |\partial \partial^{\leq s} \psi|^2 \lesssim \int_{\Sigma_0 \cap \Omega} |\partial \partial^{\leq s} \psi|^2 + \iint_{\Omega_{0,t}} |\partial^{\leq s} F|^2 + \int_{\mathcal{S}_{0,t}} |(\partial^\top)^{\leq s} G|^2. \tag{1.17}$$

Here $\mathcal{S}_{0,t}$ denotes the portion of $\mathcal{S}$ between Σ_0 and Σ_t and $\Omega_{0,t}$ is similar. The volume forms are derived from the metric h, and the derivative $\partial^\top$ emphasizes that one can only take derivatives of G tangent to $\mathcal{S}$, since G is only defined there.

Proof. We sketch the proof of the a priori estimate (1.17) in the case $s = 0$. Note that from this existence and uniqueness is standard, while the higher order estimates come from commuting (1.15)–(1.16) with well chosen vector fields. (Note one can only commute (1.16) with vector fields tangential to $\mathcal{S}$. Why is this enough?)

For the uncommuted estimate, we use the energy momentum tensor, a bilinear form $Q_{\mu\nu}$ with

$$Q(X, Y) = (X\psi)(Y\psi) - \frac{1}{2} h(X, Y) \cdot h(\nabla\psi, \nabla\psi).$$

From (1.15) it is known that $(\operatorname{div} Q)(X) = F \cdot X\psi$, so applying the divergence theorem with a multiplier vector field X, and letting $K_X = T_{\mu\nu} \nabla^\mu X^\nu$ one derives that

$$\int_{\Sigma_t \cap \Omega} Q(X, N_{\Sigma_t}) - \int_{\mathcal{S}_{0,t}} Q(X, N_{\mathcal{S}}) = \int_{\Sigma_0 \cap \Omega} Q(X, N_{\Sigma_0}) + \iint_{\Omega_{0,t}} F \cdot X\psi + K_X. \tag{1.18}$$

Recall that if X, Y are future directed and timelike then $|\partial\psi|^2 \gtrsim Q(X, Y) \gtrsim |\partial\psi|^2$. Therefore the first terms on both the left hand side and right hand side of (1.18) correspond to the the first terms on the LHS and RHS of (1.17). Using Cauchy–Schwarz (and later Grönwall's inequality) to deal with the final term in (1.18), it is thus sufficient to show that there exists some future directed timelike vector field X such that

$$-Q(X, N_{\mathcal{S}}) \geq C^{-1}|\partial\psi|^2 - C|B\psi|^2,$$

for some $C > 0$.

One such choice is the vector field $X = \hat{B} + \alpha N_{\mathcal{S}}$, where $\alpha > 0$ but is small to ensure that X is indeed future-directed timelike, in light of the condition (ii). We leave it as an exercise to the reader that choosing α small indeed provides the desired lower bound on $-Q(X, N_{\mathcal{S}})$. □

There are of course several more steps in going from the linear result of Theorem 1.2 to the fully nonlinear free boundary problem in Theorem 1.1. We outline these below:

- Firstly one must transform the free boundary problem to a (quasilinear) wave equation with a possibly nonlinear boundary condition. This is achieved in Majda–Thomann via a *partial hodograph transform*, which is fancy terminology for transforming the free boundary into a straight boundary.

  This is done by making a change of variables; we let let $y_0 = t, y_i = x_i$ for $i = 1, \dots, d-1$ but let $y_d = \tilde{\phi}^+ - \phi^-$, where $\tilde{\phi}^+$ is the maximally extended solution on the + side from before. In light of (1.13), the boundary $\mathcal{S}$ corresponds to $y_d = 0$. On the other hand, the equation (1.11) and boundary condition (1.14) are transformed to a quasilinear wave equation for the function $x_d = x_d(y)$, together with a nonlinear boundary condition.

- For this new system, one now applies the linear Theorem 1.2 together with an iteration argument to solve the quasilinear wave equation with nonlinear boundary condition on a timelike boundary. To see what is meant by the *linearized stability condition*, since $\mathcal{S} = \{\tilde{\phi}^+ - \phi^- = 0\}$, we have $\zeta \sim d(\tilde{\phi}^+ - \phi^-)$, and thus (1.14) is written purely in terms of $\tilde{\phi}^+$ and ϕ^-:

$$b(\partial\tilde{\phi}^+, \partial\phi^-) \coloneqq (\partial_\mu\tilde{\phi}^+ - \partial_\mu\phi^-) \cdot (H^\mu(\partial\tilde{\phi}^+) - H^\mu(\partial\phi^-)) = 0$$

  The linearized vector field B is now found by differentiating $b(\xi^+, \xi^-)$ with respect to its second variable ξ^- and evaluating at the initial data. Linearized stability is then simply this vector field B satisfying (i) and (ii) of Theorem 1.2 on the initial shock $\mathcal{S}_0$.

- In order to get a nonlinear result, note that one must also commute the equations with some number of vector fields, particular vector fields tangential to the shock $\mathcal{S}$. Since $\mathcal{S} = \{\tilde{\phi}^+ - \phi^-$, such vector fields $Z^\top$ tangent to $\mathcal{S}$ will also depend on ϕ^-. It is important to check that one does not lose derivatives in this process (e.g. it is crucial that the number of derivatives for the two terms on the LHS of (1.17) are equal.)

Remark. In Majda's earlier 1983 work on shock propagation for general hyperbolic systems such as the original fluid equations of (1.5), instead of using energy estimates as in Theorem 1.2 he had to use microlocal energy estimates, found after appropriately symmetrizing the system. The analogue of the conditions (i) and (ii) in Theorem 1.2 are the so-called *Lopatinski conditions.*

1.3 The Shock Development Problem

We finally explain what is meant by the shock development problem in compressible fluids, or at least the *restricted shock development* problem for irrotational shocks, associated to divergence form equations of the form (1.7).

Before doing so, we first recap the *shock formation theorem* of Christodoulou 2008, where a complete understanding of a portion of the classical maximal global hyperbolic development in a neighborhood of an initial singularity is found. This achieved via the use of an *eikonal function* u, solving the eikonal equation:

$$(h^{-1})^{\mu\nu}\partial_\mu u\,\partial_\nu u = 0, \qquad u|_{\Sigma_0} = r, \tag{1.19}$$

and then considering a function μ called the *inverse foliation density*, defined by:

$$\frac{1}{\mu} = -(h^{-1})^{\mu\nu}\partial_\mu u\,\partial_\nu t. \tag{1.20}$$

Morally, wherever $\mu > 0$ the coordinate transformation from (t, r, ϑ^A) to (t, u, ϑ^A), where ϑ^A is an angular variable, is smooth and satisfies suitable μ-weighted estimates. Moreover, the potential function ϕ is *smooth* with respect to the (t, u, ϑ^A) variables for the region $\mu \geq 0$. Then singularity formation is understood as the boundary $\mathcal{B} = \{\mu = 0\}$, where the coordinate transformation above breaks down, and thus one has blow up e.g. for $\partial^2\phi$ in the original (t, r, ϑ^A) coordinates.

It turns out that in (t, u, ϑ^A)-coordinates, where $\mathcal{B}$ is a regular surface, the intrinsic geometry of the singular hypersurface $\mathcal{B}$ is *null.* However, the backward sound cones emanating from points in $\mathcal{B}$ do not intersect $\mathcal{B}$; hence $\mathcal{B}$ is intrinsically spacelike. Finally, we note that $\mathcal{B}$ has a past boundary, which is a $d-1$-dimensional surface in $\mathbb{R}^{1+d}$, denoted $\partial_-\mathcal{B}$; this can be viewed as the set of "first singularities", and is often called the crease, or pre-shock.

To complete the local picture of the maximal development, note that there is another hypersurface $\underline{\mathcal{C}}$ whose past boundary is also $\partial_-\mathcal{B}$, which represents the *Cauchy horizon* associated to the first singularity at $\partial_-\mathcal{B}$. The terminology Cauchy horizon $\underline{\mathcal{C}}$ exactly means that any point beyond $\underline{\mathcal{C}}$ would have a past sound cone which partially lies in the future of $\partial_-\mathcal{B}$, thus cannot be predicted from initial data at Σ_0. What is perhaps more interesting is that if ϕ is smooth at initial data, then it remains smooth (and even extendible) on $\underline{\mathcal{C}} \setminus \partial_-\mathcal{B}$; however as mentioned before $\partial^2\phi$ blows up as one approaches the past boundary $\partial_-\mathcal{B}$ of $\underline{\mathcal{C}}$.

However, Christodoulou mentions that while this (local portion) of the maximal hyperbolic development is perfectly reasonable from the mathematical point of view, and also physically reasonable up to the boundary $\partial_-\mathcal{B} \cup \underline{\mathcal{C}}$, it is not the correct physical solution up to $\mathcal{B}$. Instead, paraphrasing Christodoulou, one must:

"Find a hypersurface $\mathcal{S}$ in Minkowski, lying to the past of $\mathcal{B}$ but with the same past boundary $\partial_-\mathcal{S} = \partial_-\mathcal{B}$, and the same tangent hyperplane as $\mathcal{B}$ along $\partial_-\mathcal{B}$, and a solution (u^i, ρ, s) – alternatively ϕ – of the equations of motion (1.5) – alternatively (1.7) – bounded in the past by $\underline{\mathcal{C}}$ and $\mathcal{S}$, such that on $\underline{\mathcal{C}}$ the solution coincides with the maximal hyperbolic development, while across $\mathcal{S}$ there are jumps from the maximal hyperbolic development, satisfying the associated jump conditions. Moreover, the hypersurface $\mathcal{S}$ is spacelike with respect to the acoustical metric induced by the maximal development in the past of $\mathcal{S}$, but timelike with respect to the new solution to the future of $\mathcal{S}$. That is, the shock $\mathcal{S}$ obeys the *determinism conditions*."

This is the *shock development problem*. To phrase this more mathematically, note that from the point of view of the extended solution $\underline{\mathcal{C}}$ is an incoming null hypersurface at which characteristic initial data is placed, while $\mathcal{S}$, as in the shock propagation problem, is a timelike free boundary. So we have a characteristic initial free boundary value problem with nonlinear boundary conditions.

What makes this much harder than the shock propagation problem is that the characteristic initial data at $\underline{\mathcal{C}}$ becomes singular at the past boundary $\partial_-\underline{\mathcal{C}} = \partial_-\mathcal{B}$; similarly the boundary conditions become singular as one approaches this boundary. As a result, the full shock development problem for the physical variables (u^i, ρ, s) is still open. We mention:

- **Christodoulou–Lisibach 2016** (Shock development in spherical symmetry) here the shock development problem is solved in the case of spherically symmetric initial data, where in particular no vorticity will be generated.

- **Christodoulou 2019** (Restricted shock development) here the shock development problem was solved in 3+1-dimensions for the quasilinear wave equation (1.7) and with jump conditions as in (1.13)–(1.14) of Majda–Thomann.

2 Problem Session – Formation of Shocks

We outline the formation of shocks in 3D, the aim being shock formation for the model quasilinear wave equation near Minkowski space in spherical symmetry for small and localized initial data:

$$\Box_{Mink}\phi \coloneqq -\partial_{tt}^2\phi + \Delta\phi = -\partial_t\phi \cdot \Delta\phi \tag{2.1}$$

Note that a similar mechanism will apply for more general quasilinear wave equations violating the null condition; one may for instance tackle wave equations such as

$$(h^{-1})^{\mu\nu}(\partial\phi)\partial_\mu\partial_\nu\phi = 0.$$

Various simplifications will be made when we restrict attention to our model equation (2.1).

DISCLAIMER: There may be typos, any mistakes made here are my own.

1. We first consider Burgers' equation in $1D$:

$$\partial_t\psi + (1+\psi)\partial_x\psi = 0, \qquad \psi|_{t=0} = \psi_0. \tag{2.2}$$

(Note we have made a harmless substitution $\psi \mapsto 1+\psi$ from the usual Burgers' equation.)

(a) Define L to be the vector field $L \coloneqq \frac{\partial}{\partial t} + (1+\psi)\frac{\partial}{\partial x}$. Show that:

$$L\psi = 0, \qquad L(\partial_x\psi) = -(\partial_x\psi)^2.$$

Hence show that smooth, compactly supported, initial data $\psi(t=0) = \psi_0$ of size ε leads to blow up at a time $T = O(\varepsilon^{-1})$. What is the appropriate norm on ψ_0?

(b) We now construct a portion of the maximal development using a coordinate u which is transported along characteristics:

$$Lu = 0, \qquad u|_{t=0} = x.$$

Define $\mu = (\partial_x u)^{-1}$. Then μ is the Jacobian determinant of the change of variables $(t,x) \mapsto (t,u)$. By considering the commutator $[L, \partial_x]$ or otherwise, show that

$$L\mu = \mu\partial_x\psi, \qquad L^2\mu = L(\mu\partial_x\psi) = 0.$$

(c) What is L in the (t,u) coordinate system? Show that ψ remains (globally) regular in the (t,u) coordinate system. That is, show that all derivatives $\partial_u^m \partial_t^m \psi$ remain bounded.

Show that for ψ_0 nonzero and compactly supported, $\mu \partial_x \psi$ is nonzero somewhere at $t=0$. Therefore show that for such data there exist (t,u) such that $\mu(t,u)=0$, and characterise the dependence of t on u and ψ_0 on this set.

(d) We now translate this back into the (t,x) coordinates. Consider *non-degenerate* initial data – meaning $\partial_x \psi_0$ has a unique minimum at $x=x_0$ with $\partial_x^2 \psi_0(x_0) > 0$ and $\partial_x^3 \psi_0(x_0) \neq 0$ – then show that near $u=x_0$, $\{\mu=0\}$ is a piecewise smooth curve and locally parameterised by:

$$\mathcal{B} = \left\{ \left(-(\partial_x \psi_0)^{-1}(u), x(u) \right) : u \in (x_0-\delta, x_0) \cup (x_0, x_0+\delta) \right\},$$

where $x = x(u)$ is smooth except for at $u = x_0$. (Hint: show that $u = x - (1+\psi_0(u))t$.)

Show that for $x \in (x_0-\delta, x_0) \cup (x_0, x_0+\delta)$, the tangent vector to $\mathcal{B}$ is parallel to L.

2. (*) For this question knowledge of the vector field method and bootstrap arguments is useful. If unfamiliar please ask or skip this question for now. We now consider the model equation (2.1), and first prove a lower bound on the time of existence. That is, we prove an *almost global existence* theorem due to John–Klainerman 1984. That is, consider initial data:

$$\phi(t=0) = \phi_0, \qquad \partial_t \phi(t=0) = \phi_1,$$

with ϕ_0 and ϕ_1 smooth and compactly supported in the unit ball, and of size ε in the sense that $\|\phi_0\|_{H^{s+1}}, \|\phi_1\|_s \leq \varepsilon$ for s large.

(a) Let $\Gamma = \{\partial, \Omega_{ij} = x_i \partial_j - x_j \partial_i, \Omega_{0i} = t\partial_i + x_i \partial_t, S = t\partial_t + x_i \partial_i\}$ be the standard set of commuting vector fields in Minkowski. Recall that:

$$[\Gamma, \Box_{Mink}] = c_\Gamma \Box_{Mink}, \qquad \text{where } c_\Gamma = 2 \text{ if } \Gamma = S, c_\Gamma = 0 \text{ otherwise.}$$

By considering $[\Gamma, \partial]$, show that if $N \in \mathbb{N}$ and ϕ solves (2.1) then for $|\alpha| = N$,

$$-\partial_{tt}^2 \left(\Gamma^\alpha \phi\right) + (1+\partial_t \phi)\Delta \left(\Gamma^\alpha \phi\right) = \sum_{\substack{\beta,\gamma \leq N \\ \beta+\gamma \leq N+1}} \partial \Gamma^\beta \phi \cdot \partial \Gamma^\gamma \phi. \tag{2.3}$$

where the schematic form of the RHS means that that we allow any constant coefficients in front of each summand.

(b) We use a bootstrap argument where the bootstrap assumption is:

$$\|\partial \Gamma^{\beta} \phi\|_{L^{\infty}(\Sigma_t)} \leq \varepsilon^{2/3} (1+t)^{-1} \quad \text{for all } |\beta| \leq 8. \tag{B}$$

Consider a Nth order energy of the form:

$$E_N(t) = \sum_{|\alpha| \leq N} \int_{\mathbb{R}^3} \left((\partial_t \Gamma^{\alpha} \phi)^2 + |\nabla \Gamma^{\alpha} \phi|^2 \right) dx.$$

By using the commuted equation above and a ∂_t-multiplier, show that when (B) holds and $N \leq 12$, we have the energy estimate.

$$E_N(t) \lesssim E_N(0) + \int_0^t \frac{\varepsilon^{2/3}}{s} E_N(s)\, ds.$$

(c) Using Grönwall's inequality, infer that $E_N(t) \leq \varepsilon \cdot t^{C\varepsilon^{2/3}}$. By using the Klainerman–Sobolev inequality together with the bootstrap assumption (B), show ϕ exists for $t \in [0, T]$ where

$$\log T \gtrsim \varepsilon^{-2/3} \log \varepsilon^{-1}.$$

(*) In fact, by suitably strengthening the bootstrap assumption (B) e.g. by replacing $\varepsilon^{2/3}$ with $C\varepsilon$, one may (as in John–Klainerman) that:

$$\log T \gtrsim \varepsilon^{-1}. \tag{2.4}$$

(d) (*) Let $\delta > 0$ be small, and let $p \in \mathbb{R}^{3+1}$ be a *first singularity*, i.e. so that the solution remains regular in $(J^-(p) \setminus \{p\}) \cap \{t \geq 0\}$. By localizing the above argument to $J^-(p)$, and using the extra $(1+|u|)^{-1/2}$ decay in the Klainerman–Sobolev inequality show that p lies in the wave zone:

$$W := \left\{ (t,x) : \quad t \geq 1, 1-\delta \leq \frac{|x|}{t} \leq 1+\delta \right\}.$$

3. We now consider shock formation for (2.1) in spherical symmetry, in the process showing that the almost global existence result (2.4) is sharp.

(a) Assume that ϕ solving (2.1) is spherically symmetric i.e. depends only on (t, r) and not on the angular variables ϑ^A. Show that for $\psi := \partial_t(r\phi)$, one has:

$$-\partial_{tt}^2 \psi + c^2 \partial_{rr}^2 \psi = -r^{-1} c^{-2} (\partial_t \psi)^2, \quad \text{where } c^2 = c^2(\psi) = 1 + r^{-1}\psi. \tag{2.5}$$

(Note the model equation (2.1) was chosen partially so one derives a single scalar equation for ψ where the metric depends only on ψ and not on $\partial\psi$.)

(b) Define the vector fields:

$$L = \frac{\partial}{\partial t} + c\frac{\partial}{\partial r}, \quad \underline{L} = \frac{\partial}{\partial t} - c\frac{\partial}{\partial r}.$$

By considering Lc and $\underline{L}c$, show that:

$$-L\underline{L}\psi = -\frac{1}{4}r^{-1}c^{-2}\left[(\underline{L}\psi)^2 - 3(L\psi)(\underline{L}\psi)\right] - \frac{1}{4}r^{-2}c^{-1}\psi(L\psi - \underline{L}\psi), \tag{2.6}$$

$$-\underline{L}L\psi = -\frac{1}{4}r^{-1}c^{-2}\left[(L\psi)^2 - 3(L\psi)(\underline{L}\psi)\right] - \frac{1}{4}r^{-2}c^{-1}\psi(L\psi - \underline{L}\psi). \tag{2.7}$$

The dangerous term is the $(\underline{L}\psi)^2$ term on the RHS of (2.6), since this term does not decay at the linear level.

(c) As in the case of Burgers', we now introduce a null coordinate u, obeying

$$Lu = 0, \qquad u|_{\Sigma_0} = 1 - r,$$

and an associated *inverse foliation density* $\mu := -(\partial_r u)^{-1}$. Show that:

$$L\mu = \frac{1}{4}r^{-1}c^{-2}(L\psi - \underline{L}\psi)\mu - \frac{1}{4}r^{-2}c^{-1}\psi\mu. \tag{2.8}$$

Combining with (2.6), show that:

$$-L(\mu\underline{L}\psi) = \frac{1}{2}r^{-1}c^{-2}(L\psi)(\mu\underline{L}\psi) - \frac{1}{4}r^{-2}c^{-1}\mu\psi L\psi. \tag{2.9}$$

This considering the vector field $\mu\underline{L}$ instead removes the most dangerous nonlinear term.

Also show that:

$$\mu\underline{L}u = -2c. \tag{2.10}$$

4. Now consider initial data for ψ (2.5) supported in the unit ball. We shall only consider the region:

$$\mathcal{M} = \left\{(t,r) : \quad t \geq 2,\ u \leq \frac{1}{2}\right\}.$$

By finite speed of propagation, the value of ψ in $\mathcal{M}$ is affected only by the data restricted to the annulus $\{1/2 \leq r \leq 1\}$. Roughly speaking, $\mathcal{M}$ is the wavezone.

Suppose the initial data is of size $\mathring{\varepsilon}$ in some suitable (high regularity) norm. We now use a bootstrap argument. Assume that that for some time $t_b > 2$, we have the following estimates in

$\{(t,x) \in \mathcal{M} : t \le t_b, \mu \ge 0\}$:

$$|L(\mu\underline{L}\psi)| \le \mathring{\varepsilon}^{2/3} r^{-1}, \quad |L\psi| \le \mathring{\varepsilon}^{2/3} r^{-1}, \quad |\mu\underline{L}\psi| \le \mathring{\varepsilon}^{2/3}, \quad |\psi| \le \mathring{\varepsilon}^{2/3}, \quad |\mu - 1| \le \mathring{\varepsilon}^{2/3}\log(1+r). \tag{2.11}$$

(a) Then show that in the same region $\{(t,x) \in \mathcal{M} : t \le t_b, \mu \ge 0\}$, we can improve the bootstrap assumptions. That is, assuming (2.11):

$$|L(\mu\underline{L}\psi)| \lesssim \mathring{\varepsilon} r^{-1}, \quad |L\psi| \lesssim \mathring{\varepsilon} r^{-1}, \quad |\mu\underline{L}\psi| \lesssim \mathring{\varepsilon}, \quad |\psi| \lesssim \mathring{\varepsilon}, \quad |\mu - 1| \lesssim \mathring{\varepsilon}\log(1+r). \tag{2.12}$$

Hint: The bootstrap assumptions are just the conclusions of (2.12) but with $\mathring{\varepsilon}$ replaced by $\mathring{\varepsilon}^{2/3}$. Then one can improve the bootstraps using (2.7), (2.8), (2.9), and integrating via L and $\mu\underline{L}$. Also note (2.10) means integrating via $\mu\underline{L}$ results in a finite integral with respect to du. See (†) at the end of this sheet for an example of this.

Note that (2.12) represent our *global existence-type estimates*. We now use this to show that $\mu = 0$ somewhere and that this corresponds to blow up of $\partial\psi$, and thus $\partial^2\phi$.

(b) Verify that $|u - t + r - 1| \lesssim \mathring{\varepsilon}\log(1+t)$ everywhere. By integrating (2.9), show that in (t,u) coordinates, where $L = \frac{\partial}{\partial t}\big|_u$, one has

$$\mu\underline{L}\psi(t,u) = \mu\underline{L}\psi(0,u) + O(\mathring{\varepsilon}^2).$$

Inserting this into (2.8), argue that

$$L\mu = -\frac{1}{4}\frac{1}{t-u+1}\mu\underline{L}\psi(0,u) + O\left(\frac{\mathring{\varepsilon}^2}{t}\right) + O\left(\frac{\mathring{\varepsilon}\log(1+t)}{t^2}\right). \tag{2.13}$$

(c) By integrating (2.13) with data which is compactly supported in $\{1/2 \le r \le 1\}$ (in particular so that $\mu\underline{L}\psi(u_0) \ge \mathring{\varepsilon} > 0$ somewhere on data), then there exists $t > 0$ such that $\mu(t, u_0) = 0$. Note: $\mathring{\varepsilon}$ has to be taken small such that the $O(\cdot)$ terms can be treated as error terms.

(d) Show that at a point such that $\mu(t,u) \to 0$, then in the original (t,r) coordinate system one has that

$$|\partial_t\psi|, |\partial_r\psi| \to \infty.$$

(†) For illustrative purposes we explain how to derive the estimate $L\psi \lesssim \mathring{\varepsilon} r^{-1}$. Multiplying (2.6) by

$-\mu$ we have that:

$$\mu\underline{L}L\psi = \frac{1}{4}r^{-1}c^{-2}\left[\mu(L\psi)^2 - 3(L\psi)(\mu\underline{L}\psi)\right] + \frac{1}{4}r^{-2}c^{-1}\psi(\mu L\psi - \mu\underline{L}\psi).$$

Then applying the bootstrap assumptions (2.11) and using $r^{-2}\log r \lesssim r^{-1}$, we see that

$$|\mu\underline{L}L\psi| \lesssim \mathring{\varepsilon}^{4/3}r^{-1}.$$

(We have also used that $c(\psi) = 1 + r^{-1}\psi$ satisfies $1 \lesssim c \lesssim 1$.)

Now, consider integral curves $\gamma(s)$ of the vector field $\mu\underline{L}$ with s an affine parameter; $\mu\underline{L}s = 1$. Then the fundamental theorem of calculus gives

$$\underbrace{L\psi(\gamma(s_1))}_{\text{want to estimate}} = \underbrace{L\psi(\gamma(s_0))}_{\text{`data'}} + \underbrace{\int_{s_0}^{s_1} \mu\underline{L}L\psi\, ds}_{\text{nonlinearity}}.$$

We now choose the integral curve $\gamma(s)$ so that we can understand both the data and the nonlinearity. We note that since $\mu\underline{L}u = -2c$, and $1 \lesssim c \lesssim 1$ we can change variables from s to u. The 'data' term can be set at $u_0 = 0$, where $L\psi = 0$ by the compact support assumption.

Further, the other limit of integration will be $u_1 \leq \frac{1}{2}$. So

$$\left|\int_{s_0}^{s_1} \mu\underline{L}L\psi\, ds\right| \leq \int_0^{1/2} |\mu\underline{L}L\psi|\, du \lesssim \mathring{\varepsilon}^{4/3}r^{-1}.$$

This in particular gives $L\psi \lesssim \mathring{\varepsilon}r^{-1}$.

To improve the other bootstrap assumptions, you will have to use the integral curves of L instead. Here, you might wish to change coordinates to either r or t and use the fact that $t \lesssim r \lesssim t$ in $\mathcal{M}$. Also in these cases the 'data' term is nontrivial and is size $O(\mathring{\epsilon})$.