Problem Session: Shock formation for quasilinear waves

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July 30, 2024

1 Problem Session – Formation of Shocks

We outline the formation of shocks in 3D, the aim being shock formation for the model quasilinear wave equation near Minkowski space in spherical symmetry for small and localized initial data:

$$\Box_{Mink}\phi \coloneqq -\partial_{tt}^2\phi + \Delta\phi = -\partial_t\phi \cdot \Delta\phi \tag{1.1}$$

Note that a similar mechanism will apply for more general quasilinear wave equations violating the null condition; one may for instance tackle wave equations such as

$$(h^{-1})^{\mu\nu}(\partial\phi)\partial_\mu\partial_\nu\phi = 0.$$

Various simplifications will be made when we restrict attention to our model equation (1.1).

DISCLAIMER: There may be typos, any mistakes made here are my own.

1. We first consider Burgers' equation in $1D$:

$$\partial_t\psi + (1+\psi)\partial_x\psi = 0, \qquad \psi|_{t=0} = \psi_0. \tag{1.2}$$

(Note we have made a harmless substitution $\psi \mapsto 1+\psi$ from the usual Burgers' equation.)

(a) Define L to be the vector field $L \coloneqq \frac{\partial}{\partial t} + (1+\psi)\frac{\partial}{\partial x}$. Show that:

$$L\psi = 0, \qquad L(\partial_x\psi) = -(\partial_x\psi)^2.$$

Hence show that smooth, compactly supported, initial data $\psi(t=0) = \psi_0$ of size ε leads to blow up at a time $T = O(\varepsilon^{-1})$. What is the appropriate norm on ψ_0?

(b) We now construct a portion of the maximal development using a coordinate u which is transported along characteristics:

$$Lu = 0, \qquad u|_{t=0} = x.$$

Define $\mu = (\partial_x u)^{-1}$. Then μ is the Jacobian determinant of the change of variables $(t, x) \mapsto (t, u)$. By considering the commutator $[L, \partial_x]$ or otherwise, show that

$$L\mu = \mu \partial_x \psi, \qquad L^2 \mu = L(\mu \partial_x \psi) = 0.$$

(c) What is L in the (t, u) coordinate system? Show that ψ remains (globally) regular in the (t, u) coordinate system. That is, show that all derivatives $\partial_u^m \partial_t^m \psi$ remain bounded.

Show that for ψ_0 nonzero and compactly supported, $\mu \partial_x \psi$ is nonzero somewhere at $t = 0$. Therefore show that for such data there exist (t, u) such that $\mu(t, u) = 0$, and characterise the dependence of t on u and ψ_0 on this set.

(d) We now translate this back into the (t, x) coordinates. Consider *non-degenerate* initial data – meaning $\partial_x \psi_0$ has a unique minimum at $x = x_0$ with $\partial_x^2 \psi_0(x_0) > 0$ and $\partial_x^3 \psi_0(x_0) \neq 0$ – then show that near $u = x_0$, $\{\mu = 0\}$ is a piecewise smooth curve and locally parameterised by:

$$\mathcal{B} = \left\{ \left(-(\partial_x \psi_0)^{-1}(u), x(u) \right) : u \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \right\},$$

where $x = x(u)$ is smooth except for at $u = x_0$. (Hint: show that $u = x - (1 + \psi_0(u))t$.)

Show that for $x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$, the tangent vector to $\mathcal{B}$ is parallel to L.

2. (*) For this question knowledge of the vector field method and bootstrap arguments is useful. If unfamiliar please ask or skip this question for now. We now consider the model equation (1.1), and first prove a lower bound on the time of existence. That is, we prove an *almost global existence* theorem due to John–Klainerman 1984. That is, consider initial data:

$$\phi(t = 0) = \phi_0, \qquad \partial_t \phi(t = 0) = \phi_1,$$

with ϕ_0 and ϕ_1 smooth and compactly supported in the unit ball, and of size ε in the sense that $\|\phi_0\|_{H^{s+1}}, \|\phi_1\|_s \leq \varepsilon$ for s large.

(a) Let $\Gamma = \{\partial, \Omega_{ij} = x_i \partial_j - x_j \partial_i, \Omega_{0i} = t\partial_i + x_i \partial_t, S = t\partial_t + x_i \partial_i\}$ be the standard set of commuting vector fields in Minkowski. Recall that:

$$[\Gamma, \Box_{Mink}] = c_\Gamma \Box_{Mink}, \qquad \text{where } c_\Gamma = 2 \text{ if } \Gamma = S, c_\Gamma = 0 \text{ otherwise.}$$

By considering $[\Gamma, \partial]$, show that if $N \in \mathbb{N}$ and ϕ solves (1.1) then for $|\alpha| = N$,

$$-\partial^2_{tt}\left(\Gamma^\alpha \phi\right) + (1 + \partial_t \phi)\Delta\left(\Gamma^\alpha \phi\right) = \sum_{\substack{\beta,\gamma \leq N \\ \beta+\gamma \leq N+1}} \partial\Gamma^\beta \phi \cdot \partial \Gamma^\gamma \phi. \tag{1.3}$$

where the schematic form of the RHS means that that we allow any constant coefficients in front of each summand.

(b) We use a bootstrap argument where the bootstrap assumption is:

$$\|\partial\Gamma^\beta \phi\|_{L^\infty(\Sigma_t)} \leq \varepsilon^{2/3}\,(1+t)^{-1} \qquad \text{for all } |\beta| \leq 8. \tag{B}$$

Consider a Nth order energy of the form:

$$E_N(t) = \sum_{|\alpha| \leq N} \int_{\mathbb{R}^3} \left((\partial_t \Gamma^\alpha \phi)^2 + |\nabla \Gamma^\alpha \phi|^2 \right) dx.$$

By using the commuted equation above and a ∂_t-multiplier, show that when (B) holds and $N \leq 12$, we have the energy estimate.

$$E_N(t) \lesssim E_N(0) + \int_0^t \frac{\varepsilon^{2/3}}{s} E_N(s)\, ds.$$

(c) Using Grönwall's inequality, infer that $E_N(t) \leq \varepsilon \cdot t^{C\varepsilon^{2/3}}$. By using the Klainerman–Sobolev inequality together with the bootstrap assumption (B), show ϕ exists for $t \in [0, T]$ where

$$\log T \gtrsim \varepsilon^{-2/3} \log \varepsilon^{-1}.$$

(*) In fact, by suitably strengthening the bootstrap assumption (B) e.g. by replacing $\varepsilon^{2/3}$ with $C\varepsilon$, one may (as in John–Klainerman) that:

$$\log T \gtrsim \varepsilon^{-1}. \tag{1.4}$$

(d) (*) Let $\delta > 0$ be small, and let $p \in \mathbb{R}^{3+1}$ be a *first singularity*, i.e. so that the solution remains regular in $(J^-(p) \setminus \{p\}) \cap \{t \geq 0\}$. By localizing the above argument to $J^-(p)$, and using the extra $(1+|u|)^{-1/2}$ decay in the Klainerman–Sobolev inequality show that p lies in the wave zone:

$$W := \left\{(t,x): \quad t \geq 1, 1-\delta \leq \frac{|x|}{t} \leq 1+\delta\right\}.$$

3. We now consider shock formation for (1.1) in spherical symmetry, in the process showing that the almost global existence result (1.4) is sharp.

   (a) Assume that ϕ solving (1.1) is spherically symmetric i.e. depends only on (t,r) and not on the angular variables ϑ^A. Show that for $\psi := \partial_t(r\phi)$, one has:

   $$-\partial_{tt}^2\psi + c^2\partial_{rr}^2\psi = -r^{-1}c^{-2}(\partial_t\psi)^2, \quad \text{where } c^2 = c^2(\psi) = 1 + r^{-1}\psi. \tag{1.5}$$

   (Note the model equation (1.1) was chosen partially so one derives a single scalar equation for ψ where the metric depends only on ψ and not on $\partial\psi$.)

   (b) Define the vector fields:

   $$L = \frac{\partial}{\partial t} + c\frac{\partial}{\partial r}, \quad \underline{L} = \frac{\partial}{\partial t} - c\frac{\partial}{\partial r}.$$

   By considering Lc and $\underline{L}c$, show that:

   $$-L\underline{L}\psi = -\frac{1}{4}r^{-1}c^{-2}\left[(\underline{L}\psi)^2 - 3(L\psi)(\underline{L}\psi)\right] - \frac{1}{4}r^{-2}c^{-1}\psi(L\psi - \underline{L}\psi), \tag{1.6}$$

   $$-\underline{L}L\psi = -\frac{1}{4}r^{-1}c^{-2}\left[(L\psi)^2 - 3(L\psi)(\underline{L}\psi)\right] - \frac{1}{4}r^{-2}c^{-1}\psi(L\psi - \underline{L}\psi). \tag{1.7}$$

   The dangerous term is the $(\underline{L}\psi)^2$ term on the RHS of (1.6), since this term does not decay at the linear level.

   (c) As in the case of Burgers', we now introduce a null coordinate u, obeying

   $$Lu = 0, \qquad u|_{\Sigma_0} = 1 - r,$$

   and an associated *inverse foliation density* $\mu := -(\partial_r u)^{-1}$. Show that:

   $$L\mu = \frac{1}{4}r^{-1}c^{-2}(L\psi - \underline{L}\psi)\mu - \frac{1}{4}r^{-2}c^{-1}\psi\mu. \tag{1.8}$$

   Combining with (1.6), show that:

   $$-L(\mu\underline{L}\psi) = \frac{1}{2}r^{-1}c^{-2}(L\psi)(\mu\underline{L}\psi) - \frac{1}{4}r^{-2}c^{-1}\mu\psi L\psi. \tag{1.9}$$

   This considering the vector field $\mu\underline{L}$ instead removes the most dangerous nonlinear term.

Also show that:

$$\mu \underline{L} u = -2c. \tag{1.10}$$

4. Now consider initial data for ψ (1.5) supported in the unit ball. We shall only consider the region:

$$\mathcal{M} = \left\{ (t,r) : \quad t \geq 2,\, u \leq \frac{1}{2} \right\}.$$

By finite speed of propagation, the value of ψ in $\mathcal{M}$ is affected only by the data restricted to the annulus $\{1/2 \leq r \leq 1\}$. Roughly speaking, $\mathcal{M}$ is the wavezone.

Suppose the initial data is of size $\mathring{\varepsilon}$ in some suitable (high regularity) norm. We now use a bootstrap argument. Assume that that for some time $t_b > 2$, we have the following estimates in $\{(t,x) \in \mathcal{M} : t \leq t_b, \mu \geq 0\}$:

$$|L(\mu \underline{L}\psi)| \leq \mathring{\varepsilon}^{2/3} r^{-1}, \quad |L\psi| \leq \mathring{\varepsilon}^{2/3} r^{-1}, \quad |\mu \underline{L}\psi| \leq \mathring{\varepsilon}^{2/3}, \quad |\psi| \leq \mathring{\varepsilon}^{2/3}, \quad |\mu - 1| \leq \mathring{\varepsilon}^{2/3} \log(1+r). \tag{1.11}$$

(a) Then show that in the same region $\{(t,x) \in \mathcal{M} : t \leq t_b, \mu \geq 0\}$, we can improve the bootstrap assumptions. That is, assuming (1.11):

$$|L(\mu \underline{L}\psi)| \lesssim \mathring{\varepsilon} r^{-1}, \quad |L\psi| \lesssim \mathring{\varepsilon} r^{-1}, \quad |\mu \underline{L}\psi| \lesssim \mathring{\varepsilon}, \quad |\psi| \lesssim \mathring{\varepsilon}, \quad |\mu - 1| \lesssim \mathring{\varepsilon} \log(1+r). \tag{1.12}$$

Hint: The bootstrap assumptions are just the conclusions of (1.12) *but with* $\mathring{\varepsilon}$ *replaced by* $\mathring{\varepsilon}^{2/3}$*. Then one can improve the bootstraps using* (1.7), (1.8), (1.9), *and integrating via* L *and* $\mu\underline{L}$*. Also note* (1.10) *means integrating via* $\mu\underline{L}$ *results in a finite integral with respect to du. See* (†) *at the end of this sheet for an example of this.*

Note that (1.12) represent our *global existence-type estimates.* We now use this to show that $\mu = 0$ somewhere and that this corresponds to blow up of $\partial\psi$, and thus $\partial^2\phi$.

(b) Verify that $|u - t + r - 1| \lesssim \mathring{\varepsilon} \log(1+t)$ everywhere. By integrating (1.9), show that in (t,u) coordinates, where $L = \frac{\partial}{\partial t}\big|_u$, one has

$$\mu \underline{L}\psi(t,u) = \mu \underline{L}\psi(0,u) + O(\mathring{\varepsilon}^2).$$

Inserting this into (1.8), argue that

$$L\mu = -\frac{1}{4}\frac{1}{t-u+1}\mu \underline{L}\psi(0,u) + O\left(\frac{\mathring{\varepsilon}^2}{t}\right) + O\left(\frac{\mathring{\varepsilon}\log(1+t)}{t^2}\right). \tag{1.13}$$

(c) By integrating (1.13) with data which is compactly supported in $\{1/2 \leq r \leq 1\}$ (in particular so that $\mu \underline{L}\psi(u_0) \geq \mathring{\varepsilon} > 0$ somewhere on data), then there exists $t > 0$ such that $\mu(t, u_0) = 0$. Note: $\mathring{\varepsilon}$ has to be taken small such that the $O(\cdot)$ terms can be treated as error terms.

(d) Show that at a point such that $\mu(t, u) \to 0$, then in the original (t, r) coordinate system one has that

$$|\partial_t \psi|, |\partial_r \psi| \to \infty.$$

(†) For illustrative purposes we explain how to derive the estimate $L\psi \lesssim \mathring{\varepsilon} r^{-1}$. Multiplying (1.6) by $-\mu$ we have that:

$$\mu \underline{L} L\psi = \frac{1}{4} r^{-1} c^{-2} \left[\mu (L\psi)^2 - 3(L\psi)(\mu \underline{L}\psi)\right] + \frac{1}{4} r^{-2} c^{-1} \psi(\mu L\psi - \mu \underline{L}\psi).$$

Then applying the bootstrap assumptions (1.11) and using $r^{-2}\log r \lesssim r^{-1}$, we see that

$$|\mu \underline{L} L\psi| \lesssim \mathring{\varepsilon}^{4/3} r^{-1}.$$

(We have also used that $c(\psi) = 1 + r^{-1}\psi$ satisfies $1 \lesssim c \lesssim 1$.)

Now, consider integral curves $\gamma(s)$ of the vector field $\mu \underline{L}$ with s an affine parameter; $\mu \underline{L} s = 1$. Then the fundamental theorem of calculus gives

$$\underbrace{L\psi(\gamma(s_1))}_{\text{want to estimate}} = \underbrace{L\psi(\gamma(s_0))}_{\text{`data'}} + \underbrace{\int_{s_0}^{s_1} \mu \underline{L} L\psi \, ds}_{\text{nonlinearity}}.$$

We now choose the integral curve $\gamma(s)$ so that we can understand both the data and the nonlinearity. We note that since $\mu \underline{L} u = -2c$, and $1 \lesssim c \lesssim 1$ we can change variables from s to u. The 'data' term can be set at $u_0 = 0$, where $L\psi = 0$ by the compact support assumption.

Further, the other limit of integration will be $u_1 \leq \frac{1}{2}$. So

$$\left| \int_{s_0}^{s_1} \mu \underline{L} L\psi \, ds \right| \leq \int_0^{1/2} |\mu \underline{L} L\psi| \, du \lesssim \mathring{\varepsilon}^{4/3} r^{-1}.$$

This in particular gives $L\psi \lesssim \mathring{\varepsilon} r^{-1}$.

To improve the other bootstrap assumptions, you will have to use the integral curves of L instead. Here, you might wish to change coordinates to either r or t and use the fact that $t \lesssim r \lesssim t$ in $\mathcal{M}$. Also in these cases the 'data' term is nontrivial and is size $O(\mathring{\epsilon})$.