TENSORS

FOR VECTOR CALCULUS

Review

- Vectors
- Summation representation of an n by n array
- Gradient, Divergence and Curl
- Spherical Harmonics (maybe)

Motivation

If you tape a book shut and try to spin it in the air on each independent axis you will notice that it spins fine on two axes but not on the third. That's the inertia tensor in your hands. Similar are the polarizations tensor, index of refraction tensor and stress tensor. But tensors also show up in all sorts of places that don't connect to an anisotropic material property, in fact even spherical harmonics are tensors. What are the similarities and differences between such a plethora of tensors?

> *The mathematics of tensors is particularly useful for describing properties of substances which vary in direction–although that's only one example of their use.* -Feynman II 31-1

Definitions

- q = **Scaler**
  a.k.a. Rank 0 tensor
  One real number, invariant under rotation
- A_i = **Vector**
  a.k.a Rank 1 tensor where i=1,2,3...
  Components will transform under rotations. Simply put, an arrow does not need to start at the origin or point in any specific orientation, but it's still the same arrow with the same magnitude.
- A_{ij} = **Rank 2 tensor**
  a.k.a. tensor
  Can be writhen out as a square array. Not all square arrays are tensors, there are some specific requirements. In cartesian space they must be an orthogonal norm preserving matrix.
- In N-dimensional space a tensor of rank n has N^n components.

Rank 1 Tensors (Vectors)

The definitions for contravariant and covariant tensors are inevitably defined at the beginning of all discussion on tensors. Their definitions are inviably without explanation. In that spirit we begin our discussion of rank 1 tensors. The main difference between contravaariant and covariant tensors is in how they are transformed. For example, a rotation of a vector.

Contravariant

In general for we can define a contravariant vector as one that can be transformed as A_j is in the following:

$$A'_i = \sum_j \frac{\partial x'_i}{\partial x_j} A_j$$

here might x'_i is the transformed x-axis, and x_j is the old x-axis.

Covariant

The partial derivative above may have you thinking of a gradient. The gradient is the prototype for a covariant vector which is defined as

$$\frac{\partial \varphi'}{\partial x'_i} = \sum_j \frac{\partial \varphi}{\partial x_j} \frac{\partial x_j}{\partial x'_i}$$

Covariant vectors are actually a linear form an not a vector. The linear form is a mapping of vectors into scalars which is additive and homogeneous under multiplication by scalars. [1]

[1] http://xxx.lanl.gov/PS_cache/gr-qc/pdf/9807/9807044.pdf

Notation

- **contravariant** denoted by superscript A^i took a vector and gave us a vector
- **covariant** denoted by subscript A_i
  took a scaler and gave us a vector

To avoid confusion in cartesian coordinates both types are the same so we just opt for the subscript. Thus a vector $\mathbf{x}$ would be x_1, x_2, x_3

As it turns out in an $\mathbb{R}^3$ cartesian space and other rectilinear coordinate systems there is no difference between contravariant and covariant vectors. This will *not* be the case for other coordinate systems such a curvilinear coordinate systems or in 4 dimensions. These definitions are closely related to the Jacobian.

Definitions for Tensors of Rank 2

Rank 2 tensors can be written as a square array. They have contravariant, mixed, and covariant forms. As we might expect in cartesian coordinates these are the same.

Vector Calculus and Identifers

Tensor analysis extends deep into coordinate transformations of all kinds of spaces and coordinate systems. At a rather shallow level it can be used to significantly simplify the operations and identifers of vector calculus.

Scaler Product

Take for example the Scaler or Dot Product.

$$\mathbf{A} \cdot \mathbf{B} = \sum_i A_i B_i = \overbrace{A_i B_i}^{scaler}$$

Clearly i is just a general way of specifying the components to multiply. Here we sum the product of the two components with i is the same. The equation is also written without the summation. They are equivalent because of the Einstein summation convention.

Scaler Product with an invariant tensor (Kronecker delta)

In a more thorough treatment we can also take the Scaler product using a mixed tensor of rank 2, δ_l^k more commonly recognized as the Kronecker delta δ_{ij}

$$\delta_{ij} = \begin{cases} 1 & \text{i=j} \\ 0 & \text{i} \neq \text{j} \end{cases}$$

or

$$\delta_l^k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Clearly, the Kronecker delta is a rank two tensor that can be expressed as a matrix, but rarely takes such an inherently dangerous form. The Kronecker delta is a so called invariant tensor. Now we can write our dot product as

$$\mathbf{A} \cdot \mathbf{B} = \sum_j A_i \delta_{ij} B_j = A_i \delta_{ij} B_j = A_i B_i$$

The sum is, of course, neglected because of the summation convention. Clearly the Kronecker delta facilitates the scaler product. Furthermore, the Kronecker delta can be applied to any situation where the product of orthogonal complements must be zero.

Summation Convention and Contraction

[2] **Summation Convention** When a subscript (letter, not number) appears twice on one side of an equation, summation with respect to that subscript is implied.

Contraction Consists of setting two unlike indices (subscripts) equal to each other and then summing as implied by the summation convention.

[2] From Arfkin p138

Cross Product and the invariant Levi-Civita Symbol

The rank three analog of the Kronecker delta is known as the Levi-Civita Symbol, ε_{ijk}.

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk)\text{= even permutation:} \varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} \\ -1 & \text{if } (ijk) = \text{odd permutation: } \varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} \\ 0 & \text{ if any of the two indexes are the same} \end{cases}$$

It's a three dimensional 27-element array with only six non-zero components. Levi-Civita is a completely antisymmetric unit tensor. We can use it to preform some amazing operations.

Cross product

$$\boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{C}$$
$$A_i = \varepsilon_{ijk} B_j C_k$$

Let's take a closer look

$$A_x = \varepsilon_{xjk} B_j C_k = \overbrace{\varepsilon_{xyz}}^{1} B_y C_z + \overbrace{\varepsilon_{xzy}}^{-1} B_z C_y$$
$$A_y = \varepsilon_{yjk} B_j C_k = \overbrace{\varepsilon_{yxz}}^{-1} B_x C_z + \overbrace{\varepsilon_{yzx}}^{1} B_z C_x$$
$$A_z = \varepsilon_{zjk} B_j C_k = \overbrace{\varepsilon_{zxy}}^{1} B_x C_y + \overbrace{\varepsilon_{zyx}}^{-1} B_y C_x$$

Now you can see why we just write $A_i = \varepsilon_{ijk} B_j C_k$.

Partial derivative, Grad, Div and Curl

We can write a partial derivative in general form as follows

$$\partial_i \doteq \frac{\partial}{\partial x_i}$$

Grad

Thus

$$\nabla \phi \doteq \partial_i \phi$$

We know this will generate a vector since there is nothing to sum over.

Divergence

Thus

$$\nabla \cdot \mathbf{A} = \nabla \cdot A_i \doteq \partial_i A_i = \partial_i \delta_{ij} A_j$$

Since two indexes are the same we will sum to generate at scaler.

Curl

Thus

$$\nabla \times \mathbf{A} = \nabla \times A_i \doteq \varepsilon_{ijk} \partial_j A_k = c_i$$

Now we have three indexes so we sum twice to get a vector. This is a prime example of contraction.

Rank 2 Tensors

The Kronoker delta was previously mentioned to be rank 2 mixed tensor, but without further explanation. There are three types of rank 2 tensors

$$A'^{ij} = \sum_{kl} \frac{\partial x'_i}{\partial x_k} \frac{\partial x'_j}{\partial x_l} A^{kl} \text{ contravariant}$$

$$B'^{i}_{j} = \sum_{kl} \frac{\partial x'_i}{\partial x_k} \frac{\partial x_l}{\partial x'_j} B^{k}_{l} \text{ mixed}$$

$$C'_{ij} = \sum_{kl} \frac{\partial x_k}{\partial x'_i} \frac{\partial x_l}{\partial x'_j} C_{kl} \text{ covariant}$$

Symmetry

$$A_{ij} = \frac{1}{2} \overbrace{(A_{ij} + A_{ji})}^{symetric} + \frac{1}{2} \overbrace{(A_{ij} - A_{ji})}^{antisymetric}$$

As used in P220

The functions of position are given by

$$f = x'_i$$
$$g = x'_j$$
$$h = x'_k (\text{not given in class})$$

See Griffiths Appendix A on Vector Calculus and Curvilinear Coordinates.

Vector identity examples

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{lmk} &= \delta_{jm}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \\ &= \delta_{il}\delta_{ji} - \delta_{im}\delta_{ml} \\ &= 3\delta_{il} - \delta_{il} = 2\delta_{il} \end{aligned}$$

Okay now lets derive identities

$$\begin{aligned} \mathbf{d} &= \mathbf{a} \times \underbrace{(\mathbf{b} \times \mathbf{c})}_{\mathbf{e}} \\ d_i &= \varepsilon_{ijk} a_j e_k = \varepsilon_{ijk} a_j \varepsilon_{klm} b_l c_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j b_l c_m \\ &= b_i(a_j c_j) - c_i(a_j b_j) \\ \mathbf{d} &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \end{aligned}$$

And now for another identity

$$\begin{aligned} \nabla \times (\nabla \phi) &= 0 \\ \varepsilon_{ijk} \partial_j (\partial_k \phi) &= 0 \end{aligned}$$

And another

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{A}) &= 0 \\ \partial_i \varepsilon_{ijk} \partial_j A_x &= 0 \end{aligned}$$

And now a really complicated one

$$\begin{aligned} \mathbf{c} &= \boldsymbol{\nabla} \times (\boldsymbol{a} \times \boldsymbol{b}) \\ c_i &= \varepsilon_{ijk} \partial_j (\varepsilon_{kml} a_l b_m) \\ c_i &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj}) \left[(\partial_j a_l) b_m + a_l (\partial_j b_m)\right] \\ c_i &= (\partial_j a_i) b_i + a_i (\partial_j b_j) - (\partial_j a_j) b_i - a_j (\partial_j b_j) \\ \mathbf{c} &= (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} \end{aligned}$$

References and Further Reading

Arfkin and Weber, Mathematical Methods for Physicists 5^{th}ed. (2001), Section 2.6 Tensor Analysis
Feynman, Lectures on Physics, Lecture II 31

Related topics

- Spherical Harmonics
- Group Theory

Spherical Harmonics
Group Theory