O(2)-SYMMETRY OF 3D STEADY GRADIENT RICCI SOLITONS

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In 1982, Hamilton introduced the Ricci flow equation $\partial_t g(t) = -2\text{Ric}$. Ricci solitons are Riemannian manifolds satisfying

$$\text{Ric} = \frac{1}{2} \mathcal{L}_X g + \lambda g,$$

where $X$ is a vector field and $\lambda \in \mathbb{R}$. Ricci solitons generate self-similar Ricci flows, and often arise as singularity models of compact Ricci flows. In particular, a steady gradient soliton is a soliton with $\lambda = 0$ and $X = \nabla f$ for some smooth function $f$, and thus satisfies

$$\text{Ric} = \nabla^2 f.$$

In dimension 2, the only steady gradient Ricci soliton is Hamilton’s cigar soliton, which is rotationally symmetric. In dimension $n \geq 3$, Bryant constructed a steady gradient Ricci soliton which is rotationally symmetric. In dimension 3, all steady gradient Ricci solitons are non-negatively curved and they are asymptotic to sectors of angle $\alpha \in [0, \pi]$. For instance, the Bryant soliton is asymptotic to a ray ($\alpha = 0$), and the soliton $\mathbb{R} \times \text{Cigar}$ is asymptotic to a half-plane ($\alpha = \pi$).

In dimension 3, Hamilton conjectured that there exists a 3D steady gradient Ricci soliton that is asymptotic to a sector with angle in $(0, \pi)$, which is called a 3D flying wing. We constructed a family of $\mathbb{Z}_2 \times O(2)$-symmetric flying wings, which confirmed Hamilton’s conjecture [5], and the asymptotic cone angles of these flying wings can take any value in $(0, \pi)$ [6]. In dimension $n \geq 4$, we constructed a family of non-collapsed, $\mathbb{Z}_2 \times O(n-1)$-symmetric steady gradient solitons with positive curvature operators, which are not rotationally symmetric [5]. In dimension 3, we prove

**Theorem ([7]).** All 3D steady gradient Ricci solitons are $O(2)$-symmetric.

We give a sketch of the $O(2)$-symmetry theorem. We may assume without loss of generality that it is not a Bryant soliton. Then first step is to understand the asymptotic geometry at infinity. We show there are two “edges”, two limits ($\mathbb{R} \times \text{Cigar}$ and $\mathbb{R}^2 \times S^1$), one critical point, an “almost” $\mathbb{Z}_2$-isometry, including the equality

$$\lim_{s \to \infty} R(\Gamma(s)) = \lim_{s \to -\infty} R(\Gamma(s)) = 4,$$

where $R$ is the scalar curvature and $\Gamma$ is an integral curve of $\nabla f$. In this step we used Brendle’s uniqueness of non-collapsed 3D steady solitons [1]. A corollary of this step is the uniqueness of the Bryant soliton among all 3D steady gradient solitons on $\mathbb{R}^3$ asymptotic to rays.
On the one hand, by a symmetry improvement argument, we can construct inductively an approximating \( SO(2) \)-symmetric metric \( \bar{g} \) satisfying
\[
|\bar{g} - g|_{C^{1,0}} \leq e^{- (2 + \epsilon_0) d_g (\cdot, \Gamma)},
\]
for some \( \epsilon_0 > 0 \). Let \( X \) be the killing field of the approximate metric \( \bar{g} \). Then evolve \( X \) by the heat equation \( \partial_t X(t) = \Delta_t X(t) + \text{Ric}(X(t)) \). Then the lie derivative \( \mathcal{L}_{X(t)} g(t) \) satisfies the linearized Ricci-Deturck flow \( \partial_t h(t) = \Delta_{L,g(t)} h(t) \). By Anderson-Chow estimate this implies
\[
\partial_t |h(t)| \leq \Delta_t |h(t)| + \frac{2|\text{Ric}|^2}{R} |h(t)|.
\]
The evolution equation of \( R \) implies that \( R \) is a solution to this linear heat equation, and we can moreover show that
\[
R(x) \geq C(\epsilon_0)^{-1} e^{- 2(1 + \epsilon_0) d_g (x, \Gamma)},
\]
Therefore, by combining (1) and (2), we can use \( R \) as an upper barrier function, and show by a heat kernel argument that \( |h(t)| \) decays to zero as \( t \to \infty \). So \( X(t) \) converges to a non-zero smooth vector field \( X_\infty \) satisfying \( \mathcal{L}_{X_\infty} g = 0 \). The killing field \( X_\infty \) thus generates a \( SO(2) \)-isometry, and we can furthermore show that it is also an \( O(2) \)-isometry.

We can compare 3D steady gradient Ricci solitons with convex translators in \( \mathbb{R}^3 \), under which the \( O(2) \)-symmetry is compared with the reflectional symmetry (i.e. \( \mathbb{Z}_2 \)-symmetry). By the classification for the convex translators in \( \mathbb{R}^3 \) [4], I believe:

**Conjecture 1.** If two 3D flying wings have the same asymptotic cone angle, then they are isometric modulo rescalings.

It is also interesting to see whether the \( O(2) \)-symmetry holds for 3D ancient Ricci flows, as well as higher dimensional steady gradient solitons with positive curvature operator. In dimension 4, one can ask whether the \( O(3) \)-symmetry holds for all non-collapsed 4D steady gradient solitons.
Moreover, the recent breakthrough works of Choi-Haslhofer-Hershkovits classified all non-collapsed translators in $\mathbb{R}^4$ [3, 2], which inspires the following conjecture in Ricci flow:

**Conjecture 2.** The only non-collapsed steady gradient solitons with non-negative curvature operator are the 4D Bryant soliton, and the family of $\mathbb{Z}_2 \times O(3)$-symmetric solitons I constructed in [5]. Moreover, the blow-down of each of the $\mathbb{Z}_2 \times O(3)$-symmetric soliton is a ray.

**References**


