Online Linear Programming: Applications and Extensions

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Offline and Online Linear Programming

$$\begin{array}{ll} \text{maximize}_{\mathbf{x}} & \sum_{t=1}^{n} r_t x_t \\ \text{subject to} & \sum_{t=1}^{n} \mathbf{a}_t x_t \leq \mathbf{b}, \\ & x_t \in \{0,1\} \ (0 \leq x_t \leq 1), \quad \forall t = 1, ..., n. \end{array}$$

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r_t: reward/revenue offered by the t-th customer/order

$\mathbf{a}_t \in R^m$: the bundle of resources requested by the t-th order

x_t: acceptance or rejection decision to the t-th order

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The objective $\sum_{t=1}^{n} r_t x_t$: the total collected revenue.

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- an irrevocable decision must be made as soon as an order arrives (without knowing the future data).
- Conform to resource capacity constraints at the end.

A Toy Example

Consider an auction problem:

	Bid 1($t = 1$)	Bid 2($t = 2$)	.....	Inventory($\mathbf{b}$)
Reward(r_t)	\$100	\$30	...	
Decision	x_1	x_2	...	
Pants	1	0	...	100
Shoes	1	0	...	50
T-shirts	0	1	...	500
Jackets	0	0	...	200
Hats	1	1	...	1000

where the decision for each customer/bidder is "accept" ($x_t = 1$) or "reject" ($x_t = 0$)

Price Mechanism for OLP I

The problem would be easy if there are "ideal prices":

	Bid 1($t = 1$)	Bid 2($t = 2$)	.....	Inventory($\mathbf{b}$)	$\mathbf{p}^*$
Bid(r_t)	\$100	\$30	...		
Decision	x_1	x_2	...		
Pants	1	0	...	100	\$45
Shoes	1	0	...	50	\$45
T-shirts	0	1	...	500	\$10
Jackets	0	0	...	200	\$55
Hats	1	1	...	1000	\$15

so that the online decision can be made by comparing the reward and "bundle cost" for each bid.

Primal and Dual Offline LPs

$$P: \begin{array}{ll} \max & \mathbf{r}^\top \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{e} \end{array} \qquad D: \begin{array}{ll} \min & \mathbf{b}^\top \mathbf{p} + \mathbf{e}^\top \mathbf{s} \\ \text{s.t.} & A^\top \mathbf{p} + \mathbf{s} \geq \mathbf{r} \\ & \mathbf{p} \geq \mathbf{0}, \mathbf{s} \geq \mathbf{0} \end{array}$$

where the decision variables are $\mathbf{x} \in R^n$, $\mathbf{p} \in R^m$, $\mathbf{s} \in R^n$ ($\mathbf{e}$ vector of all ones).

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where the decision variables are $\mathbf{x} \in R^n$, $\mathbf{p} \in R^m$, $\mathbf{s} \in R^n$ ($\mathbf{e}$ vector of all ones).

Denote the primal/dual optimal solution as $\mathbf{x}^*$, $\mathbf{p}^*$, $\mathbf{s}^*$, then LP duality/complementarity theory tells that for $t = 1, ..., n$,

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x_t^* = \begin{cases} 1, & r_t > \mathbf{a}_t^\top \mathbf{p}^* \\ 0, & r_t < \mathbf{a}_t^\top \mathbf{p}^* \end{cases}
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(x_t^* may take non-integer value when $r_t = \mathbf{a}_t^\top \mathbf{p}^*$).

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(x_t^* may take non-integer value when $r_t = \mathbf{a}_t^\top \mathbf{p}^*$).

Most online LP algorithms are based on learning $\mathbf{p}^*$ by dynamically solving small sample-sized LPs based on revealed data.

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We illustrate a simple Learning Algorithm:

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  and get the optimal dual solution $\hat{\mathbf{p}}$;
- Determine the future allocation x_t as:

$$
x_t = \begin{cases} 0 & \text{if } r_t \leq \hat{\mathbf{p}}^T \mathbf{a}_t \\ 1 & \text{if } r_t > \hat{\mathbf{p}}^T \mathbf{a}_t \end{cases}
$$

One may update the prices periodically and/or set $x_t = 0$ as soon as a resource is exhausted.

Data/Model Assumptions for Analyses

Stochastic Input (i.i.d) Model:

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Both assume boundedness:

(b) $|r_t| \leq \bar{r}$ and $\|\mathbf{a}_t\|_\infty \leq \bar{a}$ for all t

(c) The right-hand-side $\mathbf{b} = n \cdot \mathbf{d}(> \mathbf{0})$.

All early works also assume $r_t \geq 0, \mathbf{a}_t \geq 0$ (one-sited market).

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- What are the necessary and sufficient assumptions on the right-hand-side $\mathbf{b}$ to achieve $(1-\epsilon)$-competitive ratio of the expected online reward over the optimal offline reword?

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- What are the necessary and sufficient assumptions on the right-hand-side $\mathbf{b}$ to achieve $(1 - \epsilon)$-competitive ratio of the expected online reward over the optimal offline reword?
- If the right-hand-side $\mathbf{b}$ (such as $\mathbf{b} = O(n)$), what is the best achievable gap or regret between the two?

Competitive Ratio Summary of One-Sited Market

The journey to design $(1-\epsilon)$-competitive online algorithms against benchmark OPT-Optimal Offline Objective Value where $B = \min_i b_i$:

	Sufficient Condition
Kleinberg (2005)	$B \geq \frac{1}{\epsilon^2}$, for $m = 1$
Devanur et al (2009)	$OPT \geq \frac{m^2 \log n}{\epsilon^3}$
Feldman et al (2010)	$B \geq \frac{m \log n}{\epsilon^3}$ and $OPT \geq \frac{m \log n}{\epsilon}$
Agrawal/Wang/Y (2010,14)	$B \geq \frac{m \log n}{\epsilon^2}$ or $OPT \geq \frac{m^2 \log n}{\epsilon^2}$
Molinaro/Ravi (2013)	$B \geq \frac{m^2 \log m}{\epsilon^2}$
Kesselheim et al (2014)	$B \geq \frac{\log m}{\epsilon^2}$
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Remarks

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- The key difference between OLP and Online Convex Optimization with Constraints (OCOwC):
  - Online LP problem employs a stronger benchmark where the decision variables are allowed to take different values at each time period
  - OCOwC (Mahdavi et al., 2012; Yu et al., 2017; Yuan and Lamperski, 2018) and OCO problems usually consider a stationary benchmark where the the decision variables are required to be the same at each time period.

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  - OCOwC (Mahdavi et al., 2012; Yu et al., 2017; Yuan and Lamperski, 2018) and OCO problems usually consider a stationary benchmark where the the decision variables are required to be the same at each time period.
- Recent focuses are on dealing with two-sited markets/platforms, regret analyses, simple and fast algorithms, interior-point online algorithm, extension to bandit models and the Fisher market.

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Regret Analysis and Model

Let "offline" optimal solution be $\mathbf{x}^*$ and "online" solution of n orders be $\mathbf{x}_n$, and

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Then define

$$\Delta_n = \sup \mathbb{E}\left[R_n^* - R_n\right], \quad v(\mathbf{x}) = \sup \mathbb{E}\left[\|\left(A\mathbf{x} - \mathbf{b}\right)^+\|_2\right]$$

where the expectation is taken with respect to i.i.d distribution or random permutation, and the sup operator is over all permissible distributions and admissible data.

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Remark: A bi-criteria performance measure, but one can easily modify the algorithms such that the constraints are always satisfied at the end.

Equivalent Form of the Dual Problem

Recall the dual problem

$$\min \ \mathbf{b}^\top \mathbf{p} + \sum_{t=1}^{n} s_t \quad \text{s.t. } s_t \geq r_t - \mathbf{a}_t^\top \mathbf{p}, \forall t; \quad \mathbf{p}, \mathbf{s} \geq \mathbf{0}$$

can be rewritten as

$$\min \ \mathbf{b}^\top \mathbf{p} + \sum_{t=1}^{n} \left(r_t - \mathbf{a}_t^\top \mathbf{p} \right)^+ \quad \text{s.t. } \ \mathbf{p} \geq \mathbf{0}$$

where $(\cdot)^+$ is the positive-part or ReLU function.

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where $(\cdot)^+$ is the positive-part or ReLU function.

After normalizing the objective, it becomes

$$\min_{\mathbf{p} \geq \mathbf{0}} \mathbf{d}^\top \mathbf{p} + \frac{1}{n} \sum_{t=1}^{n} \left(r_t - \mathbf{a}_t^\top \mathbf{p} \right)^+$$

which can be viewed as a simple-sample-average (SSA) (with n sample points) of a stochastic optimization problem under an i.i.d distribution.

Convergence of $\mathbf{p}_n^*$

Theorem (Li & Y (2019, OR 2021))

Denote the n-sample SSA optimal solution by $\mathbf{p}_n^*$. Then, for the stochastic input model under moderate conditions that guarantee a local strong convexity of the underlying stochastic program $f(p)$ around its optimal solution $\mathbf{p}^*$, there exists a constant C such that

$$\mathbb{E}\|\mathbf{p}_n^* - \mathbf{p}^*\|_2^2 \leq \frac{Cm \log \log n}{n}$$

holds for all $n > m$.

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holds for all $n > m$.

This is L_2 convergence for the dual optimal solution. Heuristically,

$$\mathbf{p}_n^* \approx \mathbf{p}^* + \frac{1}{\sqrt{n}} \cdot \textbf{Noise}$$

Dual-Gradient Online Algorithm for Binary LP

1: Input: $\mathbf{d} = \mathbf{b}/n$

2: Initialize $\mathbf{p}_1 = \mathbf{0}$

3: For $t = 1, 2, ..., n$

4:

$$x_t = \begin{cases} 1, & \text{if } r_t > \mathbf{a}_t^\top \mathbf{p}_t \\ 0, & \text{if } r_t \leq \mathbf{a}_t^\top \mathbf{p}_t \end{cases}$$

5: Compute

$$\mathbf{p}_{t+1} = \mathbf{p}_t + \gamma_t \left(\mathbf{a}_t x_t - \mathbf{d} \right)$$
$$\mathbf{p}_{t+1} = \mathbf{p}_{t+1} \vee \mathbf{0}$$

6: $\mathbf{x} = (x_1, ..., x_n)$

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Line 5 performs (projected) stochastic gradient descent in the dual, where step-size $\gamma_t = \frac{1}{\sqrt{n}}$ or $\gamma_t = \frac{1}{\sqrt{t}}$.

Performance Analysis

Theorem (Li, Sun & Y (2020, NeurIPS))

With step size $\gamma_t = 1/\sqrt{n}$, the regret and expected constraint violation of the algorithm satisfy

$$\mathbb{E}[R_n^* - R_n] \leq \tilde{O}(m\sqrt{n}), \quad \mathbb{E}\left[v(\mathbf{x})\right] \leq \tilde{O}(m\sqrt{n}).$$

under both the stochastic input and the random permutation models.

- $\tilde{O}$ omits the logarithm terms and the constants related to $(\bar{a}, \bar{r})$, but the algorithm does not require any prior knowledge on the constants.
- The optimal offline value is in the range $O(mn)$.
- The algorithms runs in nm times - the time to read in the data.
- It can be implemented by posting prices: customers decide and keep r_t's private.

Adaptive Fast Online Algorithm for Binary LP

Initialize $\mathbf{b}_1 = \mathbf{b}$ and $\mathbf{p}_1 = \mathbf{0}$

For $t = 1, 2, ..., n$

$$x_t = \begin{cases} 1, & \text{if } r_t > \mathbf{a}_t^\top \mathbf{p}_t \\ 0, & \text{if } r_t \le \mathbf{a}_t^\top \mathbf{p}_t \end{cases}$$

Compute

$$\begin{aligned} \mathbf{p}_{t+1} &= \mathbf{p}_t + \gamma_t \left(\mathbf{a}_t x_t - \frac{1}{n-t+1} \mathbf{b}_t \right) \\ \mathbf{p}_{t+1} &= \mathbf{p}_{t+1} \vee \mathbf{0} \end{aligned}$$

Update remaining inventory: $\mathbf{b}_{t+1} = \mathbf{b}_t - \mathbf{a}_t x_t$.

Return $\mathbf{x} = (x_1, ..., x_n)$

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5: Update remaining inventory: $\mathbf{b}_{t+1} = \mathbf{b}_t - \mathbf{a}_t x_t$.

6: Return $\mathbf{x} = (x_1, ..., x_n)$

The average inventory vector is adaptively adjusted based on the previous realizations/decisions – this is a non-stationary approach.

Nonadaptive vs. Adaptive

The first resource (sequential) usages in 10 runs of the algorithms.

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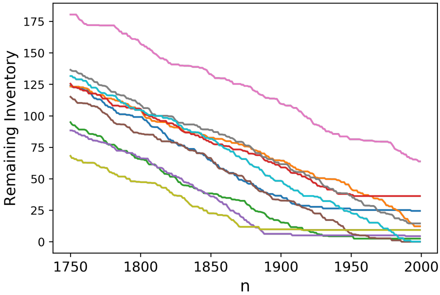

Figure: Nonadaptive

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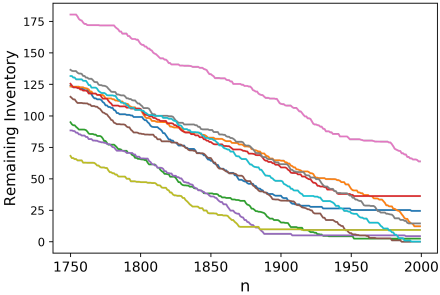

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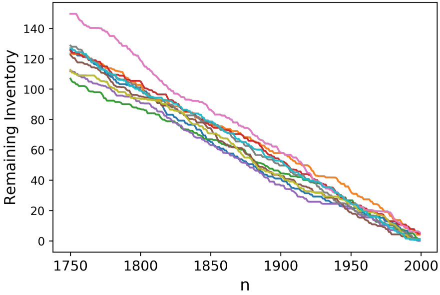

Figure: Adaptive

Fast Online LP Algorithm for Solving Offline LPs?

A crucial assumption is that the right-hand-side $\mathbf{b} = n\mathbf{d}$ scales linearly with n. Is there a remedy for this case where we do not want to compromise the computational efficiency of simple online algorithm?

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Consider a "Replicated" LP from the original LP

$$\begin{aligned} \max \ & \sum_{t=1}^{n}\sum_{h=1}^{k} r_t x_{th} \\ \text{s.t.} \ & \sum_{t=1}^{n}\sum_{h=1}^{k} \mathbf{a}_t x_{th} \le k\mathbf{b}, \ 0 \le x_t \le 1, \quad t = 1, ..., n. \end{aligned}$$

Algorithm: Solve the new LP with Simple Online Algorithm and use $x_t = \frac{1}{k}(x_{t1} + ... + x_{tk})$ as the solution to the original LP.

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Algorithm: Solve the new LP with Simple Online Algorithm and use $x_t = \frac{1}{k}(x_{t1} + ... + x_{tk})$ as the solution to the original LP.
The algorithm runs in $O(kmn)$ times.

Performance of the Variable-Replicating Algorithm

Proposition (Gao, Sun, Ye & Y (2021))

Under the random permutation model, the variable-replicating algorithm finds a solution for the original LP that achieves at least $(1-\mathcal{O}(\varepsilon))OPT$ *with the constraint violation bounded by* $(1+\mathcal{O}(\varepsilon))B$ *where* $B = \min_{i=1,\ldots,m} b_i$*, if* $\sqrt{k}B^2 \geq \frac{n^{3/2}\log kn}{\varepsilon}$ *and* $\sqrt{k}B \geq \frac{m\sqrt{n}}{\varepsilon}$ *for any* $\varepsilon > 0$*. Moreover, if* $kn \geq m$*, the relative constraint violation can be bounded by* $(1+\mathcal{O}(\frac{\varepsilon}{\sqrt{m}}))$*.*

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Takeaway: k times more computation cost for a $\sqrt{k}$ factor improvement in regret performance.

Multi-knapsack Problem Instances - Binary LP

Benchmark dataset of Chu & Beasley implementation

		V.R. Alg.	Gurobi
$m = 5, n = 500, k = 50$	Time	0.000	0.211
	Cmpt. Ratio	88.2%	95.3%
$m = 5, n = 500, k = 1000$	Time	0.007	0.211
	Cmpt. Ratio	89.2%	95.3%
$m = 8, n = 10^3, k = 50$	Time	0.004	3.800
	Cmpt. Ratio	89.9%	99.0%
$m = 8, n = 10^3, k = 1000$	Time	0.077	3.800
	Cmpt. Ratio	95.6%	99.0%
$m = 64, n = 10^4, k = 50$	Time	0.013	> 60
	Cmpt. Ratio	90.3%	98.7%
$m = 64, n = 10^4, k = 1000$	Time	0.223	> 60
	Cmpt. Ratio	96.4%	98.7%

Fast Online Algorithm as Pre-Classifier for LP

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In LP, a column generation techniques is popularly used when $n >> m$:

- Constructed a Restricted Master Problem (RMP) defined by a small subset of variables of the original problem
- Solve RMP and reselect initially unselected variables into RMP

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This is precisely where the fast online LP algorithm does a good job - classify variables being positive or zero at an optimal solution in a short time.

Implementation in LP Solvers

More precisely, the fast online LP solution can be interpreted as a "score" of how likely a variable is to be optimal basic.

We run online algorithm to obtain $\hat{\mathbf{x}}$, set a threshold ε and select the columns in $\mathbb{I}_{\{\hat{\mathbf{x}}>\varepsilon\}}$. For benchmark LP problems that have more columns than rows (such as **rail4284**, **s82**, and **scpm1** from the Mittelmann's Simplex Benchmark), the online solution identifies more than 90% of the primal optimal basis on average.

This technique has been adopted in the emerging LP solver COPT - a new state of art LP solver.

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A “Fairer” Online LP Algorithm

Recall the online LP formulation (changing n to T as in the literature)

$$\max \sum_{t=1}^{T} r_t x_t \quad \text{s.t.} \quad \sum_{t=1}^{T} \mathbf{a}_t x_t \leq \mathbf{b}, \quad x_t \in [0,1]$$

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A finite-type assumption: $\mathbb{P}((r_t, \mathbf{a}_t) = (\mu_j, \mathbf{c}_j)) = p_j$ (unknown to the decision maker) for $j = 1, ..., J$. The offline problem with the knowledge:

$$\max \sum_{j=1}^{J} p_j \mu_j y_j \quad \text{s.t.} \quad \sum_{j=1}^{J} p_j \mathbf{c}_j y_j \leq \mathbf{b}/T, \quad y_j \in [0,1]$$

where y_j is the acceptance probability for each customer type j.

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	Benchmark	Regret Bound	Key Assumption(s)
Jasin and Kumar (2012)	Fluid	Bounded	Nondeg., distrib. known
Jasin (2015)	Fluid	$\tilde{O}(\log T)$	Nondeg.
Vera et al. (2019)	Hindsight	Bounded	Distrib. known
Bumpensanti and Wang (2020)	Hindsight	Bounded	Distrib. known
Asadpour et al. (2019)	Full flex.	Bounded	Long-chain, ξ-Hall condition
Chen, Li & Y (2021)	Fluid	Bounded	Partial Nondeg.

Behavior of the Simplex and Interior-Point

The key in Chen et al. (2021) paper is to use the interior-point algorithm for solving the sample LPs with sample proportion $\hat{p}_j$

$$\max \sum_{j=1}^{J} \hat{p}_j \mu_j y_j \quad \text{s.t.} \quad \sum_{j=1}^{J} \hat{p}_j \mathbf{c}_j y_j \leq \mathbf{b}/T, \quad y_j \in [0, 1],$$

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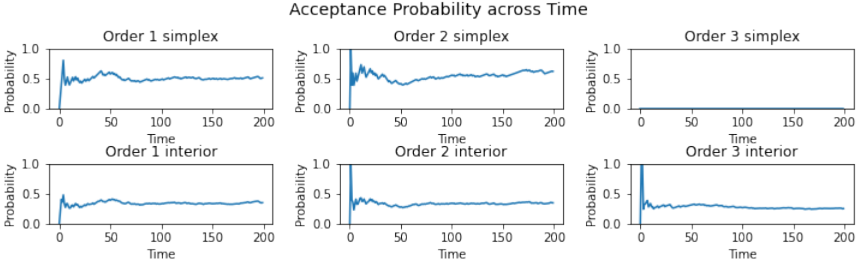

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But these individuals/groups could have different sensitive features, such as demographic, race, and gender, and areas in Hospital Admission and Hotel/Flight booking application.

Could we design an online algorithm/allocation-rule such as, while maintain the efficiency in objective value, all individual/groups get a fairer allocation shares?

Fairer Solution for the Offline Problem

We define $\mathbf{y}^*$, the fair offline optimal solution of the LP problem

$$\max \sum_{j=1}^{J} p_j \mu_j y_j, \quad \text{s.t.} \quad \sum_{j=1}^{J} p_j \mathbf{c}_j y_j \leq \mathbf{b}/T, \quad y_j \in [0, 1]$$

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Let $\mathbf{y}_t$ be allocation rule at time t which encodes the accepting probabilities under algorithm π. Then we define the cumulative unfairness of the online algorithm π as

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This definition is consistent with the definition of fair classifiers/regressors in fair machine learning.

Our Result

We develop an algorithm [Chen, Li & Y (2021)] that achieves

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Key ideas in algorithm design:

- At each time t, we use interior-point method to obtain the sample analytic-center solution $\mathbf{y}_t$, and it is necessary to achieve the performance under weak non-degeneracy assumption and maintain fairness.
- We also adjust the right-hand-side of the LP constraints properly to ensure (i) the depletion of binding resources and (ii) non-binding resources not affecting the fairness.

The use of interior-point method also relaxes a non-degeneracy assumption in previous analysis

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At each time $t \in [T]$, an arm i is selected to pull. The realized reward $\hat{r}_t$ and resources cost $\hat{\mathbf{c}}_t$ satisfying

$$\mathbb{E}[\hat{r}_t | i] = \mu_i, \quad \mathbb{E}[\hat{\mathbf{c}}_t | i] = \mathbf{c}_i.$$

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Goal: Select a subset of winning/optimal arms to maximize the total reward subject to the resource capacity constraints - pro-actively explore arms and exploit learned data.

Offline Linear Program (LP) and Regret

With mean reward $\boldsymbol{\mu} = (\mu_1, ..., \mu_k)$ and mean resource-cost $(\mathbf{c}_1, ..., \mathbf{c}_k)$ of arms, consider the following deterministic offline LP,

$$\max_{\mathbf{x}} \sum_{i=1}^{k} \mu_i x_i \quad \text{s.t.} \quad \sum_{i=1}^{k} \mathbf{c}_i x_i \leq \mathbf{b}, x_i \geq \mathbf{0}, i \in [k]$$

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Here x_i represents the optimal fractional number of playing i-th arm if everything is deterministic and known

Denote its optimal value as OPT (the benchmark) and let τ be the stopping time as soon as one of the resources is depleted. Then the problem-dependent regret

$$Regret(\mathcal{P}) = OPT - \mathbb{E}\left[\sum_{t=1}^{\tau} r_t\right],$$

where $\mathcal{P}$ encapsulates the parameters related to the underlying data distribution.

Literature and Our Result

	Paper	Result
$\mathcal{P}$-Independent	Badanidiyuru et. al. (13) Agrawal and Devanur (14)	$O(poly(m, k) \cdot \sqrt{T})$
$\mathcal{P}$-Dependent	Flajolet and Jaillet (15) Sankararaman and Slivkins (20) Li, Sun & Y (21)	$\tilde{O}(2^{m+k} \log T)$ $\tilde{O}(k \log T)$ for $m = 1$ $\tilde{O}\left(m^4 + k \log T\right)$

The problem-dependent bounds all involve parameters related to the non-degeneracy and the reduced cost of the underlying LP, while our work has the mildest assumption and requires no prior knowledge of these parameters.

Dual LP and Reduced Cost

$$
\begin{array}{llll}
\textit{Primal}: & \max & \mu^\top \mathbf{x} & \\
& \text{s.t.} & C\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge \mathbf{0} &
\end{array}
\qquad
\begin{array}{llll}
\textit{Dual}: & \min & \mathbf{b}^\top \mathbf{y} \\
& \text{s.t.} & C^\top \mathbf{y} \ge \mu, \mathbf{y} \ge \mathbf{0}
\end{array}
$$

Denote $\mathbf{x}^* \in R^k$ and $\mathbf{y}^* \in R^m$ as optimal solutions

Define reduced cost (profit) for i-th arm $\Delta_i := \mathbf{c}_i^\top \mathbf{y}^* - \mu_i$ and the non-basic variable set $\mathcal{I}' = \{i : \Delta_i > 0\}$.

Proposition (Li, Sun & Y (2021, ICML))

The regret of a BwK algorithm has the following upper bound:

$$Regret(\mathcal{P}) \le \sum_{i \in \mathcal{I}'} \Delta_i \mathbb{E}[n_i(\tau)] + \mathbb{E}[\mathbf{b}^{(\tau)}]^\top \mathbf{y}^*$$

- $\mathbf{b}^{(t)}$: remaining resource at time t
- $n_i(t)$: the number of times that i-th (non-optimal) arm is played up to time t

Implications of the Regret Upper Bound

Two tasks to accomplish to reduce the regret:

Task I: Control the number of plays $n_i(\tau)$ for non-optimal arms $i \in \mathcal{I}'$ which corresponds to the first component in the regret bound

$$\sum_{i \in \mathcal{I}'} \Delta_i \mathbb{E}[n_i(\tau)]$$

Playing each non-optimal arm will induce a cost/waste of Δ_i.

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Task II: Make sure no valuable resources $\mathbf{b}_j^{(\tau)}$ left unused, which corresponds to the second component in the regret bound

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Recall τ is the time that one of the resources is exhausted.

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Task II is often overlooked in the existing BwK literature.

Our Approach: A Two-Phase Algorithm

- Phase I: Identify the optimal arms with as fewer number of plays as possible by designing an "importance score" for arm i:

$$
\begin{aligned}
OPT_i := \max \quad & \boldsymbol{\mu}^\top \mathbf{x} \\
\text{s.t.} \quad & C\mathbf{x} \le \mathbf{b},\ x_i = 0, \mathbf{x} \ge \mathbf{0}.
\end{aligned}
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  Implication: A larger value of $OPT - OPT_i \Rightarrow x_i$ important and likely to represent an optimal arm. Our algorithm then maintains upper confidence bound (UCB)/lower confidence bound (LCB) to estimate OPT and OPT_i based are samples.

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  After $t' = O(\frac{k \log T}{\sigma^2 \delta^2})$ times of Phase I, the non-optimal arm variables are identified as set $\mathcal{I}'$ and they would be removed from further consideration, and then we start

- Phase II: Use the remaining arms to exhaust the resource through an adaptive procedure such that no valuable resources are wasted.

Adaptive Algorithm for filling the knapsacks:

For $t = t' + 1, ..., T$

1 Solve the UCB-LP and denote its optimal solution as $\tilde{\mathbf{x}}$

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{i=1}^{k} \left(\hat{\mu}_i(t) + \sqrt{\frac{2 \log T}{n_i(t)}} \right) x_i \\ \text{s.t.} \quad & \sum_{i=1}^{k} \left(\hat{\mathbf{c}}_i(t) - \sqrt{\frac{2 \log T}{n_i(t)}} \right) x_i \leq \mathbf{b}^{(t-1)} \\ & \mathbf{x} \geq \mathbf{0}, x_i = 0 \text{ for } i \in \mathcal{I}' \end{aligned}$$

2 Normalize $\tilde{\mathbf{x}}$ into a probability and play an arm accordingly

3 Update the knapsack process $\mathbf{b}^{(t)}$ (remaining resource)

Combining the Two Phases

Proposition (Li, Sun & Y 2021, ICML)

The regret of our two-phase algorithm is bounded by

$$O\left(\frac{m^4}{\sigma^2\delta^2} + \frac{k\log T}{\delta^2}\right).$$

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Here the problem-dependent conditional numbers of the deterministic BwK LP problem are:

- σ is the minimum singular value of the sub-matrix of the constraint matrix C that corresponds to the optimal basis.

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- σ is the minimum singular value of the sub-matrix of the constraint matrix C that corresponds to the optimal basis.
- δ measures the difficulty of identifying optimal basic variables:

$$\min\left\{\min\{x_i^*|x_i^*>0\}, \min\{OPT - OPT_i|x_i^*>0\}, \min\{\Delta_i|x_i^*=0\}\right\}.$$

Combining the Two Phases

Proposition (Li, Sun & Y 2021, ICML)

The regret of our two-phase algorithm is bounded by

$$O\left(\frac{m^4}{\sigma^2\delta^2} + \frac{k \log T}{\delta^2}\right).$$

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These condition numbers generalize the optimality gap for the original (unconstrained) multi-armed bandits (Lai and Robbins (1985), Auer et al. (2002)).

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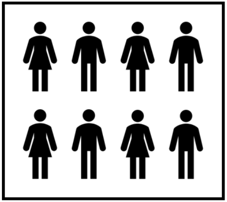
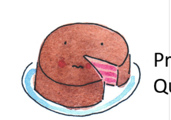
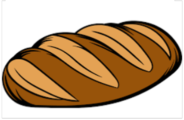
Fisher Markets for Resource-Allocation



Each agent i , with budget w_i, purchases an optimal bundle $\boldsymbol{x}_i$ given price $\boldsymbol{p}$



Price p_1
Quant.: c_1



p_2 , c_2

p_3 , c_3

How to setup "prices" for each good so that goods can be wholly allocated while keep each individual buyer/agent satisfied?

The Model: Fisher's Equilibrium Price

Buyer $i \in B$'s optimization problem for given prices p_j, $j \in G$.

$$\begin{array}{rl} \max & \mathbf{u}_i^T \mathbf{x}_i := \sum_{j \in G} u_{ij} x_{ij} \\ \text{s.t.} & \mathbf{p}^T \mathbf{x}_i := \sum_{j \in G} p_j x_{ij} \le w_i, \\ & x_{ij} \ge 0, \quad \forall j, \end{array}$$

Assume that the given amount of each good is c_j. The equilibrium price vector is the one that for all $j \in G$

$$\sum_{i \in B} x^*(\mathbf{p})_{ij} = c_j$$

where $\mathbf{x}^*(\mathbf{p})$ is a maximizer of the utility maximization problem for every buyer i.

The Aggregated Social Optimization Problem

$$
\begin{aligned}
\max \quad & \textstyle\sum_{i\in B} w_i \log(\mathbf{u}_i^T \mathbf{x}_i) \\
\text{s.t.} \quad & \textstyle\sum_{i\in B} x_{ij} = (\leq) c_j, \quad \forall j \in G \\
& x_{ij} \geq 0, \quad \forall i, j,
\end{aligned}
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Theorem (Eisenberg and Gale (1959))

Optimal dual (Lagrange) multiplier vector of equality constraints is an equilibrium price vector.

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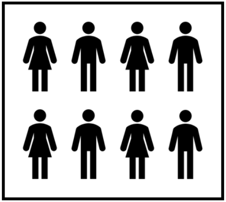
Question: how much would the aggregated social welfare be deteriorated from the offline setting? Could the algorithm be implemented by protecting privacy?

Online Fisher Markets



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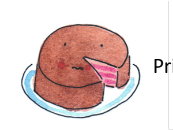
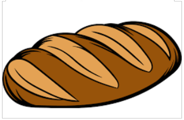


Price p_1^t



p_2^t

p_3^t

How to setup $\mathbf{p}^t$ for each good before buyer t comes so that the social welfare is maximized and capacity constraint violation is minimized?

Regret Analysis and Model

Let "offline" optimal solution be $\mathbf{x}^*$ and "online" solution of n orders be $\mathbf{x}$, and

$$R_n^* = \sum_{j=1}^{n}\sum_{i=1}^{n} w_i \log(\mathbf{u}_i^T \mathbf{x}_i^*), \quad R_n = \sum_{i=1}^{n} w_i \log(\mathbf{u}_i^T \mathbf{x}_i)$$

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Then define

$$\Delta_n = \sup \mathbb{E}\left[R_n^* - R_n\right], \quad v(\mathbf{x}) = \sup \mathbb{E}\left[\| (A\mathbf{x} - \mathbf{b})^+ \|_2\right]$$

where the expectation is taken with respect to i.i.d distribution, and the sup operator is over all permissible distributions and admissible data.

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where the expectation is taken with respect to i.i.d distribution, and the sup operator is over all permissible distributions and admissible data.

Remark: Again this a bi-criteria performance measure and, if $\Delta_n \leq o(n)$ (sublinear), then

$$\frac{(\prod_i (\mathbf{u}_i^T \mathbf{x}_i^*)^{w_i})^{1/n}}{(\prod_i (\mathbf{u}_i^T \mathbf{x}_i)^{w_i})^{1/n}} \leq e^{o(n)/n}.$$

Simple Price-Learning Algorithm

One may apply a similar primal price-learning algorithm, that is, solve the aggregated social problem based on arrived ϵ portion of buyers:

$$\begin{array}{lll} \text{maximize}_{\mathbf{x}} & \sum_{t=1}^{\epsilon n} w_t \log(\mathbf{u}_t^T \mathbf{x}_t) & \\ \text{subject to} & \sum_{t=1}^{\epsilon n} \mathbf{x}_t \leq \epsilon c_j, & j = 1, ..., m \\ & 0 \leq x_t. & \end{array}$$

One can set an initial positive price vector $\mathbf{p}^1$ and determine allocation $\mathbf{x}_t$ as the optimal solution for the individual maximization problem under price vector $\mathbf{p}^t$.

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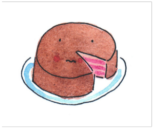
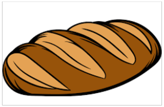
Is there an online algorithm that relies on only $\mathbf{x}_t$?

Negative Results: Two-Good Example

2 goods, each with a capacity of n





Two agent types specified by (Utility for Good 1, Utility for Good 2)

Type I: (1, 0)

Arrival Probability = 0.5

Type II: (0, 1)

Arrival Probability = 0.5

Theorem (Jelota & Y (2022))

The expect optimal social value

$$n\log(2) - 1 \le \mathbb{E}[R_n^*] \le n\log(2).$$

For any static pricing policy, either the expected regret or constraint violation is $\Omega\sqrt{n}$. Even using the optimal expected equilibrium prices Even using the optimal expect equilibrium prices for online allocation,

$$\mathbb{E}[\|(A\mathbf{x} - \mathbf{b})^+\|_2] \ge \sqrt{n}.$$

Consider the Dual of the Fisher Market

$$\min \ \mathbf{c}^\top \mathbf{p} - \sum_{t=1}^{n} w_t \log\left(\min_j \frac{p_j}{u_{tj}}\right) + \sum_{t=1}^{n} w_t(\log(w_t) - 1).$$

It can be, after removing the fixed part, equivalently rewritten as

$$\min \ (\frac{1}{n}\mathbf{c})^\top \mathbf{p} - \frac{1}{n}\sum_{t=1}^{n} w_t \log\left(\min_j \frac{p_j}{u_{tj}}\right)$$

which can be viewed as a simple-sample-average (SSA) (with n buyers) of a stochastic optimization problem under an i.i.d distribution, where $\mathbf{d} := \frac{1}{n}\mathbf{c}$ is the average resource allocation to each buyer.

Dual-Gradient Online Algorithm for Fisher-Markets

1: Initialize $\mathbf{p}^1 = \epsilon\mathbf{e}$, and for $t = 1, 2, ..., n$

2: Let $\mathbf{x}_t$ be the individual optimal bundle solution for price vector $\mathbf{p}^t$.

3: Update prices

$$\mathbf{p}_{t+1} = \mathbf{p}_t - \gamma_t \left(\mathbf{d} - \mathbf{x}_t\right)$$
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4: $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_n)$

Again, line 3 performs (projected) stochastic gradient step.

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Theorem (Jelota & Y (2022))

Under i.i.d. budget and utility parameters and when good capacities are $O(n)$, the algorithm achieves an expected regret $\Delta_n \leq O(\sqrt{n})$ and the expected constraint violation $v(\mathbf{x}) \leq O(\sqrt{n})$, where n is the number of arriving buyers.

Takeaways and Open Problems

- Geometrically aggregated welfare optimization is as easy as linear programming and more desirable in many social/economical settings.
- Tight upper and lower regret bounds for geometrically aggregated online social optimization?
- Geometrical aggregation to Bandit/MDP and other learning?
- Extensions to non-divisible goods for Fisher markets?
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OLP provides a data-driven and adaptive-learning policy/mechanism for decision making in real time...