# DRSOM: A Dimension-Reduced Second-Order Method for Nonconvex Optimization

## Distinguished Lectures to Commemorate HK PolyU and AMA Anniversary

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(Currently Visiting CUHK and HK PolyU)

# Today’s Talk

(1) **Motivation and Literature Review**

**(2) The Algorithm and Preliminary Convergence Analyses**

**(3) Computational Experiments**

# Part (1)

**Motivation and Literature Review**

# Early Complexity Analyses for Nonconvex Optimization

$$\min f(x), x \in X \ in \ \mathbb{R}^n,$$

- where $f$ is nonconvex and twice-differentiable,

$$g_k = \nabla f(x_k), H_k = \nabla^2 f(x_k)$$

- Goal: find $x_k$ such that:

$\| \nabla f(x_k) \| \leq \epsilon$ (primary)

$\lambda_{min}(H_k) \geq -\sqrt{\epsilon}$ (in active subspace, secondary)

- For the ball-constrained nonconvex QP (trust-region subproblem): O($\log\log(\epsilon^{-1})$); see Y (1989,93), Vavasis&Zippel (1990)
- For nonconvex QP with a polyhedral constraint: O($\epsilon^{-1}$); see Y (1998), Vavasis (2001)

# Standard methods for nonconvex optimization I

## First-order Method (FOM)

- Assume $f$ has $L$-Lipschitz cont. gradient
- Global convergence by, e.g., linear-search (LS)
- No guarantee for second-order conditions
- Worst-case complexity, $O(\epsilon^{-2})$; see the textbook by Nesterov (2004)

Each iteration requires $O(n^2)$ operations

# Standard methods for nonconvex optimization II

## Second-order Method (SOM): Newton-type methods

- Assume $f$ has $M$-Lipschitz cont. Hessian
- Global convergence by, e.g., linear-search (LS), Trust-region (TR), or Cubic Regularization
- Convergence to second-order points
- No better than $O(\epsilon^{-2})$, for traditional methods (steepest descent and Newton); according to Cartis et al. (2010) .

Each iteration requires $O(n^3)$ operations

# Analyses of SOM for nonconvex optimization since 2000

## Variants of SOM

- Trust-region with the fixed-radius strategy, $O(\epsilon^{-3/2})$, see the lecture notes by Ye† since 2005
- Cubic regularization, $O(\epsilon^{-3/2})$, see Nesterov and Polyak (2006), Cartis, Gould, and Toint (2011)
- A new trust-region framework, $O(\epsilon^{-3/2})$, Curtis, Robinson, and Samadi (2017)

With "slight" modification, SOM reduces from $O(\epsilon^{-2})$ to $O(\epsilon^{-3/2})$

# Other complexity analyses for some structural nonconvex optimization

- Ge, Jiang, and Y (2011), $O(\epsilon^{-1}\log(1/\epsilon))$, for $L_p$ minimization.
- Bian, Chen, and Y (2015), $O(\epsilon^{-3/2})$, for certain non-Lipschitz and nonconvex optimization.
- Bian and Chen (2013), $O(\epsilon^{-2})$, smoothing quadratic regularization for non-Lipschitzian function
- Chen et al. (2014) shows strongly NP-hardness for $L_2 - L_p$ minimization; later Ge, He, and He (2017) proposes a method with complexity of $O(\log(\epsilon^{-1}))$
- Haeser, Liu, and Y (2019) uses the first-order and second-order interior point trust-region method achieving first-order $\epsilon$-KKT points with complexity of $O(\epsilon^{-2})$ and $O(\epsilon^{-3/2})$, respectively.

# Recent efforts for nonconvex optimization

**FOM Improvements:**

- FOM with Hessian negative curvature (NC) detections, $O(\epsilon^{-7/4}\log(1/\epsilon))$
  - Carmon et al. (2018), with Hessian-vector product (HVP) and Lanczos
  - cost $O(\epsilon^{-1/4})$ for each negative curvature request
  - Also, Carmon et al. (2017), does not require HVP (only first-order condition)
- Agarwal et al. (2016), also $O(\epsilon^{-7/4})$, using accelerated methods for fast approximate matrix inversion

They are hybrid and/or randomized methods and seem difficult to be implemented

**Our approach: Reduce dimension in SOM**

# Part (2)

**The Algorithm and Preliminary Convergence Analyses**

# DRSOM : motivation from multi-directional FOM and SOM

- Recall two-direction FOM, with $d_k$ being the momentum direction $(x_k - x_{k-1})$

$$x_{k+1} = x_k - \alpha_k^1 \nabla f(x_k) + \alpha_k^2 d_k \ = x_k + d_{k+1}$$

where step-sizes are constructed; including CG, PT, AGD, Polyak, and many others.

- In SOM, a method typically minimizes a full dimensional quadratic Taylor expansion to obtain direction vector $d_{k+1}$. For example, one TR step solves for $d_{k+1}$ from

$$\min_d \quad (g_k)^T d + 0.5 d^T H_k d \quad s.t. ||\mathrm{d}||_2 \le \Delta_k$$

where $\Delta_k$ is the trust-region radius.

- DRSOM: Dimension Reduced Second-Order Method

**<u>Motivation</u>: using few directions and solving a smaller quadratic problem**

# DRSOM: a first glance

- The DRSOM constructs direction using two directions

$$d = -\alpha^1 \nabla f(x_k) + \alpha^2 d_k := d(\alpha)$$

where $g_k = \nabla f(x_k), H_k = \nabla^2 f\left(x^k\right), d_k = x_k - x_{k-1}$

- Plug the expression into the TR quadratic minimization problem, we minimize a 2-D trust-region problem to decide “two stepsizes”:

$$\min\ m_k^{\alpha}(\alpha) \coloneqq f(x_k) + (c_k)^T \alpha + \frac{1}{2}\alpha^T Q_k \alpha$$

$$||\alpha||_{G_k} \le \Delta_k$$

$$G_k = \begin{bmatrix} g_k^T g_k & -g_k^T d_k \\ -g_k^T d_k & d_k^T d_k \end{bmatrix}, Q_k = \begin{bmatrix} g_k^T H_k g_k & -g_k^T H_k d_k \\ -g_k^T H_k d_k & d_k^T H_k d_k \end{bmatrix}, c_k = \begin{bmatrix} -||g_k||^2 \\ g_k^T d_k \end{bmatrix}$$

# DRSOM: a first glance

DRSOM can be seen as:

- “Adaptive” **Accelerated Gradient Method** (Polyak’s momentum)
- A second-order method minimizing quadratic model in the reduced 2-D

  $$m_k(d) = f(x_k) + \nabla f(x_k)^T d + \frac{1}{2} d^T \nabla^2 f(x_k) d, d \in \mathrm{span}\{-g_k, d_k\}$$

  compare to, e.g., Dogleg method, 2-D Newton **Trust-Region Method**

  $$d \in \mathrm{span}\{g_k, [H(x_k)]^{-1} g_k\}$$

- A conjugate direction method exploring the Krylov **Subspace**
- For quadratic programming with no radius limit, terminates in n steps – either finds a minimal solution or detects unboundedness

# DRSOM: Computing Hessian-vector product

In the DRSOM:

$$Q_k = \begin{bmatrix} g_k^T H_k g_k & -g_k^T H_k d_k \\ -g_k^T H_k d_k & d_k^T H_k d_k \end{bmatrix}, c_k = \begin{bmatrix} -||g_k||^2 \\ g_k^T d_k \end{bmatrix}$$

How to cheaply obtain Q? Compute $H_k g_k, H_k d_k$ first.

- Finite difference:

$$H_k \cdot v \approx \frac{1}{\epsilon}[g(x_k + \epsilon \cdot v) - g_k],$$

- Analytic approach to fit modern automatic differentiation,

$$H_k g_k = \nabla(\frac{1}{2} g_k^T g_k), H_k d_k = \nabla(d_k^T g_k),$$

- or use Hessian if readily available !

# DRSOM: subproblem adaptive strategies

Recall 2-D quadratic model:

$$\min m_k^{\alpha}(\alpha) := f(x_k) + (c_k)^T \alpha + \frac{1}{2}\alpha^T Q_k \alpha$$

$$||\alpha||_{G_k} \le \Delta_k, G_k = \begin{bmatrix} g_k^T g_k & -g_k^T d_k \\ -g_k^T d_k & d_k^T d_k \end{bmatrix}, Q_k = \begin{bmatrix} g_k^T H_k g_k & -g_k^T H_k d_k \\ -g_k^T H_k d_k & d_k^T H_k d_k \end{bmatrix}, c_k = \begin{bmatrix} -||g_k||^2 \\ g_k^T d_k \end{bmatrix}$$

Apply two strategies that ensure global and convergence

- Trust-region: Adaptive radius

$$\min_{\alpha} m_k^{\alpha}(\alpha), \| \alpha \|_{G_k} \le \Delta_k, G_k = \begin{bmatrix} g_k^T g_k & -g_k^T d_k \\ -g_k^T d_k & d_k^T d_k \end{bmatrix}$$

- Radius-free

$$\min_{\alpha} m_k^{\alpha}(\alpha) + \lambda_k \|\alpha\|_{G_k}^2$$

- The subproblems can be solved efficiently.

# DRSOM: general framework

At each iteration k, the DRSOM proceeds:

- Solving 2-D Quadratic model
- Computing quality of the approximation*

$$\rho^k := \frac{f(x^k) - f(x^k + d^{k+1})}{m_p^k(0) - m_p^k(d^{k+1})} = \frac{f(x^k) - f(x^k + d^{k+1})}{m_\alpha^k(0) - m_\alpha^k(\alpha^k)}$$

- If $\rho$ is too small, increase λ (Radius-Free) or decrease Δ (trust-region)
- Otherwise, decrease λ or increase Δ

* Can be further improved by other acceptance criteria, e.g., Curtis et al. 2017

# DRSOM: key assumptions and complexity results

**Assumption**. (a) $f$ has Lipschitz continuous Hessian. (b) DRSOM iterates with a fixed-radius strategy: $\Delta_k = \epsilon/\beta)$ c) If the Lagrangian multiplier $\lambda_k < \sqrt{\epsilon}$ , assume $\| (H_k - \widetilde{H}_k)d_{k+1} \| \leq C \| d_{k+1} \|^2$ , where $\widetilde{H}_k$ is the projected Hessian in the subspace (commonly adopted for approximate Hessian)

**Theorem 1**. If we apply DRSOM to convex QP, then the iterates are the same as those by the Conjugate Gradient Method

**Theorem 2**. (Global convergence rate) For *f* with second-order Lipschitz condition, DRSOM terminates in $O(\epsilon^{-3/2})$ iterations. Furthermore, the iterate $x_k$ satisfies the first-order condition, and the Hessian is positive semi-definite in the subspace spanned by the gradient and momentum.

**Theorem 3**. (Local convergence rate) If the iterate $x_k$ converges to a strict local optimum $x^*$ such that $H(x^*) \succ 0$, and if Assumption (c) is satisfied as soon as $\lambda_k \leq C_\lambda \| d_{k+1} \|$, then DRSOM has a local superlinear (quadratic) speed of convergence, namely: $\| x_{k+1} - x^* \| = O(\| x_k - x^* \|^2)$

# DRSOM: outline of analyses

**Assumption (c):** $\| (H_k - \widetilde{H}_k) d_{k+1} \| \leq C \| d_{k+1} \|^2$

**Analysis of global convergence rate**

- Show that

$$\| g_{k+1} \| \leq \| g_{k+1} - g_k - \widetilde{H}_k d_{k+1} \| + \| \left( g_k + \widetilde{H}_k d_{k+1} \right) \|$$
$$\leq \frac{1}{2} M \| d_{k+1} \|^2 + \lambda_k \| d_{k+1} \| + \| (H_k - \widetilde{H}_k) d_{k+1} \|$$

- With the fixed-radius strategy $\| d_{k+1} \| \leq \sqrt{\epsilon}/\beta, \beta = M/2$

**Analysis of local convergence rate**

- Show that

$$\| x_{k+1} - x^* \| \leq \frac{M}{\mu} \| x_k - x^* \|^2 + \frac{1}{\mu} \| (H_k - \widetilde{H}_k) d_{k+1} \| + \left( \frac{2M}{\mu^2} + \frac{1}{\mu} \right) \lambda_k \| d_{k+1} \|,$$

**Remark**

- The analyses show that both global and local behaviors rely on Assumption (c)

# DRSOM: how to remove Assumption (c)?

**Notice**

(i) Step reduction: ( implying at most $O\left(6\beta^2(f(x_0) - f_{\inf})\epsilon^{-3/2}\right)$ iterations )

$$f(x_{k+1}) \leq f(x_k) - \frac{1}{2}\lambda_k\Delta^2 + \frac{1}{3}\beta\Delta^3$$

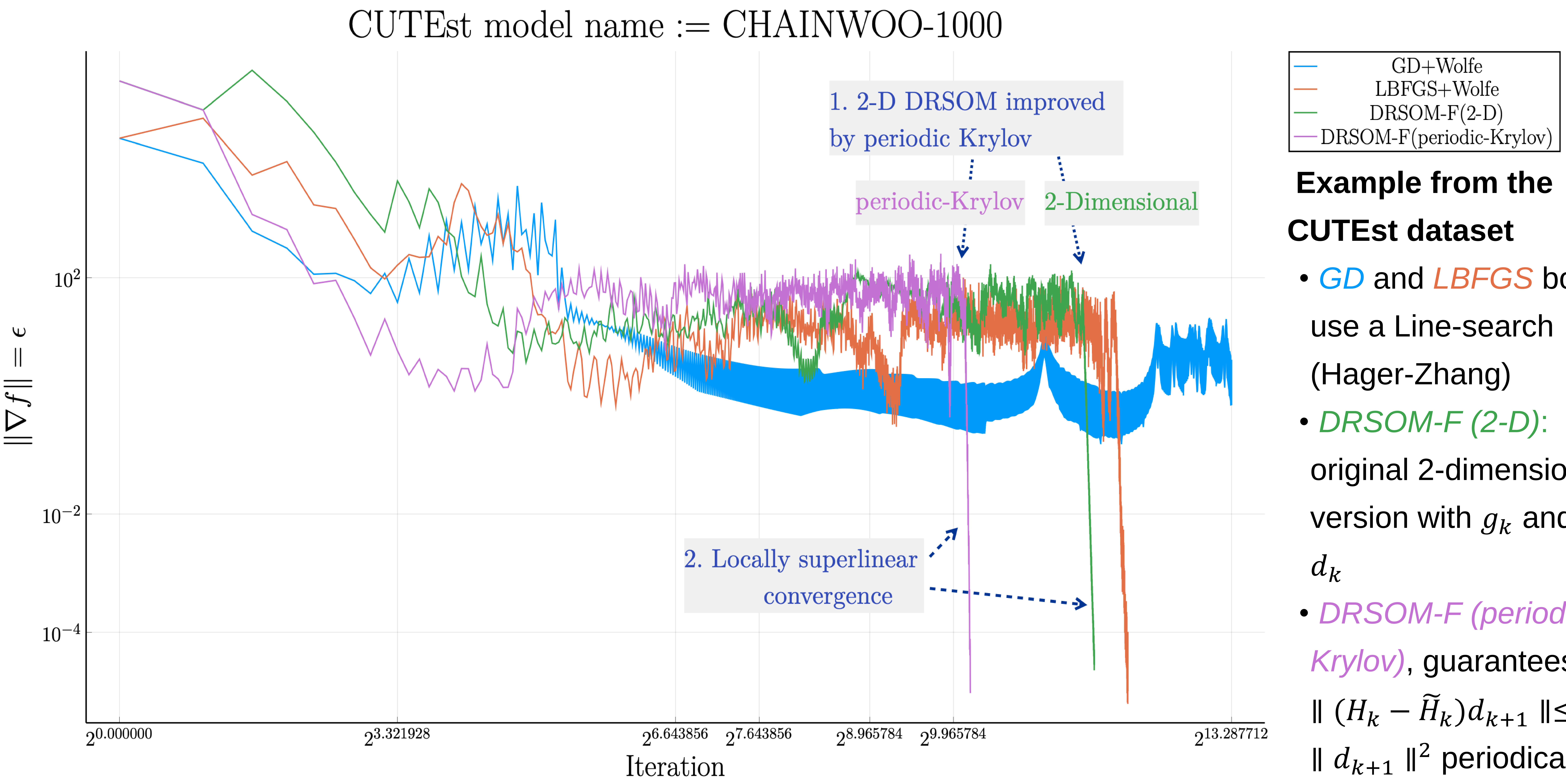
) ii) Assumption (c): $\| (H_k - \widetilde{H}_k)d_{k+1} \| \leq C \| d_{k+1} \|^2$.

- Global rate: ensure Assumption (c) holds ***periodically*** (when $\lambda_k < \sqrt{\epsilon}$) (e.g., switch to Krylov)
- Local rate: ensure Assumption (c) holds around $x^*$ ,we have the desired results.

**Expand subspace if Assumption (c) does not hold…**

- Carmon et al. (2018) Find the NC ($O(\epsilon^{-1/4}$) for each) and proceed ($\lambda_k$ increases)
- Run Lanczos (worst-case without sparsity $O(n^3)$
- Trade-off between $O(\epsilon^{-7/4})$) more dimension-free) and $O(\epsilon^{-3/2})$

# DRSOM: convergence behavior, an example




**Example from the CUTEst dataset**

- *GD* and *LBFGS* both use a Line-search (Hager-Zhang)
- *DRSOM-F (2-D)*: original 2-dimensional version with $g_k$ and $d_k$
- *DRSOM-F (periodic-Krylov)*, guarantees $\| (H_k - \widetilde{H}_k) d_{k+1} \| \leq C \| d_{k+1} \|^2$ periodically.

# Part (3)

**Computational Experiments**

# Logistic Regression

- Solve the Multinomial Logistic Regression for the MNIST dataset.
- The MLR is convex, we compare DRSOM to SAGA and LBFGS
- DRSOM is comparable to FOM and SOM (not surprisingly), but faster than full dimension SOM

| Epoch | Method | Test Error Rate |
|---|---|---|
| 10 | SAGA | 0.0754 |
| 10 | LBFGS | 0.1175 |
| 10 | DRSOM | 0.1108 |
| 40 | SAGA | 0.0754 |
| 40 | LBFGS | 0.0783 |
| 40 | DRSOM | 0.0790 |

A sample for MNIST dataset

# Nonconvex L2-Lp minimization

- Consider nonconvex L2-Lp minimization, p < 1

$$f(x) = \|Ax - b\|_2^2 + \lambda \|x\|_p^p$$

- Smoothed version

$$f(x) = \|Ax - b\|_2^2 + \lambda \sum_{i=1}^{n} s(x_i, \epsilon)^p$$

$$s(x,\epsilon) = \begin{cases} |x| & \text{if } |x| > \epsilon \\ \frac{x^2}{2\epsilon} + \frac{\epsilon}{2} & \text{if } |t| \le \epsilon \end{cases}$$

| $n$ | $m$ | DRSOM | | | AGD | | | LBFGS | | | Newton TR | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $k$ | $\|\nabla f\|$ | time | $k$ | $\|\nabla f\|$ | time | $k$ | $\|\nabla f\|$ | time | $k$ | $\|\nabla f\|$ | time |
| 100 | 10 | 28 | 5.8e-07 | 1.3e+00 | 58 | 8.5e-06 | 4.3e-01 | 21 | 8.9e-06 | 1.4e-01 | 10 | 7.1e-07 | 1.4e-02 |
| 100 | 20 | 47 | 6.0e-07 | 1.0e-03 | 150 | 8.2e-06 | 7.0e-03 | 35 | 6.2e-06 | 2.0e-03 | 9 | 4.9e-07 | 9.0e-03 |
| 100 | 100 | 98 | 1.8e-06 | 1.1e-02 | 632 | 1.0e-05 | 4.6e-01 | 106 | 9.8e-06 | 7.3e-02 | 47 | 9.9e-07 | 7.3e+00 |
| 200 | 10 | 24 | 1.3e-06 | 1.0e-03 | 37 | 7.8e-06 | 1.0e-03 | 18 | 1.4e-06 | 1.0e-03 | 13 | 5.9e-10 | 4.0e-03 |
| 200 | 20 | 47 | 9.3e-07 | 2.0e-03 | 115 | 9.4e-06 | 2.9e-02 | 33 | 6.2e-06 | 2.0e-03 | 17 | 6.7e-06 | 5.2e-02 |
| 200 | 100 | 107 | 4.3e-06 | 1.5e-02 | 814 | 9.9e-06 | 9.3e-01 | 85 | 6.2e-06 | 1.1e-01 | 36 | 1.1e-07 | 7.6e+00 |
| 1000 | 10 | 25 | 4.2e-06 | 3.0e-03 | 97 | 9.0e-06 | 3.6e-02 | 18 | 2.2e-06 | 5.0e-03 | 16 | 3.2e-07 | 5.4e-02 |
| 1000 | 20 | 27 | 5.8e-06 | 3.0e-03 | 68 | 7.6e-06 | 3.4e-02 | 27 | 4.5e-06 | 4.7e-02 | 13 | 7.8e-06 | 1.6e-01 |
| 1000 | 100 | 76 | 1.7e-05 | 2.6e-02 | 408 | 1.4e-05 | 2.6e+00 | 73 | 6.4e-06 | 6.1e-01 | 32 | 8.3e-07 | 1.3e+01 |

Iterations needed to reach ε = 10e-6

- Compare DRSOM to Accelerated Gradient Descend (AGD), LBFGS, and Newton Trust-region
- DRSOM is comparable to full-dimensional SOM in iteration number
- DRSOM is much better in computation time !

# Sensor Network Location (SNL)

- Consider Sensor Network Location (SNL)

$$N_x = \{(i,j) : \|x_i - x_j\| = d_{ij} \leq r_d\}, N_a = \{(i,k) : \|x_i - a_k\| = d_{ik} \leq r_d\}$$

where $r_d$ is a fixed parameter known as the radio range. The SNL problem considers the following QCQP feasibility problem,

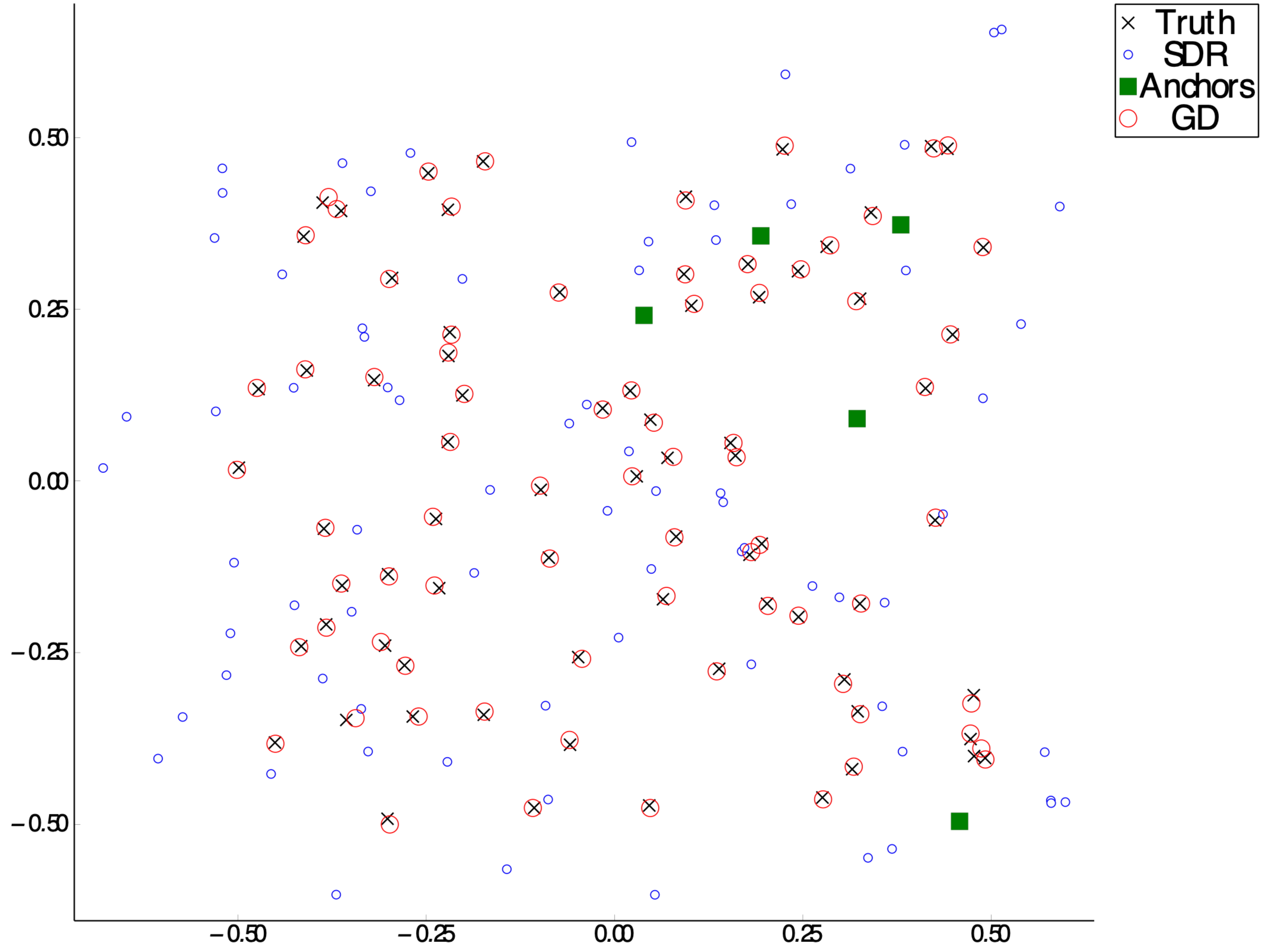
$$\|x_i - x_j\|^2 = d_{ij}^2, \forall (i,j) \in N_x$$
$$\|x_i - a_k\|^2 = \bar{d}_{ik}^2, \forall (i,k) \in N_a$$

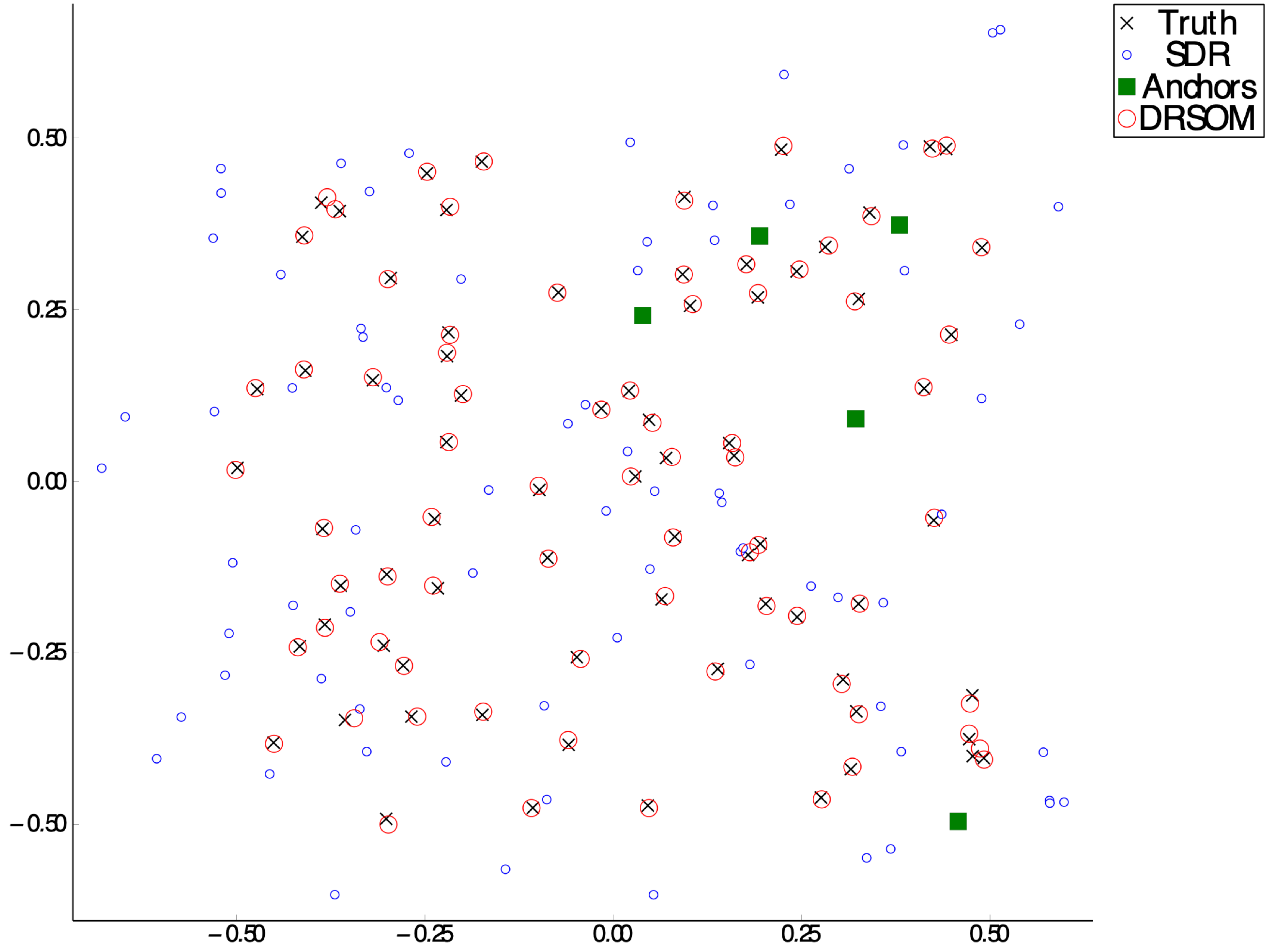
- We can solve SNL by the nonconvex nonlinear least square (NLS) problem

$$\min_X \sum_{(i<j,j)\in N_x} (\|x_i - x_j\|^2 - d_{ij}^2)^2 + \sum_{(k,j)\in N_a} (\|a_k - x_j\|^2 - \bar{d}_{kj}^2)^2.$$

# Sensor Network Location (SNL)

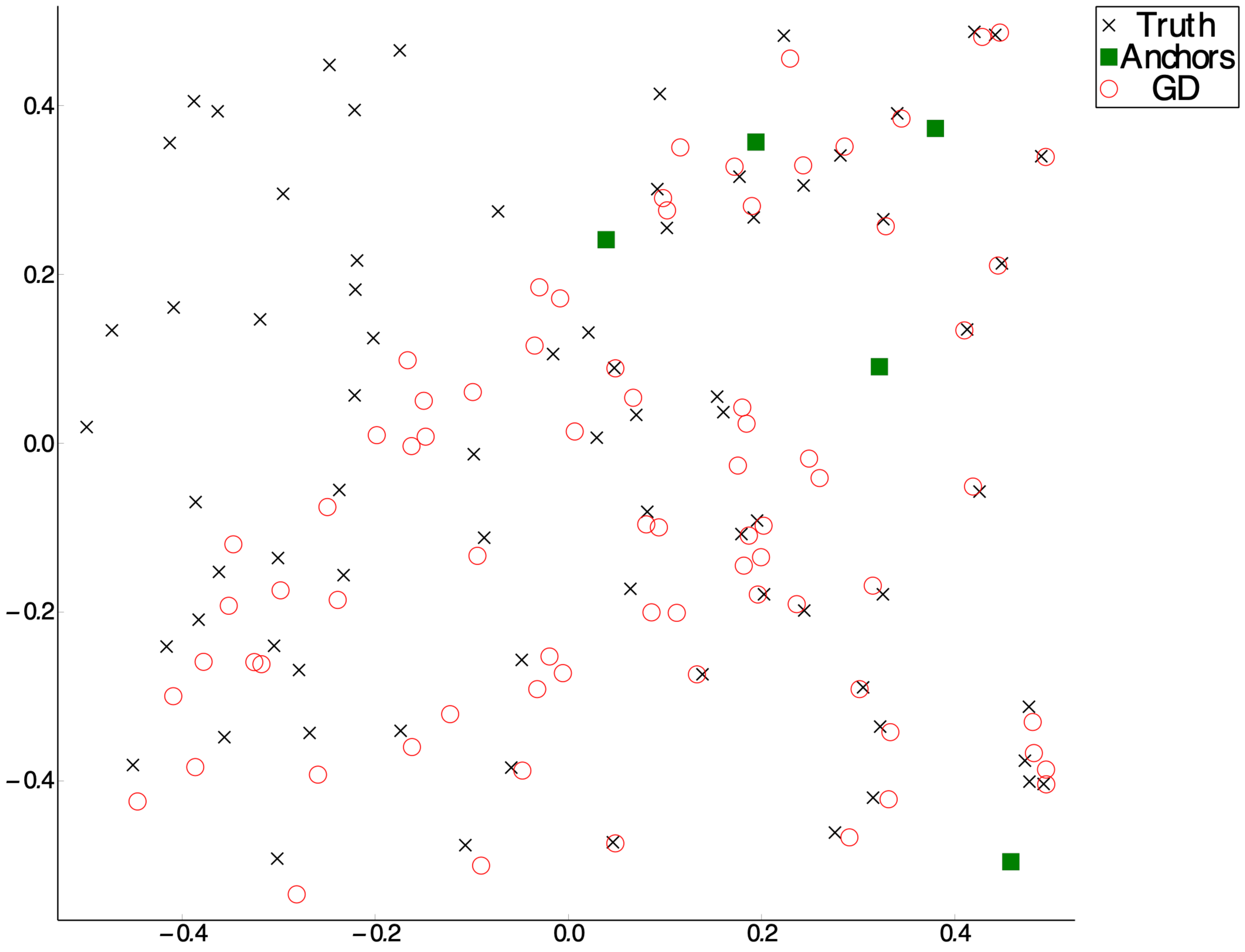
- Graphical results using SDP relaxation to initialize the NLS
- n = 80, m = 5 (anchors), radio range = 0.5, degree = 25, noise factor = 0.05
- Both Gradient Descent and DRSOM can find good solutions !

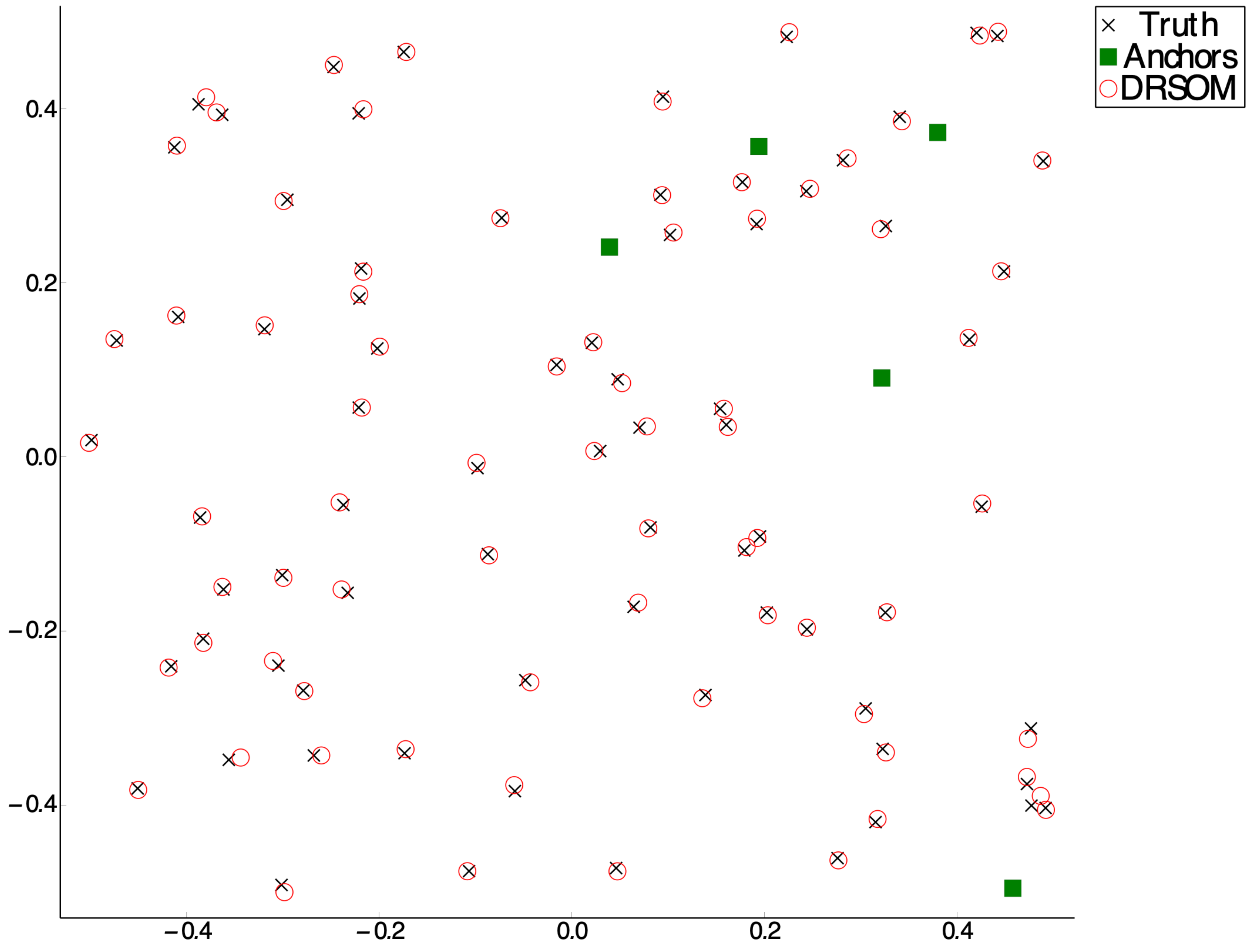


# Sensor Network Location (SNL)

- Graphical results without SDP relaxation
- DRSOM can still converge to optimal solutions

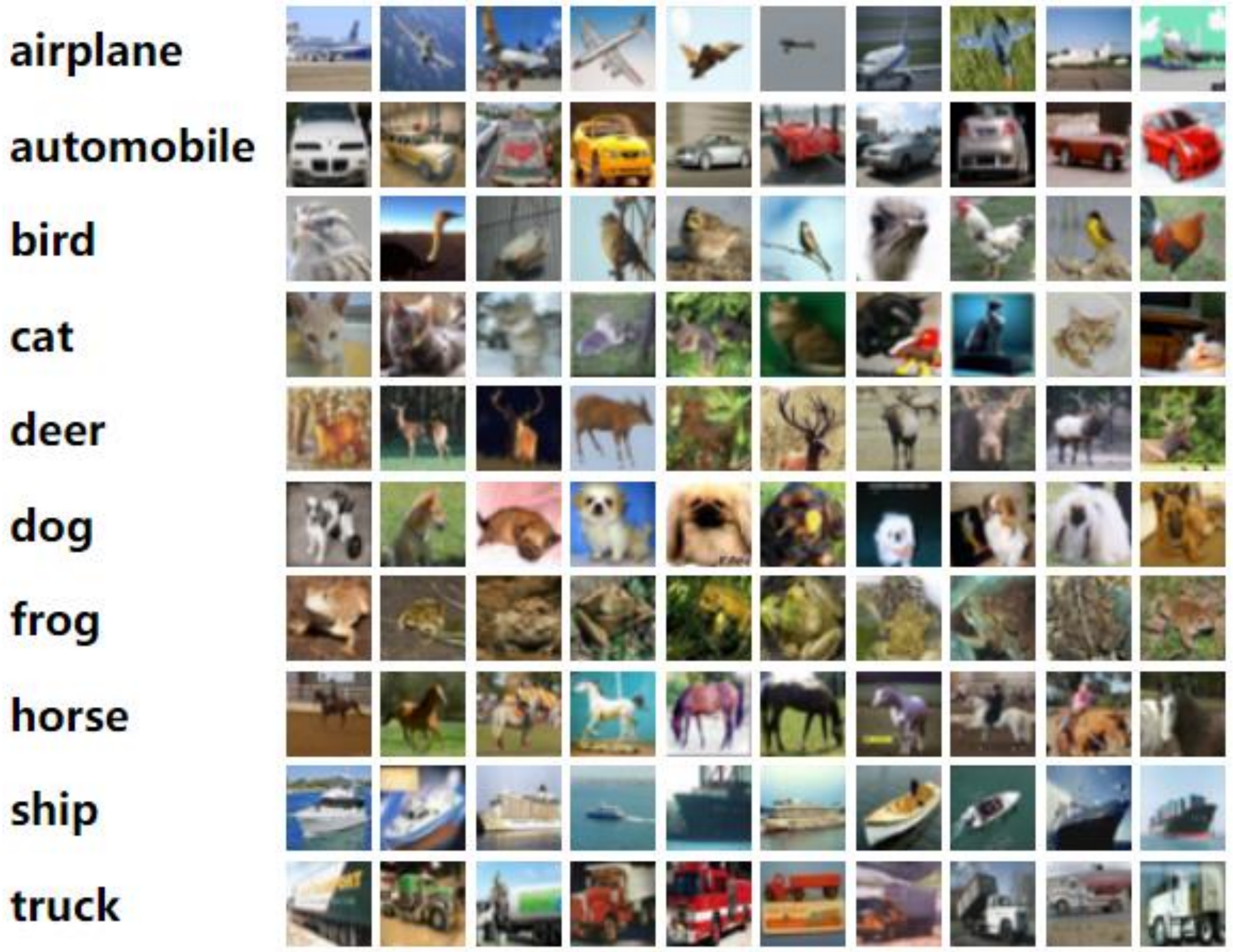


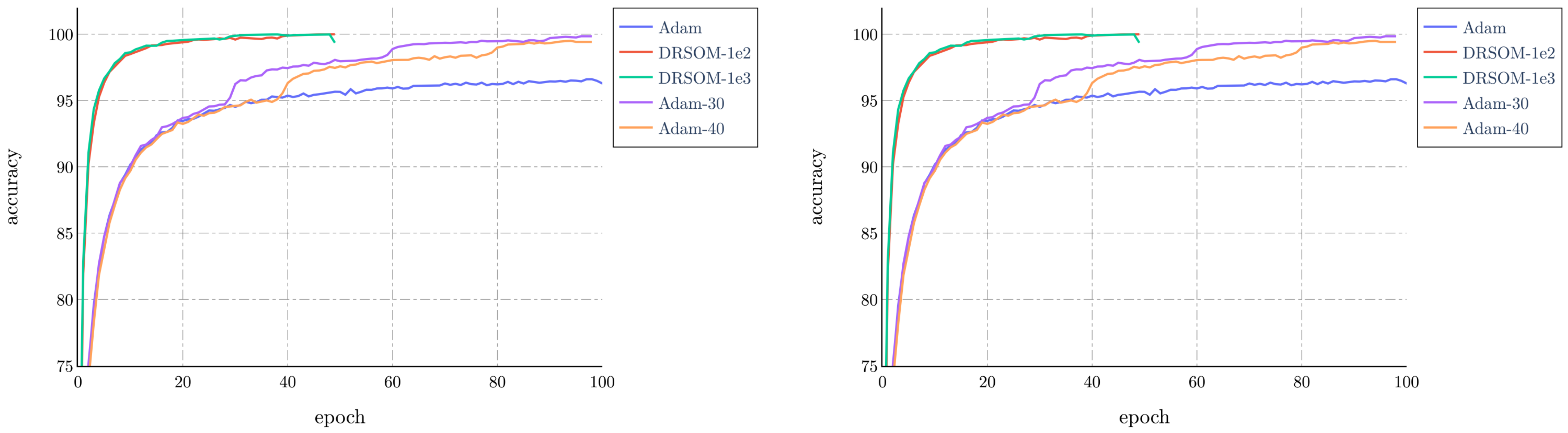
# Neural Networks and Deep Learning

To use DRSOM in machine learning problems

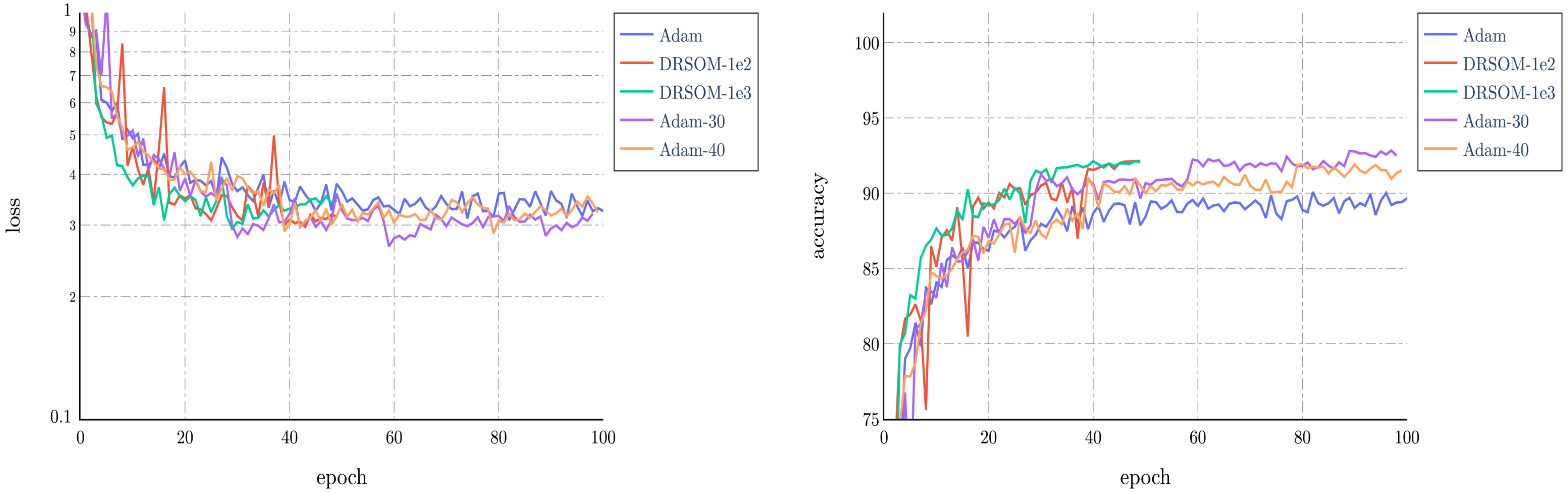
- We apply the mini-batch strategy to a vanilla DRSOM
- Use Automatic Differentiation to compute gradients
- Train ResNet18 Model with CIFAR 10
- Set Adam with initial learning rate 1e-3



# Neural Networks and Deep Learning




Training results for ResNet18 with DRSOM and Adam




Test results for ResNet18 with DRSOM and Adam

## Pros

- DRSOM has rapid convergence (30 epochs)
- DRSOM needs little tuning

## Cons

- DRSOM may overfit the models
- Needs 4~5x time than Adam to run same number of epoch

Good potential to be a standard optimizer for deep learning!

# Policy Optimization

$$\max_{\theta \in \mathbb{R}^d} J(\theta) := \mathbb{E}_{\tau \sim p(\tau|\theta)}[\mathcal{R}(\tau)] = \int \mathcal{R}(\tau) p(\tau \mid \theta) d\tau$$

- **Vanilla policy gradient**: Apply gradient descent to find the policy that maximizes the expected return:

  $$\theta_{t+1} = \theta_t + \eta_t \hat{\nabla}_\theta J(\theta)$$

  where $\hat{\nabla}_\theta J(\theta)$ is estimated stochastic gradient. Examples include:

  - REINFORCE (Williams, Simple statistical gradient-following algorithms for connectionist reinforcement learning, 1992)
  - PGT (Sutton et al., Policy gradient methods for reinforcement learning with function approximation, 1999)

- **Policy gradient based on KL divergence**
  - Trust Region Policy Optimization (TRPO): Linearize objective function and update parameter under KL constraint (J. Schulman et al. “Trust region policy optimization”, 2015)
  - Proximal Policy Optimization (PPO) : Update the parameter via KL-regularized gradient ascent (J. Schulman et al. “Proximal policy optimization algorithms”, 2017)
  - Mirror descent policy optimization (Tomar et al. 2021, Shani et al. 2020)

- **Many other recent developments**
  - Momentum policy gradient (Feihu Huang et al. 2021), Hessian-aided policy gradient (Zebang Shen et al. 2019), Variance reduced policy gradient (Papini et al. 2018)

# DRSOM for Policy Gradient (PG)

- As mentioned above, the goal is to maximize the expected discounted trajectory reward:

$$\max_{\theta \in \mathbb{R}^d} J(\theta) := \mathbb{E}_{\tau \sim p(\tau|\theta)}[\mathcal{R}(\tau)] = \int \mathcal{R}(\tau) p(\tau \mid \theta) d\tau$$

- The gradient can be estimated by:

$$\hat{\nabla} J(\theta) = \frac{1}{|\mathcal{B}|} \sum_{i \in \mathcal{B}} \nabla \log p(\tau_i \mid \theta) \mathcal{R}(\tau_i)$$

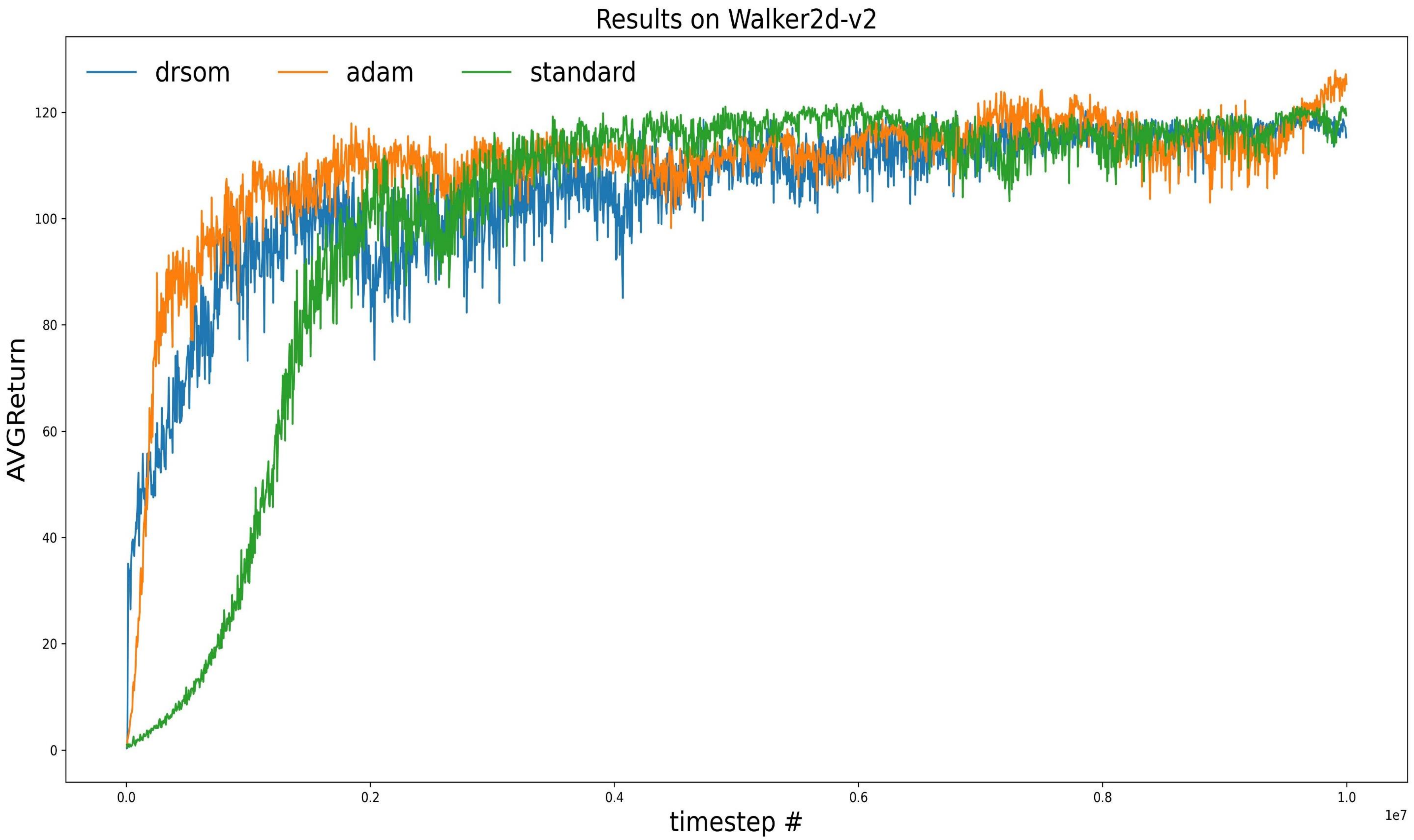
- With the estimated gradient, we can apply DRSOM to get the step size $\alpha$, and update the parameter by:

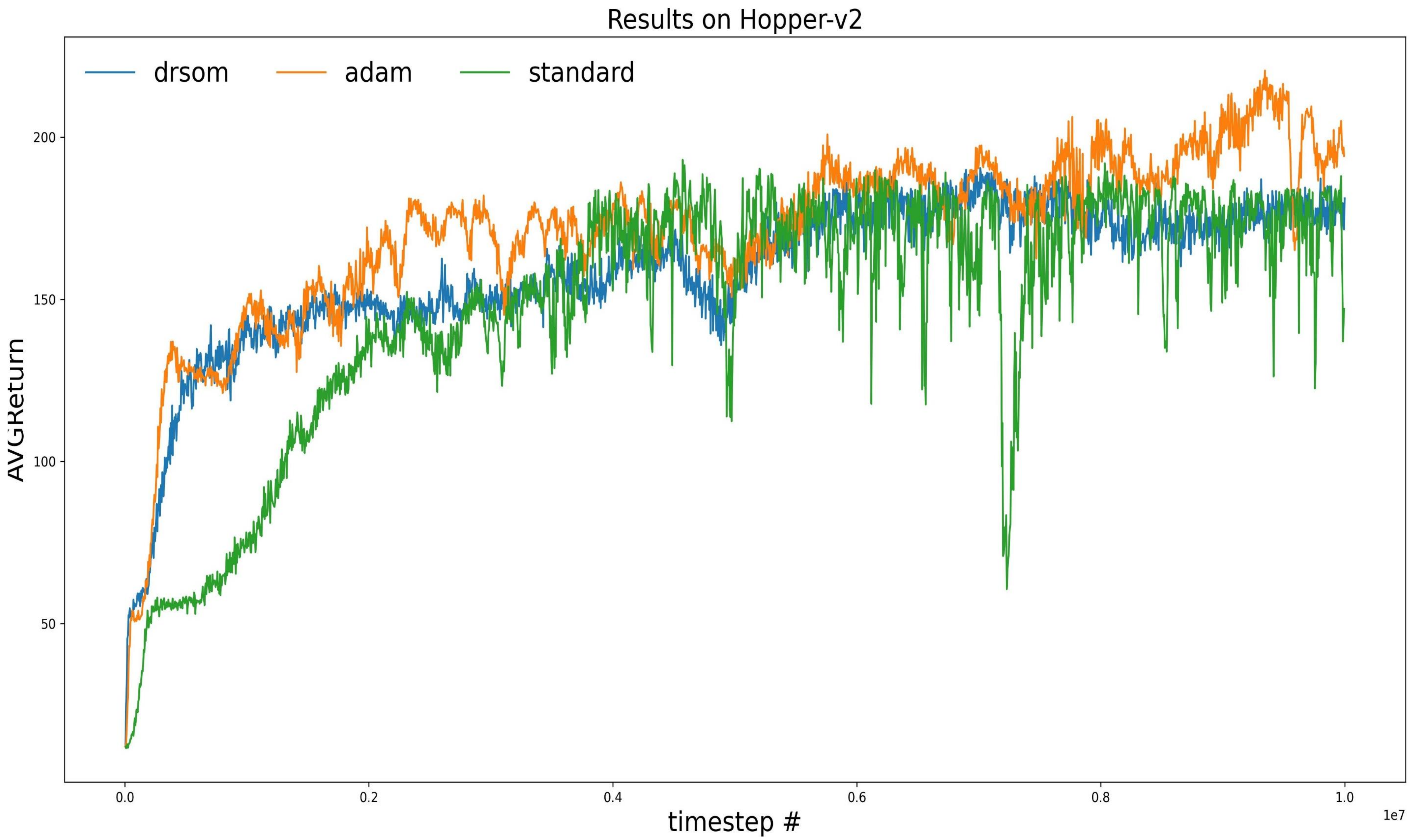
$$\theta_{t+1} = \theta_t + \alpha_t^1 \hat{\nabla} J(\theta_t) + \alpha_t^2 d_t$$

where $d_t$ is the momentum direction.

# Preliminary results

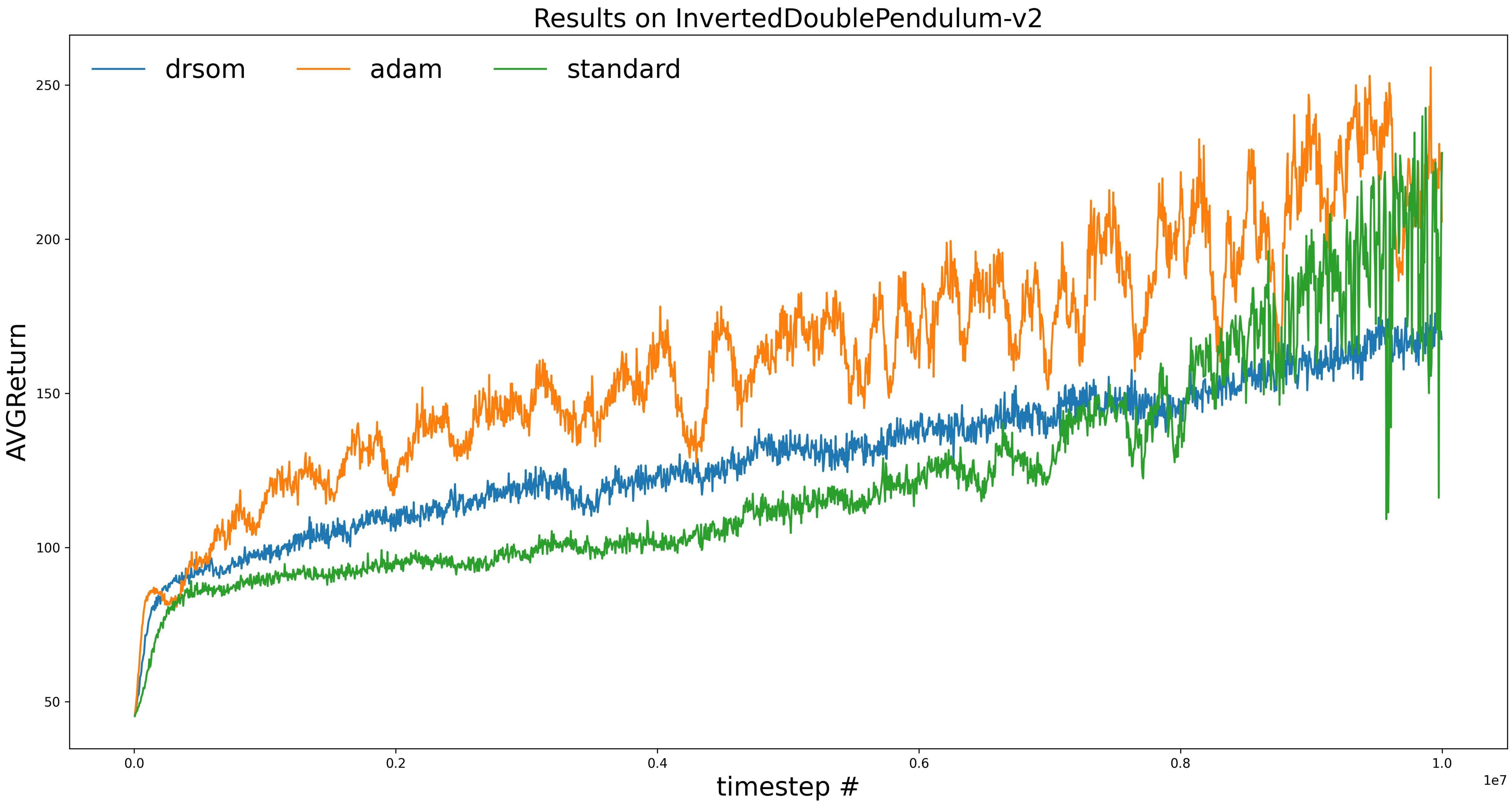
We compare the performance of DRSOM-based Reinforce with Adam-based reinforce and standard Reinforce on several GYM environments.
We set the learning rate of Adam-based and standard Reinforce both as 1e-3




In these two cases, DRSOM converges relatively faster than standard reinforce. And the performance of DRSOM is similar to ADAM.

# Preliminary Results



# DRSOM for TRPO

- TRPO attempts to optimize a surrogate function (based on the current iterate) of the objective function while keep a KL divergence constraint

$$\begin{aligned} \max_\theta \quad & L_{\theta_k}(\theta) \\ \text{s.t.} \quad & \mathrm{KL}\left(\mathrm{Pr}_\mu^{\pi_{\theta_k}} \,\|\, \mathrm{Pr}_\mu^{\pi_\theta}\right) \le \delta \end{aligned}$$

- In practice, it linearizes the surrogate function, quadratizes the KL constraint, and obtain

$$\begin{aligned} \max_\theta \quad & g_k^T(\theta - \theta_k) \\ \text{s.t.} \quad & \frac{1}{2}(\theta - \theta_k)^T F_k (\theta - \theta_k) \le \delta \end{aligned}$$

where $F_k$ is the Hessian of the KL divergence.

# DRSOM for TRPO

- The problem admits a closed form solution, but requires solving a full dimension linear system,

$$F_k x = g_k$$

  leading to high computational cost !

- With the idea of DRSOM, we restrict $\theta_{k+1} \in \text{span}\{g_k, d_k\}$, then update $\theta_{k+1} = \theta_k + \alpha_k^1 g_k + \alpha_k^2 d_k$. To choose the step size, we consider the following optimization problem:

$$\max_{\alpha \in \mathbb{R}^2} \; c_k^T \alpha$$

$$\text{s.t. } \frac{1}{2}\alpha^T G_k \alpha \leq \delta$$

  where

$$c_k = \begin{pmatrix} \|g_k\|^2 \\ g_k^T d_k \end{pmatrix} \in \mathbb{R}^2 \text{ and } G_k = \begin{pmatrix} g_k^T H_k g_k & d_k^T H_k g_k \\ d_k^T H_k g_k & d_k^T H_k d_k \end{pmatrix} \in \mathcal{S}^2$$

Still has a closed form solution, but we only need to solve a 2 dimension linear system!

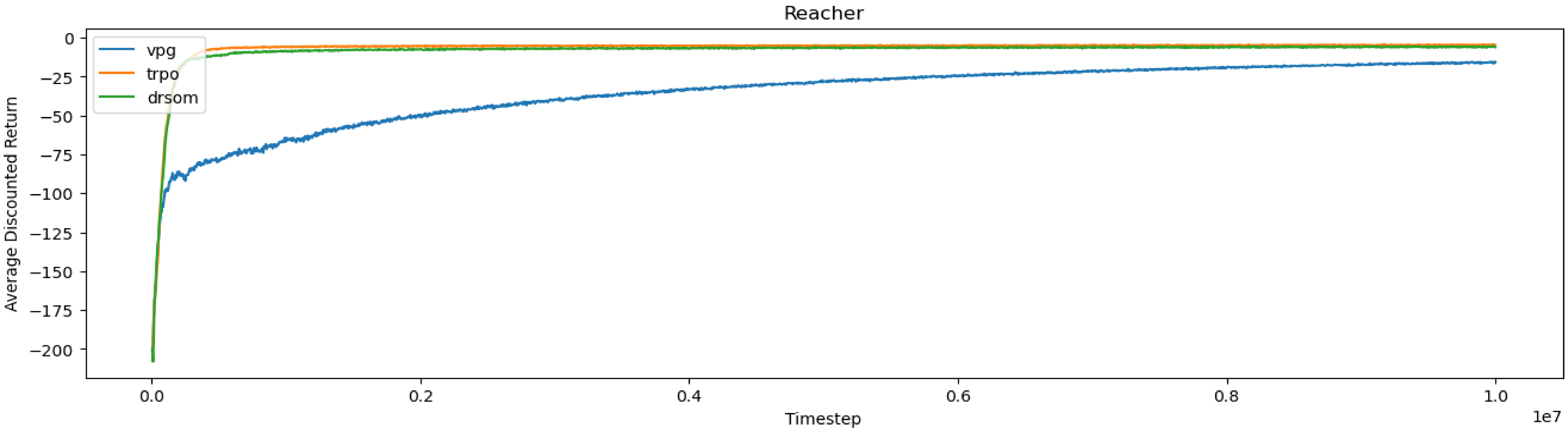
# Preliminary Results I

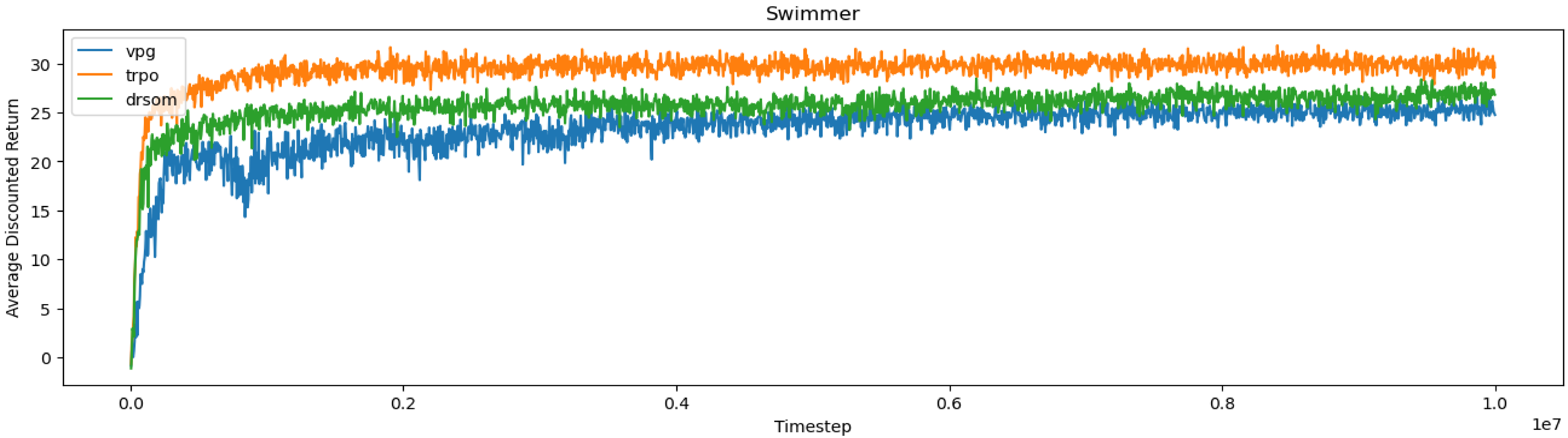
- Although we only maintain the linear approximation of the surrogate function, surprisingly the algorithm works well in some RL environments (the green line, better than VPG)

# Preliminary Results II

- Sometimes even better than TRPO !

# Preliminary Results III



# Linear Programing using DR-Potential Reduction

We consider a simplex-constrained QP model

$$\min_x \quad \frac{1}{2}\|Ax\|^2 \quad =: f(x)$$
$$\text{subject to} \quad e^\top x = 1$$
$$x \geq 0$$

$$\begin{aligned} Ax - b\tau &= 0 \\ -A^\top y - s + c\tau &= 0 \\ b^\top y - c^\top x - \kappa &= 0 \\ e_n^\top x + e_n^\top s + \kappa + \tau &= 1 \end{aligned}$$

We wish to solve a standard LP (and its dual)

$$\min_x \quad c^\top x$$
$$\text{subject to} \quad Ax = b$$
$$x \geq 0$$

$$\max_{y,s} \quad b^\top y$$
$$\text{subject to} \quad A^\top y + s = c$$
$$s \geq 0$$

The self-dual embedding builds a bridge

- The homogeneous QP seems so restrictive!

Then we define the (nonconvex) potential function and apply DRSOM to it

- How to solve much more general LPs?

$$\phi(x) := \rho \log(f(x)) - \sum_{i=1}^{n} \log x_i$$

$$\nabla \phi(x) = \frac{\rho \nabla f(x)}{f(x)} - X^{-1} e$$

$$\nabla^2 \phi(x) = -\frac{\rho \nabla f(x) \nabla f(x)^\top}{f(x)^2} + \rho \frac{A^\top A}{f(x)} + X^{-2}$$

Combined with scaled gradient(Hessian) projection, the method solves LPs

# DR-Potential Reduction: Preliminary Results

One feature of the DR-Potential reduction is the use of negative curvature of

$$\nabla^2\phi(x) = -\frac{\rho\nabla f(x)\nabla f(x)^\top}{f(x)^2} + \rho\frac{A^\top A}{f(x)} + X^{-2}$$

- Computable using Lanczos iteration
- Getting LPs to high accuracy $10^{-6} \sim 10^{-8}$ if negative curvature is efficiently computed

| Problem | PInfeas | DInfeas. | Compl. | Problem | PInfeas | DInfeas. | Compl. |
|---|---|---|---|---|---|---|---|
| ADLITTLE | 1.347e-10 | 2.308e-10 | 2.960e-09 | KB2 | 5.455e-11 | 6.417e-10 | 7.562e-11 |
| AFIRO | 7.641e-11 | 7.375e-11 | 3.130e-10 | LOTFI | 2.164e-09 | 4.155e-09 | 8.663e-08 |
| AGG2 | 3.374e-08 | 4.859e-08 | 6.286e-07 | MODSZK1 | 1.527e-06 | 5.415e-05 | 2.597e-04 |
| AGG3 | 2.248e-05 | 1.151e-06 | 1.518e-05 | RECIPELP | 5.868e-08 | 6.300e-08 | 1.285e-07 |
| BANDM | 2.444e-09 | 4.886e-09 | 3.769e-08 | SC105 | 7.315e-11 | 5.970e-11 | 2.435e-10 |
| BEACONFD | 5.765e-12 | 9.853e-12 | 1.022e-10 | SC205 | 6.392e-11 | 5.710e-11 | 2.650e-10 |
| BLEND | 2.018e-10 | 3.729e-10 | 1.179e-09 | SC50A | 1.078e-05 | 6.098e-06 | 4.279e-05 |
| BOEING2 | 1.144e-07 | 1.110e-08 | 2.307e-07 | SC50B | 4.647e-11 | 3.269e-11 | 1.747e-10 |
| BORE3D | 2.389e-08 | 5.013e-08 | 1.165e-07 | SCAGR25 | 1.048e-07 | 5.298e-08 | 1.289e-06 |
| BRANDY | 2.702e-05 | 7.818e-06 | 1.849e-05 | SCAGR7 | 1.087e-07 | 1.173e-08 | 2.601e-07 |
| CAPRI | 7.575e-05 | 4.488e-05 | 4.880e-05 | SCFXM1 | 4.323e-06 | 5.244e-06 | 8.681e-06 |
| E226 | 2.656e-06 | 4.742e-06 | 2.512e-05 | SCORPION | 1.674e-09 | 1.892e-09 | 1.737e-08 |
| FINNIS | 8.577e-07 | 8.367e-07 | 1.001e-05 | SCTAP1 | 5.567e-07 | 8.430e-07 | 5.081e-06 |
| FORPLAN | 5.874e-07 | 2.084e-07 | 4.979e-06 | SEBA | 2.919e-11 | 5.729e-11 | 1.448e-10 |
| GFRD-PNC | 4.558e-05 | 1.052e-05 | 4.363e-05 | SHARE1B | 3.367e-07 | 1.339e-06 | 3.578e-06 |
| GROW7 | 1.276e-04 | 4.906e-06 | 1.024e-04 | SHARE2B | 2.142e-04 | 2.014e-05 | 6.146e-05 |
| ISRAEL | 1.422e-06 | 1.336e-06 | 1.404e-05 | STAIR | 5.549e-04 | 8.566e-06 | 2.861e-05 |
| STANDATA | 5.645e-08 | 2.735e-07 | 5.130e-06 | STANDGUB | 2.934e-08 | 1.467e-07 | 2.753e-06 |
| STOCFOR1 | 6.633e-09 | 9.701e-09 | 4.811e-08 | VTP-BASE | 1.349e-10 | 5.098e-11 | 2.342e-10 |

- Now solving small and medium Netlib instances in 10 seconds

  within 1000 iterations
- In MATLAB and getting transferred into C for acceleration

# Ongoing Research and Future Directions

- How to enforce or remove assumption c) in analyses
- How to design an adaptive-radius mechanism with the same complexity bound, e.g., Curtis trust-region framework [Curtis et al., 2017]
- Incorporate the second-order steepest-descent direction, the eigenvector of the most negative Hessian eigenvalue
- Indefinite Hessian rank-one updating vs BFGS
- Dimension Reduced Non-Smooth/Semi-Smooth Newton [Qi, Sun et al. ]
- Dimension Reduced Second-Order Methods for optimization on manifolds