Distributionally Robust Optimization, Online Linear Programming and Markets for Public-Good Allocations

Models/Algorithms for Learning and Decision Making Driven by Data/Samples

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(Joint work with many others ...)
Outline

- Introduction to Stochastic and Distributionally Robust Optimization (DRO)
- DRO under Moment, Likelihood and Wasserstein Bounds (Probability Theory)
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Develop tractable and provable models and algorithms for decision-making and optimization with uncertain, online/dynamic and massive data.
We start from considering a **stochastic optimization** problem as follows:

\[
\text{maximize}_{\mathbf{x} \in \mathcal{X}} \quad \mathbb{E}_{F_{\xi}} [h(\mathbf{x}, \xi)]
\]  

(1)

where \( \mathbf{x} \) (typically called \( \beta \) in data science with minimizing a loss function) are decision variables with feasible region \( \mathcal{X} \), \( \xi \) represents random variables satisfying joint distribution \( F_{\xi} \).
We start from considering a stochastic optimization problem as follows:

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where $x$ (typically called $\beta$ in data science with minimizing a loss function) are decision variables with feasible region $\mathcal{X}$, $\xi$ represents random variables satisfying joint distribution $F_\xi$.

- **Pros:** In many cases, the expected value is a good measure of performance; simply apply simple sample average (SSA) approach.
- **Cons:** One has to know the exact distribution of $\xi$ to perform the stochastic optimization. Deviant from the assumed distribution may result in sub-optimal solutions. Even know the distribution, the solution/decision is generically risky.
Learning with Noises

```
+ ε
```

“panda”
57.7% confidence

“gibbon”
99.3% confidence
Learning with Noises

“panda”
57.7% confidence

+ \epsilon

= “gibbon”
99.3% confidence

Goodfellow et al. [2014]
In order to overcome the lack of knowledge on the distribution, people proposed the following (static) robust optimization approach:

\[
\maximize_{x \in X} \min_{\xi \in \Xi} h(x, \xi) \tag{2}
\]

where $\Xi$ is the support of $\xi$. 

Pros: Robust to any distribution; only the support of the parameters are needed.

Cons: Too conservative and it ignores observed/training data information/statistics. The decision that maximizes the worst-case pay-off may perform badly in usual cases; e.g., Ben-Tal and Nemirovski [1998, 2000], etc.
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Motivation for a Middle Ground

In practice, although the exact distribution of the random variables may not be known, people usually know certain observed samples or training data and other statistical information.
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- In practice, although the exact distribution of the random variables may not be known, people usually know certain observed samples or training data and other statistical information.

- Thus we could choose an intermediate approach between stochastic optimization, which has no robustness in the error of distribution; and the robust optimization, which admits vast unrealistic single-point distribution on the support set of random variables.
A solution to the above-mentioned question is to take the following Distributionally Robust Optimization/Learning (DRO) model:

$$\max_{x \in X} \min_{F_{\xi} \in \mathcal{D}} \mathbb{E}_{F_{\xi}}[h(x, \xi)]$$

In DRO, we consider a set of distributions $\mathcal{D}$ and choose one to maximize the expected value for any given $x \in X$. 
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In DRO, we consider a set of distributions \( \mathcal{D} \) and choose one to maximize the expected value for any given \( x \in X \).

When choosing \( \mathcal{D} \), we need to consider the following:

- **Tractability** (fast algorithm available)
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\max_{x \in X} \min_{F_\xi \in D} \mathbb{E}_{F_\xi}[h(x, \xi)]
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When choosing $\mathcal{D}$, we need to consider the following:

- **Tractability** (fast algorithm available)
- **Practical (Statistical) Meanings** (utilize observed/training data)
- **Performance** (the potential loss comparing to the benchmark cases)
Brief History of DRO

- First introduced by Scarf [1958] in the context of inventory control problem with a single random demand variable.
- Distribution set based on moments: Dupacova [1987], Prekopa [1995], Bertsimas and Popescu [2005], Delage and Y [2007,2010], etc
- Distribution set based on Likelihood/Divergences: Nilim and El Ghaoui [2005], Iyanger [2005], Wang, Glynn and Y [2012], etc
- Distribution set based on Wasserstein ambiguity set: Mohajerin Esfahani and Kuhn [2015], Blanchet, Kang, Murthy [2016], Duchi, Glynn, Namkoong [2016]
- Axiomatic motivation for DRO: Delage et al. [2017]; Ambiguous Joint Chance Constraints Under Mean and Dispersion Information: Hanasusanto et al. [2017]
Table of Contents

1. Introduction to Stochastic and Distributionally Robust Optimization

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Define

\[ \mathcal{D} = \left\{ F_\xi \mid P(\xi \in \Xi) = 1, \quad (\mathbb{E}[\xi] - \mu_0)^T \Sigma_0^{-1} (\mathbb{E}[\xi] - \mu_0) \leq \gamma_1, \quad \mathbb{E}[(\xi - \mu_0)(\xi - \mu_0)^T] \leq \gamma_2 \Sigma_0 \right\} \]

That is, the distribution set is defined based on the support, first and second order moments constraints, where \( \mu_0 \) and \( \Sigma_0 \) are sample mean vector and variance matrix.
DRO with Moment Bounds

Define

\[ \mathcal{D} = \left\{ F_\xi \middle| \begin{array}{l} P(\xi \in \Xi) = 1 \\ (\mathbb{E}[\xi] - \mu_0)^T\Sigma_0^{-1}(\mathbb{E}[\xi] - \mu_0) \leq \gamma_1 \\ \mathbb{E}[(\xi - \mu_0)(\xi - \mu_0)^T] \leq \gamma_2 \Sigma_0 \end{array} \right\} \]

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Theorem

Under mild technical conditions, the DRO model can be solved to any precision \( \epsilon \) in time polynomial in \( \log(1/\epsilon) \) and the sizes of \( x \) and \( \xi \)

Delage and Y [2010]
Confidence Region on $F_\xi$

Does the construction of $D$ make a statistical sense?
Does the construction of $D$ make a statistical sense?

**Theorem**

Consider

$$D(\gamma_1, \gamma_2) = \left\{ F_\xi \left\vert \begin{array}{l} P(\xi \in \Xi) = 1 \\ (E[\xi] - \mu_0)^T \Sigma_0^{-1} (E[\xi] - \mu_0) \leq \gamma_1 \\ E[(\xi - \mu_0)(\xi - \mu_0)^T] \leq \gamma_2 \Sigma_0 \end{array} \right. \right\}$$

where again $\mu_0$ and $\Sigma_0$ are point estimates from the empirical data (of size $m$) and $\Xi$ lies in a ball of radius $R$ such that $\|\xi\|_2 \leq R$ a.s..

Then for $\gamma_1 = O\left(\frac{R^2}{m} \log \left(\frac{4}{\delta}\right)\right)$ and $\gamma_2 = O\left(\frac{R^2}{\sqrt{m}} \sqrt{\log \left(\frac{4}{\delta}\right)}\right)$,

$$P(F_\xi \in D(\gamma_1, \gamma_2)) \geq 1 - \delta$$
DRO with Likelihood Bounds

Define the distribution set by the constraint on the likelihood ratio. With observed Data: $\xi_1, \xi_2, \ldots, \xi_N$, we define

$$D_N = \left\{ F_\xi \mid \frac{P(\xi \in \Xi)}{L(\xi, F_\xi)} = 1 \right\}$$

where $\gamma$ adjusts the level of robustness and $N$ represents the sample size.
Define the distribution set by the constraint on the likelihood ratio. With observed Data: $\xi_1, \xi_2, \ldots, \xi_N$, we define

$$
\mathcal{D}_N = \left\{ F_\xi \bigg| \begin{array}{l}
P(\xi \in \Xi) = 1 \\
L(\xi, F_\xi) \geq \gamma
\end{array} \right\}
$$

where $\gamma$ adjusts the level of robustness and $N$ represents the sample size.

For example, assume the support of the uncertainty is finite $\xi_1, \xi_2, \ldots, \xi_n$ and we observed $m_i$ samples on $\xi_i$. Then, $F_\xi$ has a finite discrete distribution $p_1, \ldots, p_n$ and the likelihood/entropy function

$$
L(\xi, F_\xi) = \sum_{i=1}^{n} \frac{m_i}{n} \log p_i \quad \text{or} \quad L(\xi, F_\xi) = \sum_{i=1}^{n} -p_i \log \left( p_i \frac{n}{m_i} \right).
$$
Theory on Likelihood Bounds

The model is a convex optimization problem, and connects to many statistical theories:

- **Statistical Divergence theory**: provide a bound on KL divergence
- **Bayesian Statistics with the threshold $\gamma$ estimated by samples**: confidence level on the true distribution
- **Non-parametric Empirical Likelihood theory**: inference based on empirical likelihood by Owen
- **Asymptotic Theory of the likelihood region**
- **Possible extensions to deal with Continuous Case**

Wang, Glynn and Y [12,16]
By the Kantorovich-Rubinstein theorem, the Wasserstein distance between two distributions can be expressed as the minimum cost of moving one to the other, which is a semi-infinite transportation LP.
DRO using Wasserstein Ambiguity Set

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**Theorem**

When using the Wasserstein ambiguity set

\[ \mathcal{D}_N := \{ F_\xi \mid P(\xi \in \Xi) = 1 \& d(F_\xi, \hat{F}_N) \leq \varepsilon_N \}, \]

where \( d(F_1, F_2) \) is the Wasserstein distance function and \( N \) is the sample size, the DRO model satisfies the following properties:

- **Finite sample guarantee**: the correctness probability \( \bar{P}^N \) is high
- **Asymptotic guarantee**: \( \bar{P}^\infty(\lim_{N \to \infty} \hat{x}_{\varepsilon_N} = x^*) = 1 \)
- **Tractability**: DRO is in the same complexity class as SAA

Mohajerin Esfahani & Kuhn [15, 17], Blanchet, Kang, Murthy [16], Duchi, Glynn, Namkoong
DRO with Wasserstein Ambiguity for Logistic Regression

Let \( \{ (\hat{\xi}_i, \hat{\lambda}_i) \}_{i=1}^{N} \) be a feature-label training set i.i.d. from \( P \), and consider applying logistic regression:

\[
\min_{x} \frac{1}{N} \sum_{i=1}^{N} \ell(x, \hat{\xi}_i, \hat{\lambda}_i) \text{ where } \ell(x, \xi, \lambda) = \ln(1 + \exp(-\lambda x^T \xi))
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DRO suggests solving $\min_x \sup_{F \in D_N} \mathbb{E}_F[\ell(x, \xi_i, \lambda_i)]$ with the Wasserstein ambiguity set.
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\]

DRO suggests solving \( \min_x \sup_{F \in \mathcal{D}_N} \mathbb{E}_F[\ell(x, \xi_i, \lambda_i)] \) with the Wasserstein ambiguity set.

When labels are considered to be error free, DRO with \( \mathcal{D}_N \) reduces to regularized logistic regression:

\[
\min_x \frac{1}{N} \sum_{i=1}^N \ell(x, \hat{\xi}_i, \hat{\lambda}_i) + \varepsilon \|x\|_*
\]

Shafieezadeh Abadeh, Mohajerin Esfahani, & Kuhn, NIPS, [2015]
Results of the DRO Learning

Original  ERM  FGM
IFGM  PGM  WRM

Sinha, Namkoong and Duchi [2017]
Medical Application

Ref: Filtered Back Projection reconstructions of noise-free data
FBP: FBP reconstructions of noisy data
TV: TV-based reconstruction
DL: Dictionary Learning-based reconstruction
DL+DRO: DL+DRO to encourage low-rankness and robustness

Liu at all. [2017]
Conditional estimation given specific covariate values (i.e., local conditional estimation or functional estimation) is ubiquitously useful with applications in engineering, social and natural sciences. We may

\[ \min_{\psi} \mathbb{E}[\|Y - \psi(X)\|_2^2], \]

where the \( \min \) is taken over all infinite measurable functions.

Or it can be cast into solving a family of finite-dimensional optimization problems parametrically in \( x_0 \):

\[ \min_{\beta} \mathbb{E}_F[\ell(Y, \beta) | X = x_0] \tag{4} \]

with an appropriately chosen statistical loss function \( \ell \) and a response variable \( Y \), given the value or observation over a covariate \( X \).
Local Conditional Estimator

But it may happen that there is no or little training data with the exact covariate $X = x_0$. Then one can consider solving a Local conditional estimation problem:

$$\min_{\beta} \mathbb{E}_F[\ell(Y, \beta) | X \in N_\gamma(x_0)]$$  \hspace{0.5cm} (5)

where the expectation is conditioned on a neighborhood $N_\gamma(x_0)$ around $x_0$ of radius $\gamma$. Under reasonable regularity assumptions, problem (4) can be viewed as the limit of (5) as the radius $\gamma$ of the neighborhood shrinks to zero.

There are always added errors which may be due to finite sampling, noise and corrupted data. This motivates us to investigate potential strategies to robustify the local conditional estimation problem (5) under the framework of distributionally robust optimization.
We compare to DRLE $k$-nearest neighbour (KNN), Nadaraya-Watson (N-W), and Nadaraya-Epanechnikov (N-E) estimators on a digit estimation problem using the MNIST database. We executed 100 experiments where training and test sets were randomly drawn without replacement from the 60,000 training examples of this dataset.

<table>
<thead>
<tr>
<th>Method</th>
<th>$N = 50$</th>
<th>$N = 100$</th>
<th>$N = 500$</th>
</tr>
</thead>
<tbody>
<tr>
<td>KNN</td>
<td>25% ± 2%</td>
<td>30% ± 2%</td>
<td>59% ± 2%</td>
</tr>
<tr>
<td>N-W</td>
<td>29% ± 2%</td>
<td>39% ± 2%</td>
<td>63% ± 1%</td>
</tr>
<tr>
<td>N-E</td>
<td>26% ± 1%</td>
<td>34% ± 1%</td>
<td>48% ± 1%</td>
</tr>
<tr>
<td>DRLE</td>
<td>34% ± 2%</td>
<td>47% ± 2%</td>
<td>70% ± 1%</td>
</tr>
</tbody>
</table>

**Table**: Comparison of expected out-of-sample classification accuracy when rounding the estimators’ outputs under different training set sizes. The 90% confidence interval are estimated based on 100 experiments.
The DRO models yield a solution with a guaranteed confidence level to the possible distributions. Specifically, the confidence region of the distributions can be constructed upon the historical data and sample distributions.
Summary of DRO under Moment, Likelihood or Wasserstein Ambiguity Set

- The DRO models yield a solution with a guaranteed confidence level to the possible distributions. Specifically, the confidence region of the distributions can be constructed upon the historical data and sample distributions.

- The DRO models are tractable, and sometimes maintain the same computational complexity as the stochastic optimization models with known distribution.
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- The DRO models are tractable, and sometimes maintain the same computational complexity as the stochastic optimization models with known distribution.

- This approach can be applied to a wide range of problems, including inventory problems (e.g., newsvendor problem), portfolio selection problems, image reconstruction, machine learning, etc., with reported superior numerical results.
# Table of Contents

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Background

Consider a store that sells a number of goods/products
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- There is a fixed inventory of goods
- Customers come and require a bundle of goods and bid for certain prices
- Decision: To sell or not to each individual customer?
- Objective: Maximize the revenue.
### An Example

<table>
<thead>
<tr>
<th>Reward ($r_t$)</th>
<th>Bid 1 ($t = 1$)</th>
<th>Bid 2 ($t = 2$)</th>
<th>.....</th>
<th>Inventory ($b$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decision</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>.....</td>
<td></td>
</tr>
<tr>
<td>Pants</td>
<td>1</td>
<td>0</td>
<td>.....</td>
<td>100</td>
</tr>
<tr>
<td>Shoes</td>
<td>1</td>
<td>0</td>
<td>.....</td>
<td>50</td>
</tr>
<tr>
<td>T-shirts</td>
<td>0</td>
<td>1</td>
<td>.....</td>
<td>500</td>
</tr>
<tr>
<td>Jackets</td>
<td>0</td>
<td>0</td>
<td>.....</td>
<td>200</td>
</tr>
<tr>
<td>Hats</td>
<td>1</td>
<td>1</td>
<td>.....</td>
<td>1000</td>
</tr>
</tbody>
</table>
Online Linear Programming Model

The classical offline version of the above program can be formulated as a linear (integer) program as all information data would have arrived: compute $x_t$, $t = 1, \ldots, n$ and

\[
\begin{align*}
\text{maximize}_x & \quad \sum_{t=1}^{n} r_t x_t \\
\text{subject to} & \quad \sum_{t=1}^{n} a_{it} x_t \leq b_i, \quad \forall i = 1, \ldots, m \\
& \quad x_t \in \{0, 1\} \ (0 \leq x_t \leq 1), \quad \forall t = 1, \ldots, n.
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Now we consider the online or streamline and data-driven version of this problem:

- We only know \( b \) and \( n \) at the start
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\]

Now we consider the online or streamline and data-driven version of this problem:

- We only know $b$ and $n$ at the start
- the bidder information is revealed sequentially along with the corresponding objective coefficient.
- an irrerevocable decision must be made as soon as an order arrives without observing or knowing the future data.
Model Assumptions

Main Assumptions
- $0 \leq a_{it} \leq 1$, for all $(i, t)$;
- $r_t \geq 0$ for all $t$;
- The bids $(a_t, r_t)$ arrive in a random order (rather than from some iid distribution).
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- The bids $(a_t, r_t)$ arrive in a random order (rather than from some iid distribution).

Denote the offline LP maximal value by $OPT(A, \pi)$. We call an online algorithm $A$ to be $c$-competitive if and only if

$$E_\sigma \left[ \sum_{t=1}^{n} r_t x_t(\sigma, A) \right] \geq c \cdot OPT(A, \pi) \ \forall (A, \pi),$$

where $\sigma$ is the permutation of arriving orders.

In what follows, we let

$$B = \min_i b_i (> 0).$$
Main Results: Necessary and Sufficient Conditions

Theorem

For any fixed $0 < \epsilon < 1$, there is no online algorithm for solving the linear program with competitive ratio $1 - \epsilon$ if

$$B < \frac{\log(m)}{\epsilon^2}.$$
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The problem would be easy if there are "ideal prices":

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</tr>
<tr>
<td>Jackets</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>200</td>
</tr>
<tr>
<td>Hats</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1000</td>
</tr>
</tbody>
</table>
Pricing the bid: The optimal dual price vector $p^*$ of the offline LP problem can play such a role, that is $x_t^* = 1$ if $r_t > a_t^T p^*$ and $x_t^* = 0$ otherwise, yields a near-optimal solution.
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Based on this observation, our online algorithm works by learning a threshold price vector $\hat{\mathbf{p}}$ and using $\hat{\mathbf{p}}$ to price the bids.
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One-time learning algorithm: learn the price vector once using the initial $\epsilon n$ input.
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One-time learning algorithm: learn the price vector once using the initial $\epsilon n$ input.

Dynamic learning algorithm: dynamically update the prices at a carefully chosen pace.
We illustrate a simple One-Time Learning Algorithm:

- Set $x_t = 0$ for all $1 \leq t \leq \epsilon n$;
One-Time Learning Algorithm

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- Set $x_t = 0$ for all $1 \leq t \leq \epsilon n$;
- Solve the $\epsilon$ portion of the problem

\[
\begin{align*}
\text{maximize}_x & \quad \sum_{t=1}^{\epsilon n} r_t x_t \\
\text{subject to} & \quad \sum_{t=1}^{\epsilon n} a_{it} x_t \leq (1 - \epsilon) \epsilon b_i \quad i = 1, \ldots, m \\
& \quad 0 \leq x_t \leq 1 \quad t = 1, \ldots, \epsilon n
\end{align*}
\]

and get the optimal dual solution $\hat{p}$;
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& \quad 0 \leq x_t \leq 1 \quad t = 1, \ldots, \epsilon n
\end{align*}
$$

and get the optimal dual solution $\hat{p}$;
- Determine the future allocation $x_t$ as:

$$x_t = \begin{cases} 
0 & \text{if } r_t \leq \hat{p}^T a_t \\
1 & \text{if } r_t > \hat{p}^T a_t
\end{cases}$$

as long as $a_{it} x_t \leq b_i - \sum_{j=1}^{t-1} a_{ij} x_j$ for all $i$; otherwise, set $x_t = 0$. 
Theorem

For any fixed $\epsilon > 0$, the one-time learning algorithm is $(1 - \epsilon)$ competitive for solving the linear program when

$$B \geq \Omega \left( \frac{m \log (n/\epsilon)}{\epsilon^3} \right)$$

This is one $\epsilon$ worse than the optimal lower bound.
Dynamic Learning Algorithm

In the dynamic price learning algorithm, we update the price at time $\epsilon n, 2\epsilon n, 4\epsilon n, \ldots$, till $2^k \epsilon \geq 1$. 

Ye, Yinyu (Stanford) 
Robust and Online Optimization 
June 2, 2020 33 / 60
Dynamic Learning Algorithm

In the dynamic price learning algorithm, we update the price at time $\epsilon n$, $2\epsilon n$, $4\epsilon n$, ..., till $2^k\epsilon \geq 1$.

At time $\ell \in \{\epsilon n, 2\epsilon n, \ldots\}$, the price vector is the optimal **dual solution** to the following linear program:

$$\begin{align*}
\text{maximize}_x & \quad \sum_{t=1}^{\ell} r_t x_t \\
\text{subject to} & \quad \sum_{t=1}^{\ell} a_{it} x_t \leq (1 - h_\ell) \frac{\ell}{n} b_i, \quad i = 1, \ldots, m \\
& \quad 0 \leq x_t \leq 1, \quad t = 1, \ldots, \ell
\end{align*}$$

where

$$h_\ell = \epsilon \sqrt{\frac{n}{\ell}};$$

and this price vector is used to determine the allocation for the next **immediate** period.
Geometric Pace/Grid of Price Updating

$t_1 = \epsilon n$

$t_2 = 2\epsilon n$

$t_3 = 4\epsilon n$

$t_4 = 8\epsilon n$
In the dynamic algorithm, we update the prices $\log_2(1/\epsilon)$ times during the entire time horizon. One can also update prices or resolve an LP problem after every bid to achieve a lightly better competitive ratio.
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Does the price vector converge?
Comments on Dynamic Learning Algorithm

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- Can the model handle the double-market?
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Does the price vector converges?

Can the model handle the double-market?

Could the online algorithm avoid solving LPs?
Recent Results: Dual Convergence I

The offline dual problem again

$$\begin{align*}
\min & \quad \sum_{i=1}^{m} b_i p_i + \sum_{j=1}^{n} y_j \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} p_i + y_j \geq r_j, \quad j = 1, \ldots, n. \\
& \quad p_i, y_j \geq 0 \text{ for all } i, j.
\end{align*}$$

can be rewritten as:

$$\begin{align*}
\min & \quad \sum_{i=1}^{m} b_i p_i + \sum_{j=1}^{n} (r_j - \sum_{i=1}^{m} a_{ij} p_i)^+ \\
\text{s.t.} & \quad p_i \geq 0, \quad i = 1, \ldots, m.
\end{align*}$$

Here, entries of $(r_j, a_j)$ can be either positive or negative!
Recent Results: Dual Convergence II

Normalize the objective,

\[
\min f_n(p) := \sum_{i=1}^{m} b_ip_i + \frac{1}{n} \sum_{j=1}^{n} (r_j - \sum_{i=1}^{m} a_{ij}p_i)^+
\]
\[
\text{s.t.} \quad p_i \geq 0, \quad i = 1, \ldots, m.
\]

Define the stochastic program

\[
\min f(p) := b^\top p + \mathbb{E}_{(r,a) \sim \mathcal{P}} [(\pi - a^\top p)^+]
\]
\[
\text{s.t.} \quad p \geq 0,
\]

Observation: \( f_n(p) \) is a sample average approximation for \( f(p) \)

The optimal dual solution \( p_n^* \) converges to the (unique) optimal solution \( p^* \) of the stochastic program. But at what rate?
(Li and Y 2019) Define the binding index set $B = \{ i : p_i^* = 0 \}$ and the non-binding index set $N = \{ i : p_i^* \neq 0 \}$. Under standard stochastic assumptions (iid...), there exists a universal constant $C$ such that

$$
P(\|p_n^*(B) - p^*(B)\|_2 \geq \epsilon) \leq m \exp \left( -\frac{Cn\epsilon^2}{m} \right)
$$

$$
P(\|p_n^*(N) - p^*(N)\|_2 \geq \epsilon) \leq m \exp \left( -\frac{Cn\epsilon}{m} \right)
$$

for any $\epsilon > 0$, $m, n \in \mathbb{N}^+$ and $P \in \Xi$. Here $p_n^*(B)$, $p^*(B)$, $p_n^*(N)$, $p^*(N)$ denote the sub-vectors corresponding to the indexes in $B$, $N$. 
“Offline” optimal solution $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ and “online” solution $\mathbf{x}$.

$$R_n^* = \sum_{j=1}^{n} r_j x_j^*$$ and $$R_n = \sum_{j=1}^{n} r_j x_j.$$ 

Objective: Minimize the worst-case gap, Regret, between the offline and online objective values

$$\Delta_n = \sup_{\mathcal{P} \in \Xi} \mathbb{E}_{\mathcal{P}} [R_n^* - R_n] \quad \text{(Stochastic Input)}$$

$$\delta_n = \sup_{\mathcal{D} \in \Xi_{\mathcal{D}}} R_n^* - \mathbb{E} [R_n] \quad \text{(Random Permutation)}$$

Also, we measure the constraint violation of the online solution

$$v(\mathbf{x}) = \| (A\mathbf{x} - \mathbf{b})^+ \|_2$$

Remark: A bi-objective performance measure
Recent Results: Action-Dependent Learning

Earlier Online Algorithms

At each time $t$, compute the dual optimal solution for the problem:

$$\max \sum_{j=1}^{t-1} r_j x_j$$

subject to

$$\sum_{j=1}^{t-1} a_{ij} x_j \leq b_i (t-1)/n, \quad \forall i$$

$$0 \leq x_j \leq 1, \quad j = 1, \ldots, t-1$$

The algorithms are estimating $p^*$ purely using data.

Action-Dependent Learning

At each time $t$, compute the dual optimal solution for the problem:

$$\max \sum_{j=1}^{t-1} r_j x_j$$

subject to

$$\sum_{j=1}^{t-1} a_{ij} x_j \leq \tilde{b}_i (t-1) \frac{n}{n-t+1}, \quad \forall i$$

$$0 \leq x_j \leq 1, \quad j = 1, \ldots, t-1$$

where $\tilde{b}_i = b_i - \sum_{j=1}^{t-1} a_{ij} \tilde{x}_j$ – the new algorithm also considers actions already taken.
Recent Results: Action-Dependent Learning II

Provable regret performance curves with $m = 4$ when $b = o(n)$, where A1, A2, and A3 stand for Algorithm 1 (Early Dynamic Learning), Algorithm 2 (Action-Dependent), and Algorithm 3 (Using $p^*$), respectively.

Ye, Yinyu (Stanford) Robust and Online Optimization June 2, 2020 41 / 60
Fast LP-Free Algorithm

Input: \( \mathbf{d} = \mathbf{b}/n \) and Initialize \( \mathbf{p}^1 = \mathbf{0} \)

For \( t = 1, \ldots, n \)

Set

\[
x_t = \begin{cases} 
1, & r_t > \mathbf{a}_t^\top \mathbf{p}_t \\
0, & r_t \leq \mathbf{a}_t^\top \mathbf{p}_t 
\end{cases}
\]

Compute

\[
\mathbf{p}_{t+1} = \mathbf{p}_t + \gamma_t (\mathbf{a}_t x_t - \mathbf{d})
\]

\[
\mathbf{p}_{t+1} = \mathbf{p}_{t+1} \vee \mathbf{0}
\]

Output: \( \mathbf{x} = (x_1, \ldots, x_n) \).

Stochastic subgradient descent for the dual equivalent form!
Theorem (Li, Sun, & Ye (2020))

With step size $\gamma_t = 1/\sqrt{n}$, the regret and expected constraint violation of the algorithm satisfy

$$\mathbb{E}[R_n^* - R_n] \leq O(m\sqrt{n})$$

$$\mathbb{E}[\nu(x)] \leq O(m\sqrt{n}).$$

hold for all $m, n \in \mathbb{N}^+$ and distribution $P \in \Xi$.

Remark: The proof utilizes the structure of the LP and largely mimics the analyses of the online gradient descent algorithm.
Theorem (Li, Sun, & Ye (2020))

With the step size $\gamma_t = \frac{1}{\sqrt{n}}$, the regret and expected constraint violation of the algorithm satisfy

$$R_n^* - \mathbb{E}[R_n] \leq O((m + \log n)\sqrt{n})$$

$$\mathbb{E}[\nu(x)] \leq O(m\sqrt{n}).$$

for all $m, n \in \mathbb{N}^+$ and $D \in \Xi_D$.

An extra $\log(n)\sqrt{n}$ in the Regret for the Permutation Input Model compared with that in the Stochastic iid Input Model. The proof builds upon the notion of *Permutational Rademacher Complexity* (Tolstikhin et al. 2015) which is originally used for analyzing transductive learning.
Theorem (Li, Sun, & Ye (2020))

Under the random permutation model, the regret and expected constraint violation of the two algorithms (Agrawal et al. 2014, Kesselheim et al. 2014) both satisfy

$$R_n^* - \mathbb{E}[R_n] \leq O(\sqrt{mn})$$

$$\mathbb{E}[v(x)] \leq O(\sqrt{mn \log n})$$

for all $m, n \in \mathbb{N}^+$ and $\mathcal{D} \in \Xi_D$.

Remark: Compared with the fast algorithm, the regret and constraint violation are reduced by a factor of $\sqrt{m}$ with the price of more computation cost.
## Numerical Experiments I


<table>
<thead>
<tr>
<th></th>
<th>Gurobi</th>
<th>Fast Alg.</th>
<th>Alg. 1</th>
<th>Alg. 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 5, n = 500$</td>
<td>CPU time</td>
<td>0.260</td>
<td>0.039</td>
<td>487</td>
</tr>
<tr>
<td></td>
<td>Cmpt. Ratio</td>
<td>100%</td>
<td>94.6%</td>
<td>95.0%</td>
</tr>
<tr>
<td>$m = 10, n = 500$</td>
<td>CPU time</td>
<td>0.350</td>
<td>0.029</td>
<td>373</td>
</tr>
<tr>
<td></td>
<td>Cmpt. Ratio</td>
<td>100%</td>
<td>95.74%</td>
<td>94.09%</td>
</tr>
<tr>
<td>$m = 30, n = 500$</td>
<td>CPU time</td>
<td>0.310</td>
<td>0.039</td>
<td>491</td>
</tr>
<tr>
<td></td>
<td>Cmpt. Ratio</td>
<td>100%</td>
<td>95.6%</td>
<td>93.9%</td>
</tr>
</tbody>
</table>

- Alg. 1: Agrawal et al. (2014); Alg. 2: Kesselheim et al. (2014)
- Gurobi computes the optimal solution in an offline fashion while the other three algorithms are online.
- Gurobi is set to solve the **binary** LP problem. The optimality ratio is reported against the objective value of the Gurobi’s **binary** solution.
Numerical Experiments II

Large-scale multi-knapsack problem:

<table>
<thead>
<tr>
<th></th>
<th>Gurobi</th>
<th>Fast Alg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 100, n = 10,000$</td>
<td>CPU time</td>
<td>34.7</td>
</tr>
<tr>
<td></td>
<td>Cmpt. Ratio</td>
<td>100%</td>
</tr>
<tr>
<td>$m = 500, n = 10,000$</td>
<td>CPU time</td>
<td>267</td>
</tr>
<tr>
<td></td>
<td>Cmpt. Ratio</td>
<td>100%</td>
</tr>
<tr>
<td>$m = 1000, n = 10,000$</td>
<td>CPU time</td>
<td>764</td>
</tr>
<tr>
<td></td>
<td>Cmpt. Ratio</td>
<td>100%</td>
</tr>
</tbody>
</table>

- The other two algorithms are too slow to finish while Gurobi is set to solve the relaxed LP problem.
- The optimality ratio is reported against the objective value of the relaxed LP, and it decreases as $m$ grows with fixed $n$. 
Summary and Future Questions on Online Optimization

- \( B = \frac{\log m}{\epsilon^2} \) is now a necessary and sufficient condition (differing by a constant factor).
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- Thus, they are optimal online algorithms for a very general class of online linear programs.

Better to do dynamic learning, that is, "learning-while-doing". Action-Dependent dynamic learning is even better.

Multi-item price-posting market? More general online optimization? Approximately solve large-scale offline binary LPs with the proposed fast algorithm?
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# Table of Contents

1. Introduction to Stochastic and Distributionally Robust Optimization

2. DRO under Moment, Likelihood and Wasserstein Bounds

3. Online Linear Optimization and Dynamic Learning

4. Markets for Efficient Public Good Allocation
A pandemic adds capacity restrictions to shops, gyms, schools, and public spaces such as parks and beaches, which seems limitless under normal times.

Either open: An overcrowded open beach
Capacity Constraints on “Public Goods”

Either open: An overcrowded open beach

Or closed: a completely empty beach generating no value to society
Capacity Constraints on “Public Goods”

A pandemic adds capacity restriction to shops, gyms, schools, and public spaces such as parks and beaches, which seems limitless under normal times.
To achieve an intermediate between the two extreme scenarios, open or closed, through Time-of-Use goods. More precisely, create different time periods and people “book/purchase” permits to use the public spaces at one time-period so that the population density on spaces can be upper limited. (Jalota, Pavone and Y 2020)
Market-Based Mechanism Design

Requirement of Strict Capacity Constraints

Presence of Public Good Alternatives

A Prioritization scheme for Public Goods Allocation

But does Pricing not defeat the purpose of a public good?

Pricing?

Artificial Currencies
How to setup “prices” for each time-period/good so that resources can be efficiently allocated while keep each individual satisfied?
The Model: Fisher’s Equilibrium Price

Buyer $i \in B$’s optimization problem for given prices $p_j, j \in G$.

$$\begin{align*}
\text{max} & \quad u^T_i x_i := \sum_{j \in G} u_{ij} x_{ij} \\
\text{s.t.} & \quad p^T x_i := \sum_{j \in G} p_j x_{ij} \leq w_i, \\
& \quad x_{ij} \geq 0, \quad \forall j,
\end{align*}$$

Assume that the given amount of each good is $\bar{s}_j$. The equilibrium price vector is the one that for all $j \in G$

$$\sum_{i \in B} x^*(p)_{ij} = \bar{s}_j$$

where $x^*(p)$ is a maximizer of the utility maximization problem for every buyer $i$. 

The Fisher Market Illustration

Goods

Buyers

1 \hspace{1cm} 2 \hspace{1cm} \ldots \hspace{1cm} n \hspace{1cm} m

\begin{align*}
\text{s}_1, & \text{p}_1 \\
\text{s}_2, & \text{p}_2 \\
\text{s}_3, & \text{p}_3 \\
\vdots & \\
\text{s}_n, & \text{p}_n \\
\end{align*}

\begin{align*}
\text{w}_1, & \text{u}_1(\cdot) \\
\text{w}_2, & \text{u}_2(\cdot) \\
\vdots & \\
\text{w}_m, & \text{u}_m(\cdot) \\
\end{align*}
The Aggregated Social Optimization Problem

\[
\begin{align*}
\max & \quad \sum_{i \in B} w_i \log(u_i^T x_i) \\
\text{s.t.} & \quad \sum_{i \in B} x_{ij} = \bar{s}_j, \quad \forall j \in G \\
& \quad x_{ij} \geq 0, \quad \forall i, j,
\end{align*}
\]

Theorem

(Eisenberg and Gale 1959) Optimal dual (Lagrange) multiplier vector of equality constraints is an equilibrium price vector.

Proof: The optimality conditions of the social problem are identical to the equilibrium conditions.
The Model with Individual Physical Constraints

Buyer $i \in B$’s optimization problem for given prices $p_j, j \in G$.

$$\begin{align*}
\text{max} & \quad u_i^T x_i := \sum_{j \in G} u_{ij} x_{ij} \\
\text{s.t.} & \quad p^T x_i := \sum_{j \in G} p_j x_{ij} \leq w_i, \\
& \quad A_i x_i \leq b_i, \\
& \quad x_{ij} \geq 0, \quad \forall j,
\end{align*}$$

Could we still solve (?)

$$\begin{align*}
\text{max} & \quad \sum_{i \in B} w_i \log(u_i^T x_i) \\
\text{s.t.} & \quad \sum_{i \in B} x_{ij} = \bar{s}_j, \quad \forall j \in G \\
& \quad A_i x_i \leq b_i, \forall i \in B, \\
& \quad x_{ij} \geq 0, \quad \forall i, j,
\end{align*}$$
The Budget Adjust Mechanism

\[
\begin{align*}
\text{max} \quad & \sum_{i \in B} (w_i + \delta_i) \log(u_i^T x_i) \\
\text{s.t.} \quad & \sum_{i \in B} x_{ij} = \bar{s}_j, \quad \forall j \in G \\
& A_i x_i \leq b_i, \forall i \in B, \\
& x_{ij} \geq 0, \quad \forall i, j,
\end{align*}
\]

by adding \( \delta_i \geq 0 \)'s at ideal levels, which itself becomes a fixed-point computation problem.

**Theorem**

*There is one-one correspondence of an equilibrium price vector \( p^* \) and a fixed-point solution of \( \delta_i^* \)'s, that is, \( \delta_i^* = b_i^T y_i^*, \forall i \) where \( y_i^* \) is the optimal multiplier vector of constraint \( A_i x_i \leq b_i \) with input \( \delta_i^* \)'s.*

**Implementation**: There is no need to solve the problem to the individual level but a clusters of buyers who have similar behaviors/preferences.
The Budget Adjustment Iterative Process

Adjust $\delta_{i}^{k+1} = b_{i}^{T}y_{i}^{k}$ for all $i$ iteratively where $y_{i}^{k}$ is the optimal multiplier vector of constraint $A_{i}x_{i} \leq b_{i}$ with input $\delta_{i}^{k}$.

![Graph showing the difference in $\lambda$ from dual variables over iterations](image-url)
Summary of the Model with Physical Constraints

- The equilibrium price vector exist under mild technical conditions.
- The problem seems able to be solved efficiently from a simple iterative procedure as described.
Summary of the Model with Physical Constraints

- The equilibrium price vector exist under mild technical conditions.
- The problem seems able to be solved efficiently from a simple iterative procedure as described.

Open questions
- Is the equilibrium price vector unique?
- Is the simple iterative procedure provably convergent?
- Is the fixed-point problem Tarski’s type?
- Is the problem in the class of PPAD or there exists a polynomial time algorithm?