Multi-Block Alternating Direction Method of Multipliers

Are there Alternative Algorithms for Linear Programming?

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Table of Contents

1. Linear Programming and Algorithms
2. Advances in Simplex and Interior-Point Algorithms
3. First-Order and Alternating Direction Method of Multipliers
4. Convergence of RP-ADMM in Expectation
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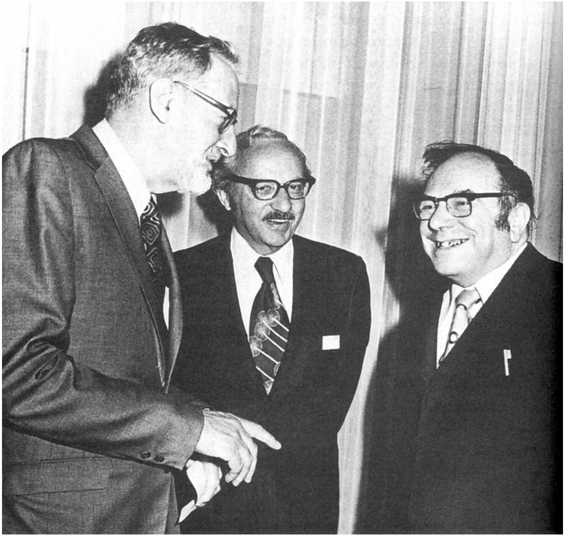
Linear Programming (LP)



Algebra of Linear Programming

$$\begin{array}{lll} \text{minimize}_{\mathbf{x}} & \mathbf{c}^T\mathbf{x} & \\ \\ \text{subject to} & A\mathbf{x} & (<=>)\mathbf{b}, \\ & \mathbf{x} & \geq \mathbf{0}, \end{array}$$

where given constraint matrix A is an $m \times n$ matrix, the right-hand $\mathbf{b}$ is an m-dimensional vector, the objective coefficients $\mathbf{c}$ is an n-dimensional vector.

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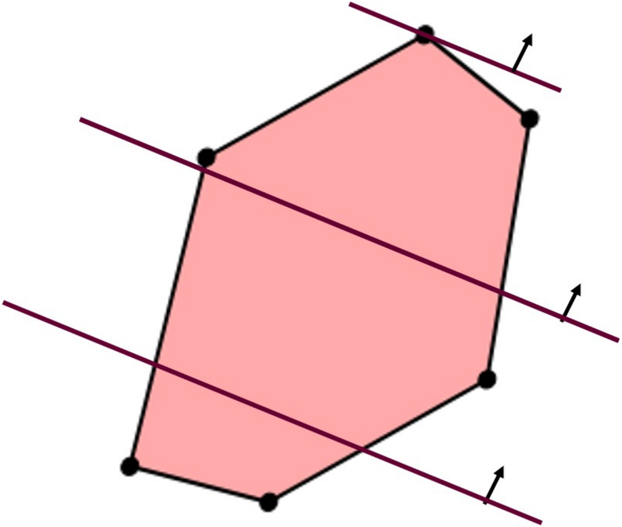
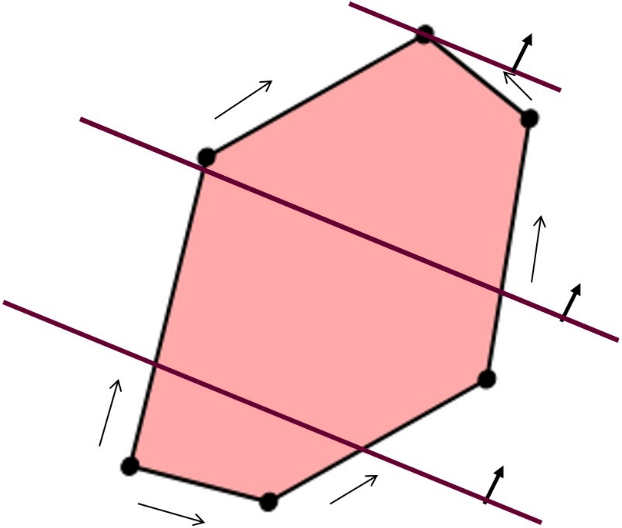
LP is a data-driven computation/decision model that has wide applications so that we like to decide the optimal $\mathbf{x}$ fast in theory and practice.

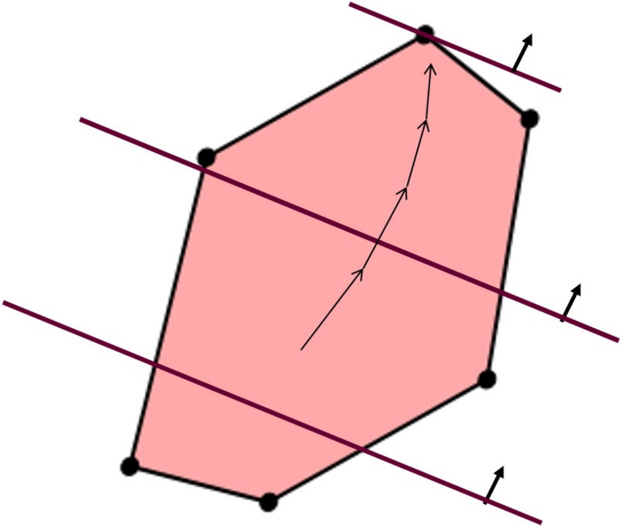
Geometry of Linear Programming



LP Algorithms: the Simplex Method



LP Algorithms: the Interior-Point Method



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- MDPs are useful for studying a wide range of optimization problems solved via dynamic programming, where it was known at least as early as the 1950s (cf. Shapley 53, Bellman 57).
- Modern applications include dynamic planning, reinforcement learning, social networking, and almost all other dynamic/sequential decision making problems in Mathematical, Physical, Management and Social Sciences.

The Markov Decision Process/Game continued

- At each time step, the process is in some state $i \in \{1, ..., m\}$, and the decision maker chooses an action j_i among a set of actions, $\mathcal{A}_i$, for state i.

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- The MDP is to find a policy to optimiza the expected discounted sum over infinite horizon with a discount factor $0 \leq \gamma < 1$.

The Fixed-Point of Cost-to-Go Values

An optimal policy is associated with m cost-to-go values for every state, $\mathbf{y} \in R^m$, such that it is a fixed point:

$$y_i^* = \min\{c_{j_i} + \gamma \mathbf{p}_{j_i}^T \mathbf{y},\ j_i \in \mathcal{A}_i\},\ \forall i,$$

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$$\begin{array}{lccl}
\text{maximize}_{\mathbf{y}} & \sum_{i=1}^m y_i & & \\
\text{subject to} & y_1 & \leq & \mathbf{c}_{j_1} + \gamma \mathbf{p}_{j_1}^T \mathbf{y},\ j_1 \in \mathcal{A}_1 \\
 & \ldots & \ldots & \ldots \\
 & y_i & \leq & \mathbf{c}_{j_i} + \gamma \mathbf{p}_{j_i}^T \mathbf{y},\ j_i \in \mathcal{A}_i \\
 & \ldots & \ldots & \ldots \\
 & y_m & \leq & \mathbf{c}_{j_m} + \gamma \mathbf{p}_{j_m}^T \mathbf{y},\ j_m \in \mathcal{A}_m.
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Algorithmic Events of the MDP Methods

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- de Ghellinck 60, D'Epenoux 60 and Manne 60 showed that the MDP has an LP representation, so that it can be solved by the simplex method of Dantzig 47, which is a special policy-iteration method that each step only actions in one state is switched.

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- But most analyses of these methods are negative: the simplex method with smallest index rule (Melekopoglou/Condon 90), random pivoting rule (Friedman et al 12), and policy-iteration for undiscounted finite-horizon MDP (Fearnley 10) are all exponential.

Positive Results for MDP of m states and n actions

- The classic simplex method, with the Dantzig pivoting rule, terminates in
$$\frac{m(n-m)}{1-\gamma} \cdot \log\left(\frac{m^2}{1-\gamma}\right)$$
iterations, and each iteration uses at most $O(mn)$ arithmetic operations (Y MOR10).

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- The policy-iteration or block-pivoting method
$$\frac{n}{1-\gamma} \cdot \log\left(\frac{m}{1-\gamma}\right),$$
iterations and each iteration uses at most m^2n arithmetic operations (Hansen/Miltersen/Zwick ACM12).

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- The result was extended to general LP by Kitahara and Mizuno 2012; also see renewed exciting research work on the simplex method, e.g., Feinberg/Huang 2013, Lee/Epelman/Romeijn/Smith 2013, Scherrer 2014, Fearnley/Savani 2014, Adler/Papadimitriou/Rubinstein 2014, etc.

The Turn-Based Two-Person Zero-Sum Game

- The states is partitioned to two sets where one is to max and the other is to min, i.e., the fixed point:

$$
\begin{aligned}
y_i^* &= \min\{\mathbf{c}_{j_i} + \gamma \mathbf{p}_{j_i}^T \mathbf{y},\ j_i \in \mathcal{A}_i\},\ \forall i \in I^+,\\
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- It does not admit a convex programming formulation, and it is unknown if it can be solved in polynomial time in general.
- Hansen/Miltersen/Zwick ACM12 proved that the strategy iteration method solves it in strongly polynomial time when discount factor is fixed – the first strongly polynomial time algorithm.

Advances in Interior-Point Methods

In theory, the best interior-point algorithms converge in $\tilde{O}(\sqrt{\max\{m,n\}})$ iterations, where notation $\tilde{O}$ includes some constant and logarithmic factors of dimensions and accuracy ϵ.

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- Efficient Inverse Maintenance and Faster Algorithms for Linear Programming (Lee and Sidford, FOCS15), where the author also have also reduced the operation complexity for linear programming.

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- Use subgradient method for a transformed problem, assuming knowing a strict feasible point (Reneger 14, Freund 15) with iteration complexity $O(L^2 D^2 \frac{1}{\epsilon^2})$, where L and D are condition numbers on the optimal solution structure and feasible region.

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- First-order Karmarkar potential reduction reduction algorithm...

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- First order method for packing and covering LP (Allen-Zhu/Orecchia 14): penalize the constraint, then use stochastic coordinate descent and several techniques. The iteration complexity: $\tilde{O}(\frac{1}{\epsilon})$ for packing LP and $\tilde{O}(\frac{1}{\epsilon^{1.5}})$ for covering LP.

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- Two-Block ADMM for solve primal-LP (He/Yuan 11, Monteiro/Svaiter 13, Monteiro/Ortiz/Svaiter 14) with iteration complexity $\tilde{O}(\frac{1}{\epsilon})$ (a matrix needs to be inversed once).

First-Order Potential Reduction I

$$\begin{array}{ll} \text{Minimize} & f(\mathbf{x}) \\ \text{Subject to} & \mathbf{e}^T\mathbf{x} = 1;\ \mathbf{x} \geq \mathbf{0}, \end{array}$$

where $\mathbf{e}$ is the vector of all ones.

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Assume that $f(\mathbf{x})$ is a convex function in $\mathbf{x} \in R^n$ and $f(\mathbf{x}^*) = 0$ where $\mathbf{x}^*$ is a minimizer of the problem. Furthermore,

$$f(\mathbf{x}+\mathbf{d}) - f(\mathbf{x}) \leq \nabla f(\mathbf{x})^T\mathbf{d} + \frac{\gamma}{2}\|\mathbf{d}\|^2,$$

where positive γ is the Lipschitz parameter.

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We consider the potential function (e.g., Karmarkar 84,Todd/Ye 87)

$$\phi(\mathbf{x}) = \rho \ln(f(\mathbf{x})) - \sum_j \ln(x_j),$$

where $\rho \ge n$ over the simplex. If we start from $\mathbf{x}^0 = \frac{1}{n}\mathbf{e}$, and generate a sequence of points $\mathbf{x}^k$, $k = 1, ...,$, whose potential value is strictly decrease by a fixed amount.

First-Order Potential Reduction II

Update $\mathbf{x}$ by solving

$$\begin{array}{ll} \text{Minimize} & \nabla\phi(\mathbf{x})^T\mathbf{d} \\ \text{Subject to} & \mathbf{e}^T\mathbf{d} = 0,\ \|X^{-1}\mathbf{d}\| \leq \beta; \end{array}$$

where $\beta < 1$ is yet to be determined; and $\mathbf{x}^+ = \mathbf{x} + \mathbf{d}$.

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$$\phi(\mathbf{x}^+) - \phi(\mathbf{x}) \le -\beta + \frac{\rho\gamma}{2f(\mathbf{x})}\beta^2 + \frac{\beta^2}{2(1-\beta)}$$

so that one can choose a β to make

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Theorem

The steepest descent potential reduction algorithm generates a $\mathbf{x}^k$ with $f(\mathbf{x}^k)/f(\mathbf{x}^0)\le\epsilon$ in no more than $4(n+\sqrt{n})\frac{\max\{1,2(n+\sqrt{n})\gamma/f(x^0)\}}{\epsilon}\ln(\frac{1}{\epsilon})$ steps.

Alternating Direction Method of Multipliers

- Consider the convex optimization problem

$$\begin{aligned}
\min_{\mathbf{x}\in R^n} \quad & f_1(\mathbf{x}_1) + \cdots + f_p(\mathbf{x}_p), \\
s.t. \quad & A\mathbf{x} \triangleq A_1\mathbf{x}_1 + \cdots + A_p\mathbf{x}_p = \mathbf{b}, \\
& \mathbf{x}_i \in \mathcal{X}_i \subset R^{d_i},\ \ i = 1, \ldots, p.
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- Augmented Lagrangian function:

$$\begin{aligned} L_\gamma(\mathbf{x}_1, .., \mathbf{x}_p; \mathbf{y}) = \ & \textstyle\sum_i f_i(\mathbf{x}_i) - \mathbf{y}^T(\sum_i A_i\mathbf{x}_i - \mathbf{b}) \\ & + \tfrac{\gamma}{2}\|\textstyle\sum_i A_i\mathbf{x}_i - \mathbf{b}\|^2. \end{aligned}$$

Alternating Direction Method of Multipliers continued

- Multi-block ADMM:

$$\begin{cases} \mathbf{x}_1^{k+1} \longleftarrow \arg\min_{\mathbf{x}_1 \in \mathcal{X}_1} L_\gamma(\mathbf{x}_1, \mathbf{x}_2^k \ldots, \mathbf{x}_p^k; \mathbf{y}^k), \\ \qquad\qquad \vdots \\ \mathbf{x}_p^{k+1} \longleftarrow \arg\min_{\mathbf{x}_p \in \mathcal{X}_p} L_\gamma(\mathbf{x}_1^{k+1}, \ldots, \mathbf{x}_{p-1}^{k+1}, \mathbf{x}_p; \mathbf{y}^k), \\ \mathbf{y}^{k+1} \longleftarrow \mathbf{y}^k - \gamma(\sum_i A_i \mathbf{x}_i^{k+1} - \mathbf{b}). \end{cases}$$

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- Convergence was well established when $p = 1$ or $p = 2$ (..., Glowinski/Marrocco 75, Gabay/Mercier '76, Eckstein/Bertsekas 92,...)
- But what about $p > 2$?

Convergence of Multi-Block ADMM

- Recent Discovery: When $p \geq 3$, ADMM can diverge [Chen/He/Y/Yuan MP14]:

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$$A\mathbf{x} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{0}, \text{ where } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

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- The ADMM would be a linear mapping of matrix M

$$\mathbf{z}^{k+1} = M\mathbf{z}^k, \text{ where } \mathbf{z}^k = (x_1^k; x_2^k; x_3^k; \mathbf{y}^k)$$

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$$\mathbf{z}^{k+1} = M\mathbf{z}^k, \text{ where } \mathbf{z}^k = (x_1^k; x_2^k; x_3^k; \mathbf{y}^k)$$

- Why diverges? $\rho(M) > 1$ for above A and any choice of γ.

Convergence of Multi-Block ADMM

- Recent Discovery: When $p \geq 3$, ADMM can diverge [Chen/He/Y/Yuan MP14]: $p = 3$.

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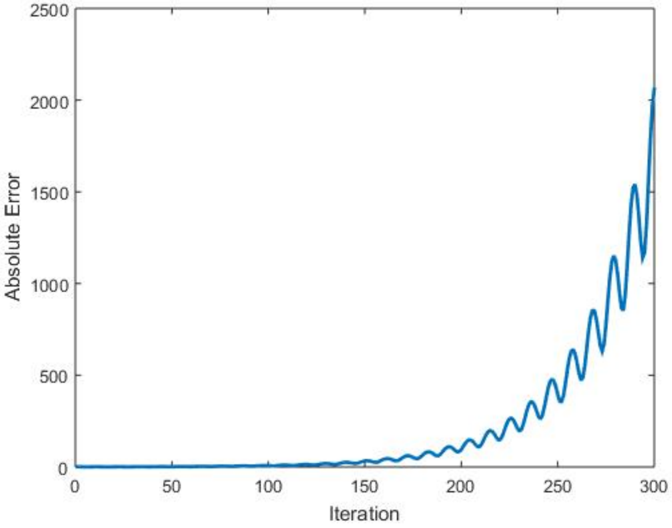
- Why diverges? $\rho(M) > 1$ for above A and any choice of γ.
- When you randomly choose an initial solution, it diverges with probability one!

Almost Always Diverges



Strong Convexity Does not Help

$$\begin{array}{ll} \min & 0.05x_1^2 + 0.05x_2^2 + 0.05x_3^2 \\ \text{s.t.} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \end{array}$$

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- The mapping matrix M in the ADMM ($\gamma = 1$) has

$$\rho(M) = 1.0087 > 1$$

- That is, even for strongly convex programming, the ADMM is not necessarily convergent for a range of $\gamma > 0$.

The Stepsize of ADMM

For some step-size $0 < \beta < 1$, update the Lagrangian multiplier:

$$\mathbf{y}^{k+1} := \mathbf{y}^k - \beta\gamma(\sum_i A_i \mathbf{x}_i^{k+1} - \mathbf{b}).$$

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- $p = 1$; (Augmented Lagrangian Method) for $\beta \in (0, 2)$, (Hestenes 69, Powell 69).
- $p = 2$; (Alternating Direction Method of Multipliers) for $\beta \in (0, \frac{1+\sqrt{5}}{2})$, (Glowinski 84).
- $p \geq 3$; for β sufficiently small provided additional conditions on the problem, (Hong/Luo 12)

Is there a Problem-Data-Independent β?

For any positive β the following linear system makes ADMM diverge

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There is no practical problem-data-independent β such that the small-step size variant would work.

Many other alternatives were proposed, but somehow they all make ADMM converge slower in practice...

Randomly Permuted ADMM

Random-Permuted ADMM (RP-ADMM): each round, draw a random permutation $\sigma = (\sigma(1), \ldots, \sigma(p))$ of $\{1, \ldots, p\}$, and

$$\text{Update } \mathbf{x}_{\sigma(1)} \to \mathbf{x}_{\sigma(2)} \to \cdots \to \mathbf{x}_{\sigma(p)} \to \mathbf{y}.$$

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- Force "absolute fairness" among blocks or a mixed strategy on updating order.
- Random permutation has been effectively used in many areas but little theoretical analysis is known.
- Simulation Test Result: almost always converges and fast!

Almost Always Converges when Randomize the Update Order

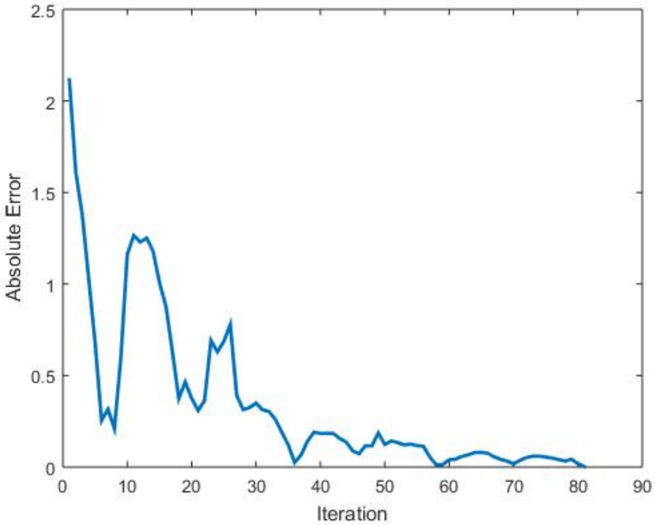

Table of Contents

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Any Theory Behind the Success?

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Consider a square system of linear equation:

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(S) & \min\limits_{\mathbf{x}\in R^n} & 0, \\
 & s.t. & A_1\mathbf{x}_1 + \cdots + A_p\mathbf{x}_p = \mathbf{b},
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where $A = [A_1, \ldots, A_p] \in R^{n\times n}$, $\mathbf{x}_i \in R^{d_i}$ and $\sum_i d_i = n$.

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After k rounds, RP-ADMM generates $\mathbf{z}^k$, an r.v. depending on

$$\boldsymbol{\xi}_k = (\sigma_1, \ldots, \sigma_k),$$

where σ_j is the randomly picked permutation at j-th round. Let

$$\phi^k \triangleq E_{\xi_k}(\mathbf{z}^k).$$

Convergence in Expectation (Sun/Luo/Y 15)

Theorem 1

The expected iterate $\phi^k \triangleq E_{\xi_k}(\mathbf{z}^k)$ converges to the solution linearly for any $1 \leq p \leq n$ if A is invertible, i.e.

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Remark: Expected convergence $\neq$ convergence, but is a strong evidence for convergence for solving most problems, e.g., when iterates are bounded.

The Average Mapping is a Contraction

- The update equation of RP-ADMM for (S) (with $\mathbf{b} = \mathbf{0}$) is

$$\mathbf{z}^{k+1} = M_\sigma \mathbf{z}^k,$$

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Theorem 2

The spectral radius of M is strictly less than 1, i.e. $\rho(M) < 1$, for any $1 \le p \le n$ if A is invertible.

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  - Though $\rho(M) < \|M\|$, it turns out $\|M\| > 2.3$ for the counterexample.
- RP is not independent so that M is a complicated function of A.
- Techniques: Symmetrization and Mathematical Induction.

Two Main Lemmas to Prove Theorem 2

- **Step 1**: Relate M to a symmetric matrix AQA^T.

Lemma 1

$$\lambda \in \text{eig}(M) \Longleftrightarrow \frac{(1-\lambda)^2}{1-2\lambda} \in \text{eig}(AQA^T).$$

Sicne Q is symmetric, we have

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- Remark: $4/3$ is "almost" tight; for $m = 3$, maximum ≈ 1.18. Increase to $4/3$ as m increases.

Fixed Cyclic ADMM vs RP-ADMM

Solving weakly Laplacian linear systems:

$$A(i,i) = 1, \quad A(i,j) = A(j,i) = \tau \cdot \text{rand}(1,1).$$

(n,τ) - (iter,time)	CycADMM	RP-ADMM
$(500, 0.01)$	diverge	(10.1, 1.3)
$(500, 0.05)$	diverge	(67, 0.61)
$(5000, 0.002)$	diverge	(7.5, 21.1)
$(5000, 0.01)$	diverge	(22, 38)
$(5000, 0.02)$	diverge	(205, 251)

Further Result on Randomized Permutation for Convex Optimization

Consider the non-separable convex quadratic problem

$$\begin{array}{ll} \min_{\mathbf{x} \in R^n} & \mathbf{x}^T H \mathbf{x} + \mathbf{c}^T \mathbf{x}, \\ s.t. & A\mathbf{x} \triangleq A_1\mathbf{x}_1 + \cdots + A_p\mathbf{x}_p = \mathbf{b}, \\ & \mathbf{x}_i \in \mathcal{X}_i \subset R^{d_i}, \; i = 1, \ldots, p. \end{array}$$

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\end{aligned}$$

Theorem 3

If each block subproblem possesses a unique solution, the expected iterate $\phi^k \triangleq E_{\xi_k}(\mathbf{z}^k)$ converges to an optimal solution of the original problem (Chen, Li, Liu and Y 15).

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Universal Decomposition and Block Coordinate Descent

Consider the homogeneous and self-dual linear program to find feasible solution $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ such that

$$\begin{array}{rcl} A\mathbf{x} - \mathbf{b}\tau & = & \mathbf{0}, \\ -A^T\mathbf{y} - \mathbf{s} + \mathbf{c}\tau & = & \mathbf{0}, \\ \mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} - \kappa & = & 0, \\ \mathbf{e}^T\mathbf{x} + \tau + \mathbf{e}^T\mathbf{s} + \kappa & = & 1, \\ (\mathbf{x}, \tau, \mathbf{s}, \kappa) & \geq & \mathbf{0}, \end{array}$$

where three blocks $(\mathbf{x}, \tau)$, $\mathbf{y}$ and $(\mathbf{s}, \kappa)$ are alternatively updated (Donoghue/Chu/Parikh/Boyd 15), also see Sun and Toh 14 for SDP.

Combine ADMM and Interior-Point

Consider the logarithmic barrier function as objective:

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\min & -\mu \ln(\tau\kappa) - \mu \sum_j \ln(x_j s_j) & & \\
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Then μ is gradually reduced to 0 as in interior-point methods – initial computational results are encouraging (Lin/Ma 15)!

Implementation on Distributed Platforms I

Reformulate

$$\begin{array}{lll} \text{minimize}_{\mathbf{x}} & \mathbf{c}^T\mathbf{x} & \\ \text{subject to} & A\mathbf{x} & = \mathbf{b}, \\ & \mathbf{x} & \geq \mathbf{0}, \end{array} \quad \text{as}$$

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Then apply Multi-Block ADMM to the new formulation (He and Yuan 15, Sun/Y 15).

Implementation on Distributed Platforms II

In each round of ADMM, for $i = 1, ..., m$, for a simple quadratic objective one independently solves

$$\begin{array}{lll} \text{minimize}_{\mathbf{x}_i} & q_i(\mathbf{x}_i) & \\ \text{subject to} & \mathbf{a}_i\mathbf{x} & = b_i; \end{array}$$

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- Linear Programming Research Continues ...