

Online Equilibrium Pricing for Stochastic Fisher Markets

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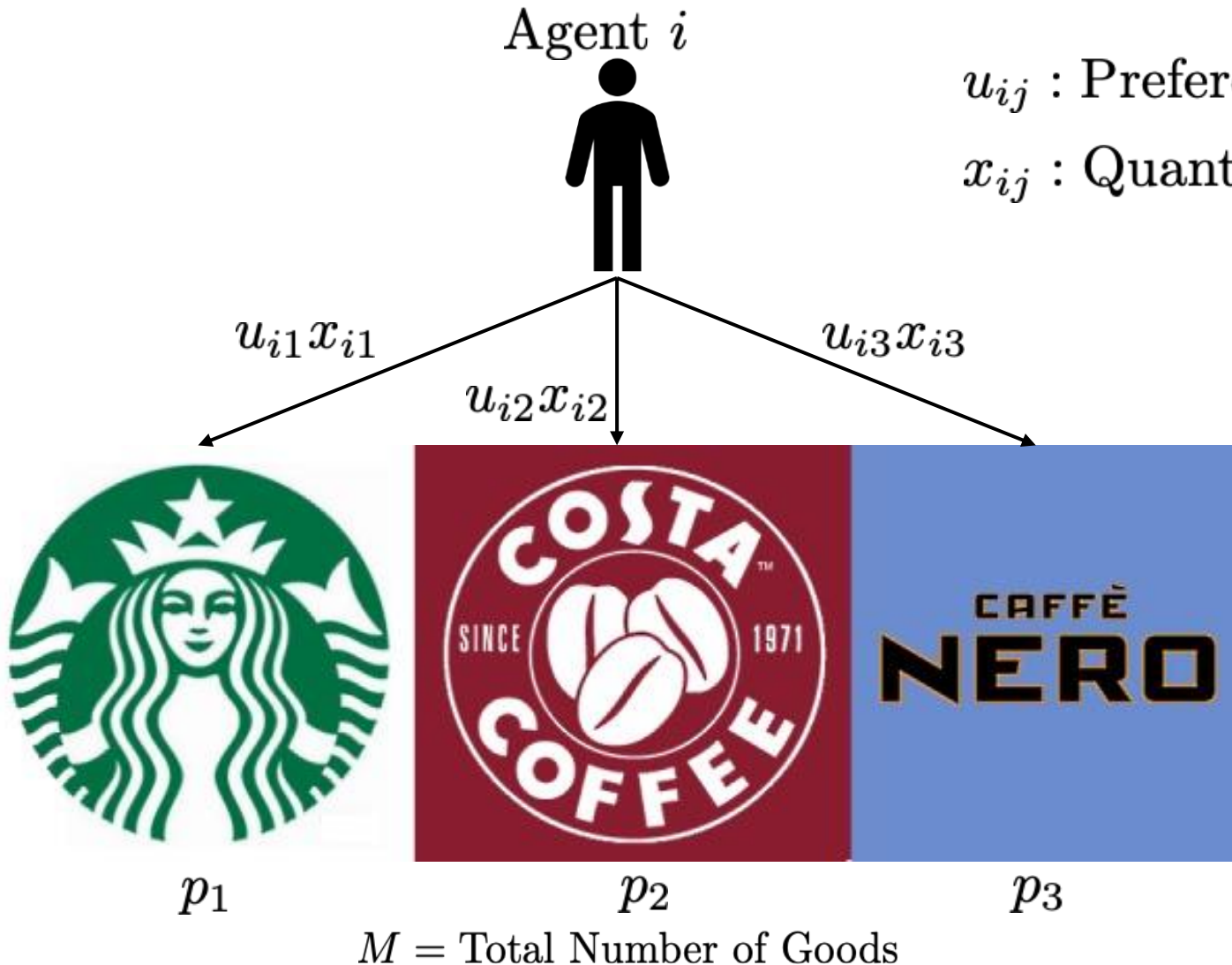
**Stanford University and CUHKSZ (Sabbatical Leave)
(Joint work with many)**

Organization

- **Introduction**

- Review of Online Linear Programming
- Performance Metrics and Limitations of Static Pricing for Stochastic Markets
- Adaptive Sample-Based Pricing Algorithm and Privacy-Preserving Algorithm
- Conclusion

One of the key resource allocation frameworks to achieve a fair allocation is that of Fisher Markets



u_{ij} : Preference of Agent i for one unit of good j

x_{ij} : Quantity of good j purchased by person i

p_j : Price of Good j

w_i : Budget of Agent i

Individual Optimization Problem:

$$\max_{\mathbf{x}_i} \sum_j u_{ij}x_{ij}$$

$$\text{s.t. } \mathbf{p}^T \mathbf{x}_i \leq w_i$$

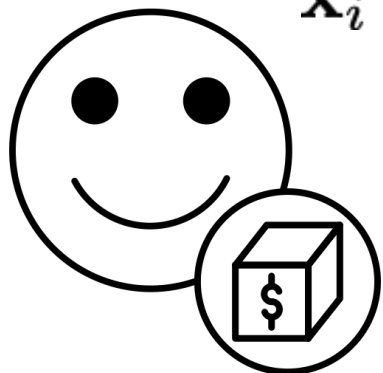
$$\mathbf{x}_i \geq \mathbf{0}$$

Prices are posted and known to all agents so that they have freedom to choose

The prices can be derived from a centralized optimization problem with a budget weighted social-welfare objective (Eisenberg-Gale)

Individual Optimization Problem:

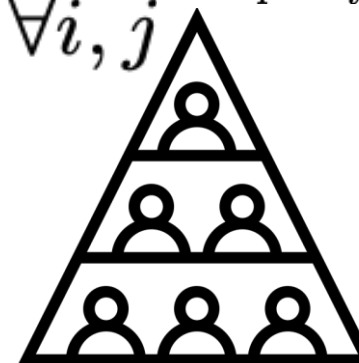
$$\begin{aligned} \max_{\mathbf{x}_i} \quad & \sum_j u_{ij} x_{ij} \\ \text{s.t.} \quad & \mathbf{p}^T \mathbf{x}_i \leq w_i \\ & \mathbf{x}_i \geq \mathbf{0} \end{aligned}$$



Social Optimization Problem:

$$\begin{aligned} \max_{\mathbf{x}_i, \forall i \in [N]} \quad & \sum_i w_i \log \left(\sum_j u_{ij} x_{ij} \right) \\ \text{s.t.} \quad & \sum_i x_{ij} \leq c_j, \forall j \in [M] \\ & x_{ij} \geq 0, \forall i, j \end{aligned}$$

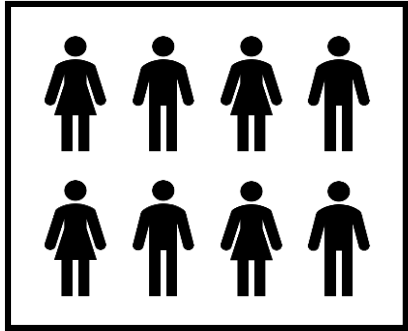
Capacity Constraints



p_j : Price of Good j = Dual Variable of Constraint j

Theorem: The social problem can be solved in polynomial time (Jain, Vazirani, Ye 2005; Jalota, Qi et al. GEB 2023)

We study an online and incomplete information variant of Fisher markets



Buyers arrive sequentially with utility and budget parameters drawn i.i.d. from a distribution



Establish performance limits of static pricing algorithms, including one that sets expected equilibrium prices

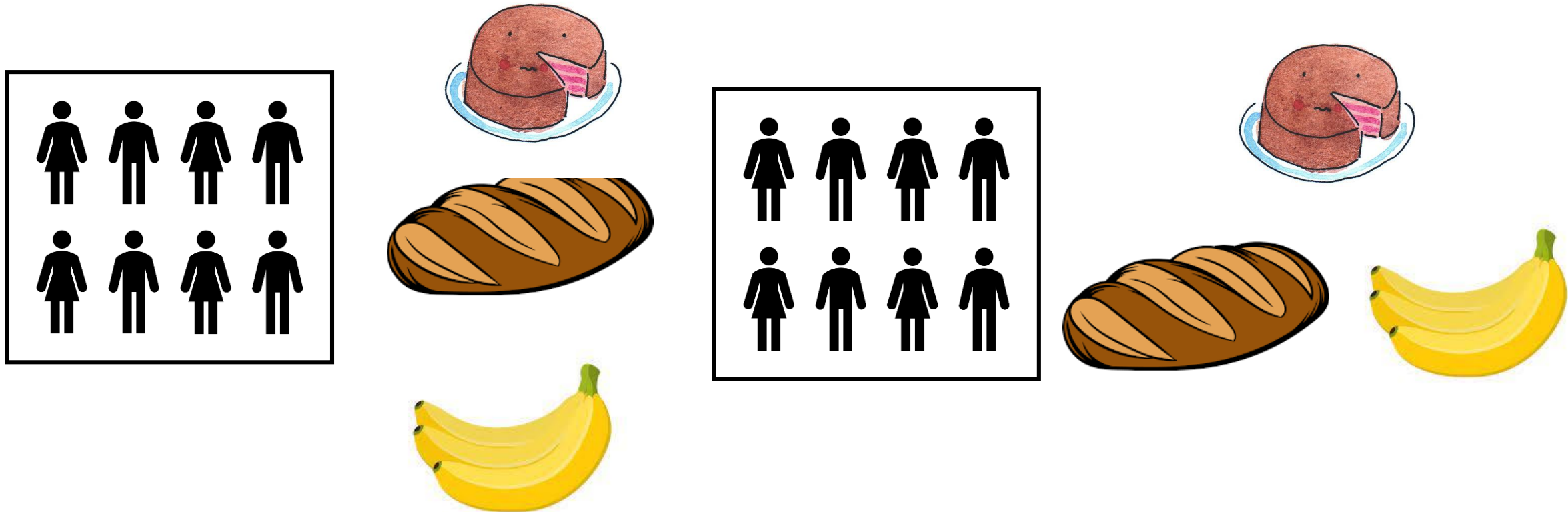


Develop an adaptive expected equilibrium pricing approach with strong performance guarantees



Develop a revealed preference-revealing and privacy-preserving algorithm with sub-linear regret and capacity violation

Prior work on online variants of Fisher markets have considered the setting of goods arriving sequentially



Prior Work: Goods Arrive Online
[Gorokh, Banerjee, Iyer, 2021]

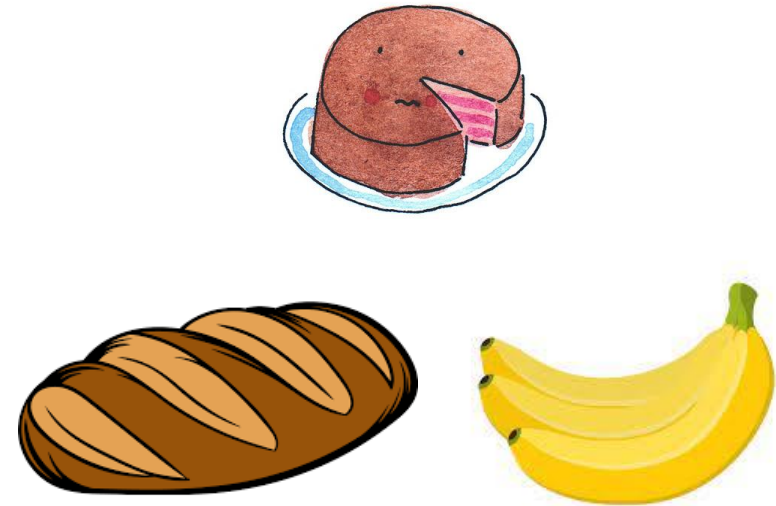
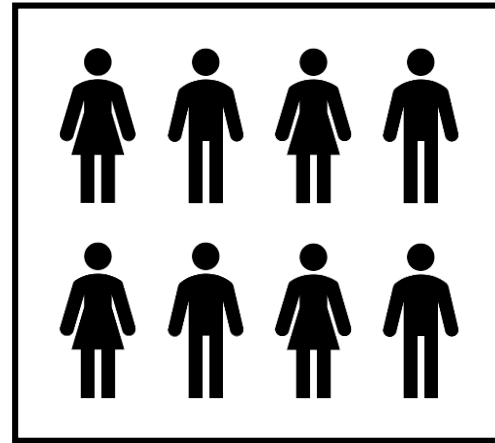
This Work: Agents Arrive Online

The setting of agents arriving online has been studied in online linear programming (OLP)



$$\text{Utility} = \sum_{j=1}^m u_{tj} x_{tj}$$

Objective: Maximize $\sum_{t=1}^n \sum_{j=1}^m u_{tj} x_{tj}$
Subject to resource constraints



Performance of online algorithm measured with respect to regret on offline linear objective

[Mehta et al. 2007], [Agrawal et al. 2010, 2014], [Kesselheim et al 2014]

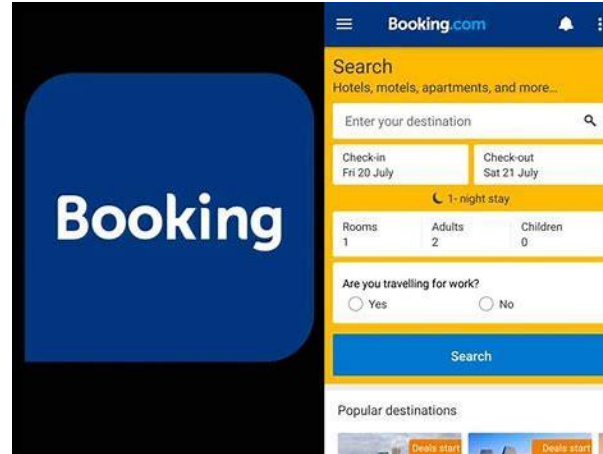
[Li/Ye, 2019], [Li et al. 2020],

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Online Resource Allocation & Revenue Management

- m type of resources; T customers
- Decision maker needs to decide whether and how much resources are allocated to each customer
- Resources are limited!
- **Online setting:**
 - Customers arrive sequentially and the decision needs to be made instantly upon the customer arrival: **Sell or No-sell?**



$$\begin{aligned} \max \quad & \sum_{t=1}^T r_t x_t \\ \text{s.t.} \quad & \sum_{t=1}^T a_{it} x_t \leq b_i, \quad i = 1, \dots, m \\ & 0 \leq x_t \leq 1 \quad \text{or} \quad x_t \in \{0, 1\}, \quad t = 1, \dots, T \end{aligned}$$

Performance of online algorithm measured with respect to regret from the offline linear objective

[Agrawal et al. 2010, 2014], [Kesselheim et al 2014]

[Li/Ye, 2019], [Li et al. 2020],

Auction Markets: An Illustration Example

Bid #	\$100	\$30	Inventory	
Decision	X1=?	X2=?					
Pants	1	0	100	
Shoes	1	0				50	
T-Shirts	0	1				500	
Jackets	0	0				200	
Hats	1	1	1000	

- Agents/Traders come one by one **sequentially**, **buy or sell**, or combination, with a combinatorial order/bid (\mathbf{a}_t, π_t)
- The seller/market-maker has to make an order-fill decision **as soon as an order arrives**
- Optimal Policy/Mechanism?

Regret-Ratio for Online Algorithm/Mechanism vs Offline

$$\begin{aligned} \text{OPT}(A, \pi) = \max \quad & \sum_k \pi_k x_k \\ \text{s.t.} \quad & \sum_k a_{ik} x_k \leq b_i \quad \forall i \in S \\ & 0 \leq x_k \leq 1 \quad \forall k \in N \end{aligned}$$

- We know the total number of customers, say n ;
- Assume customers arrive in a **random order or with i.i.d distributions.**
- For a given online algorithm/decision-policy/mechanism

$$Z(A, \pi) = E_{\sigma} \left[\sum_1^n \pi_k x_k \right]$$

$$R(A, \pi) = 1 - \frac{Z(A, \pi)}{\text{OPT}(A, \pi)}$$

$$R = \sup_{(A, \pi)} R(A, \pi)$$

Impossibility Result on Regret-Ratio

Theorem: There is no online algorithm/decision-policy/mechanism such that

$$R \leq O\left(\sqrt{\log(m)/B} \right), \quad B = \min_i b_i.$$

Corollary: If $B \leq \log(m)/\varepsilon^2$, then it is impossible to have a decision policy/mechanism such that $R \leq O(\varepsilon)$.

Agrawal, Wang and Y, "A Dynamic Near-Optimal Algorithm for Online Linear Programming," 2010.

Possibility Result on Regret-Ratio

Theorem: There is an online algorithm/decision-policy/mechanism such that

$$R \leq O\left(\sqrt{m \log(n)/B} \right), \quad B = \min_i b_i.$$

Corollary: If $B > m \log(n)/\varepsilon^2$, then there is an online algorithm/decision-policy/mechanism such that $R \leq O(\varepsilon)$.

Agrawal, Wang and Y, "A Dynamic Near-Optimal Algorithm for Online Linear Programming," 2010.

Theorem: If $B > \log(mn)/\varepsilon^2$, then there is an online algorithm/decision-policy/mechanism such that $R \leq O(\varepsilon)$.

Kesselheim et al. "Primal Beat the Dual...", 2014, ...

Online Algorithm and Price-Mechanism

- Learn “ideal” itemized-prices
- Use the prices to price each bid
- Accept if it is an over bid, and reject otherwise

Bid #	\$100	\$30	Inventory	Price?
Decision	x1	x2					
Pants	1	0	100	45
Shoes	1	0				50	45
T-Shirts	0	1				500	10
Jackets	0	0				200	55
Hats	1	1	1000	15

Such ideal prices exist and they are shadow/dual prices of the offline LP

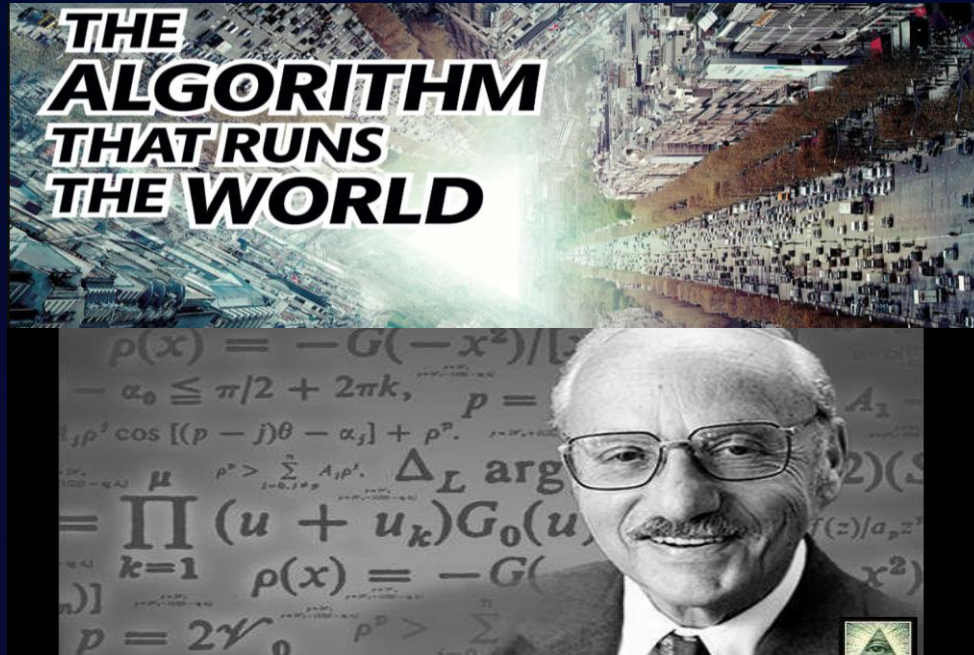
How to Learn “Shadow Prices” Online

At time t , solve the sample LP at $t=\varepsilon n, 2\varepsilon n, 4\varepsilon n, \dots$; and use the new shadow prices for the decision in the coming period (**the resource allocations can be adaptively adjusted**).



$$\begin{aligned} \max \quad & \sum_{k=1}^t \pi_k x_k \\ \text{s.t.} \quad & \sum_{k=1}^t a_{ik} x_k \leq (1 - h_t) \frac{t}{n} b_i \quad \forall i \in S \\ & 0 \leq x_k \leq 1 \quad \forall k \in N \end{aligned}$$

Differences of Math/OR and AI Models



- Based on Science/Logic
- Physical/Economical Principles
 - Objective
 - Explainable Insights
- Online Training&Decision-Making

- Based on Cases/Experience
- Observation/Behavior
 - Subjective
 - Black-box
- Offline Training

$$\max \sum \pi_j x_j \quad \text{s.t.} \quad \sum_j \mathbf{a}_j x_j \leq \mathbf{b}, \quad x_j \geq 0 \quad \forall j = 1, \dots, J$$

- The decision variable x_j represents the **total-times of pulling** the j -th arm.
- We have developed a two-phase algorithm
 - **Phase I**: Distinguish the optimal **super-basic** variables/arms from the optimal **non-basic** variables/arms with as fewer number of plays as possible
 - **Phase II**: Use the arms in the optimal face to exhaust the resource through an adaptive procedure and achieve **fairness**
- The algorithm achieves a problem dependent regret that bears a **logarithmic** dependence on the horizon T . Also, it identifies a number of LP-related parameters as the **bottleneck or condition-numbers** for the problem
 - Minimum non-zero **reduced cost**
 - Minimum **singular-values** of the optimal basis matrix.
- **First algorithm** to achieve the $O(\log T)$ regret/gap bound [Li, Sun & Y 2021 ICML] (<https://proceedings.mlr.press/v139/li21s.html>)

阿里巴巴在2019年云栖大会上提到在智能履行决策上使用OLP的算法

2018 杭州·云栖大会 Alibaba Group

智能履行决策

商家

杭州-上海 杭州-广州 杭州-北京 杭州-武汉 ...

YTO ZTO YUNDA

→

商家

菜鸟智能发货引擎

时效	服务	成本	单量平衡	...
线路容量	网点容量	局部优化	全局优化	...

最优快递

智能决策
ML & Optimization

$C_{ij} = c1 * \text{成本} + c2 * \text{服务} + c3 * \text{时效}$

决策变量

max_x $\sum_{i=1}^n \sum_{j=1}^m C_{i,j} x_{i,j}$ 将订单 I 匹配给快递公司 j 与否

s.t. $\sum_{j=1}^m x_{i,j} \leq 1$

$\sum_{i=1}^n x_{i,j} * a_j \leq u_j$ 商家发货CP总单量比例约束

$\sum_{i=1}^n \sum_{j=1}^m x_{i,j} b_{k,i,j} \leq B_k$ 全局约束值, 比如总成本

订单的履行是带有全局约束的序列执行决策

- Online assignment problem
- Control based method
- Online linear programming

Ref: Agrawal, Shipra, Zizhuo Wang, and Yinyu Ye. "A dynamic near-optimal algorithm for online linear programming." *Operations Research* 62.4 (2014): 876-890.

阿里巴巴团队在2020年CIKM会议论文Online Electronic Coupon Allocation based on Real-Time User Intent Detection上提到他们设计的发红包的机制也使用了OLP的方法[2]

Spending Money Wisely: Online Electronic Coupon Allocation based on Real-Time User Intent Detection

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$$\begin{aligned} & \max \sum_{i=1}^M \sum_{j=1}^N v_{ij} x_{ij} \\ & \text{s.t. } \sum_{i=1}^M \sum_{j=1}^N c_j x_{ij} \leq B, \\ & \sum_j x_{ij} \leq 1, \quad \forall i \\ & x_{ij} \geq 0, \quad \forall i, j \end{aligned} \quad (5)$$

3.3 MCKP-Allocation

We adopt the primal-dual framework proposed by [2] to solve the problem defined in Equation 5. Let α and β_j be the associated dual variables respectively. After obtaining the dual variables, we can solve the problem in an online fashion. Precisely, according to the principle of the primal-dual framework, we have the following allocation rule:

$$x_{ij} = \begin{cases} 1, & \text{where } j = \arg \max_i (v_{ij} - \alpha c_j) \\ 0, & \text{otherwise} \end{cases} \quad (9)$$

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Online for Geometric Objective: evaluate algorithms through the absolute regret of social welfare and capacity violation

Regret (Optimality Gap)

Difference in the Optimal Social Objective of the online policy π to that of the optimal offline social value

$$R_n(\pi) =$$

$$\sum_i w_i \log \left(\sum_j u_{ij} x_{ij}^* \right) - \sum_i w_i \log \left(\sum_j u_{ij} x_{ij}(\pi) \right)$$

Optimal Offline Objective

Objective of online policy

Prior Work on concave objectives [Lu, Balserio, Mirrkoni, 2020] assume non-negativity and boundedness of utilities, none of which are true for the log objective

Constraint Violation

Norm of the violation of capacity constraints of the online policy π

$$V_j(\pi) = \sum_j x_{ij}(\pi) - c_j$$

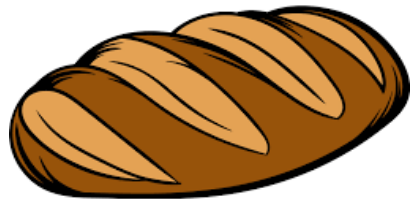
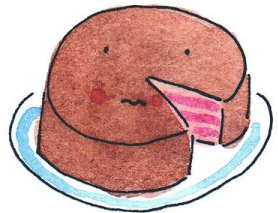
Violation of Capacity Constraint of good j

$$V_n(\pi) = \|\mathbb{E}[V(\pi)^+]\|_2$$

Norm of the expected constraint violation

Using the optimal expected prices, the capacity violation must be $\Omega(\sqrt{n})$, where n is the number of total agents

2 goods, each with a capacity of n



Two agent types specified by (Utility for Good 1, Utility for Good 2)

Type I: (1, 0)



Arrival Probability = 0.5

Type II: (0, 1)



Arrival Probability = 0.5

Expected Optimal Objective $\approx n \log(2)$

Since Type I users receive two units of good one, while type two receive two units of good two

While $\frac{n}{2}$ users of Type I arrive in expectation, the realized arrivals of type I users deviates by $O(\sqrt{n})$

Theorem: More generally, any static pricing algorithm achieves either a regret or capacity violation of $\Omega(\sqrt{n})$

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- **Adaptive Sample-Based Pricing Algorithm and Privacy-Preserving Algorithm**
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Our adaptive expected equilibrium pricing approach achieves constant constraint violation and log regret

Algorithm 1: Adaptive Expected Equilibrium Pricing

Input : Initial Good Capacities \mathbf{c} , Number of Users n , Threshold Parameter Vector Δ , Support of Probability Distribution $\{\tilde{w}_k, \tilde{\mathbf{u}}_k\}_{k=1}^K$, Occurrence Probabilities $\{q_k\}_{k=1}^K$

Initialize $\mathbf{c}_1 = \mathbf{c}$ and the average remaining good capacity to $\mathbf{d}_1 = \frac{\mathbf{c}}{n}$;

for $t = 1, 2, \dots, n$ **do**

Phase I: Set Price

if $\mathbf{d}_{t'} \in [\mathbf{d} - \Delta, \mathbf{d} + \Delta]$ **for all** $t' \leq t$ **then**

 Set price \mathbf{p}^t as the dual variables of the capacity constraints of the certainty equivalent problem $CE(\mathbf{d}_t)$ with capacity \mathbf{d}_t ;

else

 Set price \mathbf{p}^t using the dual variables of the capacity constraints of the certainty equivalent problem $CE(\mathbf{d})$ with capacity $\mathbf{d} = \mathbf{d}_1$;

end

Phase II: Observed User Consumption and Update Available Good Capacities

 User purchases optimal bundle of goods \mathbf{x}_t given price \mathbf{p}^t ;

 Update the available good capacities $\mathbf{c}_{t+1} = \mathbf{c}_t - \mathbf{x}_t$;

 Compute the average remaining good capacities $\mathbf{d}_{t+1} = \frac{\mathbf{c}_{t+1}}{n-t}$;

end

} Set price based on dual variable of capacity constraints of certainty equivalent problem as the same as the OLP

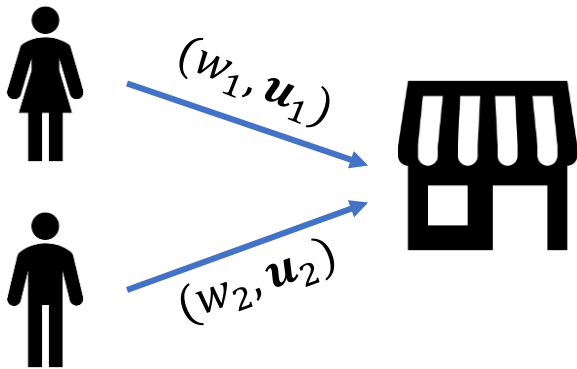
} Users consume optimal bundle of goods

} Update average remaining resource capacities

Theorem: Under i.i.d. budget and utility parameters with a discrete probability distribution and when good capacities are $O(n)$, Algorithm 1 achieves an expected regret of $R_n(\boldsymbol{\pi}) \leq O(\log(n))$ and expected constraint violation of $V_n(\boldsymbol{\pi}) \leq O(1)$

The sample-based algorithms are often computationally expensive and do not preserve user privacy

User parameters (w, \mathbf{u}) are revealed



Such algorithms require information on user parameters, which may not be known in practice

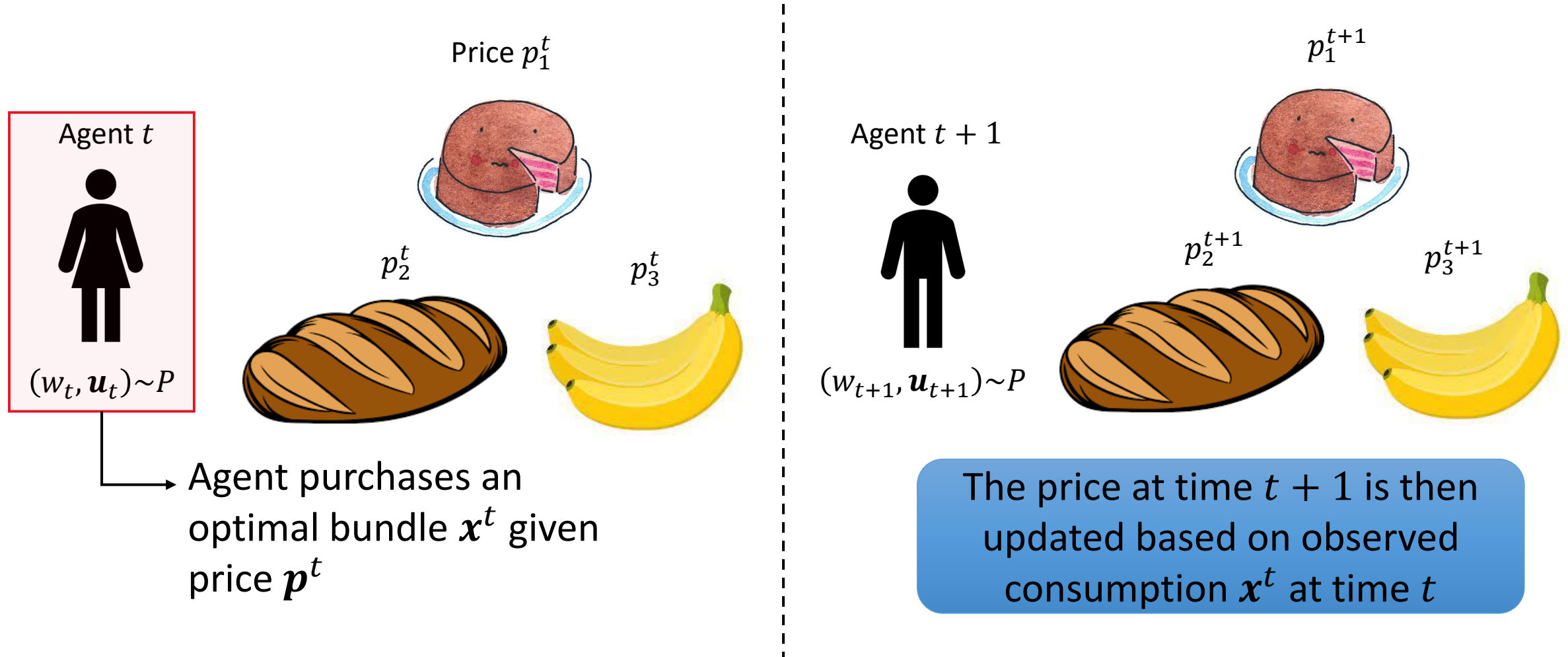
With parameters until user t arrives, we can solve the following primal problem

$$\begin{aligned} \max_{\mathbf{x}_i \in \mathbb{R}^m, \forall i \in [t]} \quad & \sum_{i=1}^t w_i \log \left(\sum_{j=1}^m u_{ij} x_{ij} \right) \\ \text{s.t.} \quad & \sum_{i=1}^t x_{ij} \leq \frac{t}{n} c_j, \quad \forall j \in [m] \\ & x_{ij} \geq 0, \quad \forall i \in [t], j \in [m] \end{aligned}$$

Prices can be set based on dual of capacity constraints

At each time instance, we solve a larger convex program, which may become computationally expensive in real time

We design a dual based algorithm, wherein users see prices at each time they arrive



Applying gradient descent to the dual of the social optimization problem motivates a natural algorithm

Dual of social optimization problem with Lagrange multiplier of the capacity constraints p_j

$$\min_{\mathbf{p}} \sum_{t=1}^n w_t \log(w_t) - \sum_{t=1}^n w_t \log\left(\min_{j \in [m]} \frac{p_j}{u_{tj}}\right) + \sum_{j=1}^m p_j c_j - \sum_{t=1}^n w_t$$

Equivalent Sample Average Approximation (SAA) of Dual Problem

$$\min_{\mathbf{p}} D_n(\mathbf{p}) = \sum_{j=1}^m p_j \frac{c_j}{n} + \frac{1}{n} \sum_{t=1}^n \left(w_t \log(w_t) - w_t \log\left(\min_{j \in [m]} \frac{p_j}{u_{tj}}\right) - w_t \right)$$

(Sub)-gradient descent of dual problem for each agent: $O(m)$ complexity of price update

$$\partial_{\mathbf{p}} \left(\sum_{j \in [m]} p_j \frac{c_j}{n} + w \log(w) - w \log\left(\min_{j \in [m]} \frac{p_j}{u_j}\right) - w \right) \Big|_{\mathbf{p}=\mathbf{p}^t} = \frac{1}{n} \mathbf{c} - \mathbf{x}_t$$

Difference between market share of each agent and goods purchased

We develop a revealed preference algorithm with sub-linear regret and constraint violation guarantees

Algorithm 2: Revealed Preference Algorithm for Online Fisher Markets

Input : Number of users n , Vector of good capacities per user $\mathbf{d} = \frac{\mathbf{c}}{n}$

Initialize $\mathbf{p}^1 > \mathbf{0}$;

for $t = 1, 2, \dots, n$ **do**

Phase I: ;

 User purchases an optimal bundle of goods \mathbf{x}_t given the price \mathbf{p}^t ;

Phase II (Price Update): ;

$\mathbf{p}^{t+1} \leftarrow \mathbf{p}^t - \gamma_t (\mathbf{d} - \mathbf{x}_t)$;

Difference between market share of
each agent and goods purchased

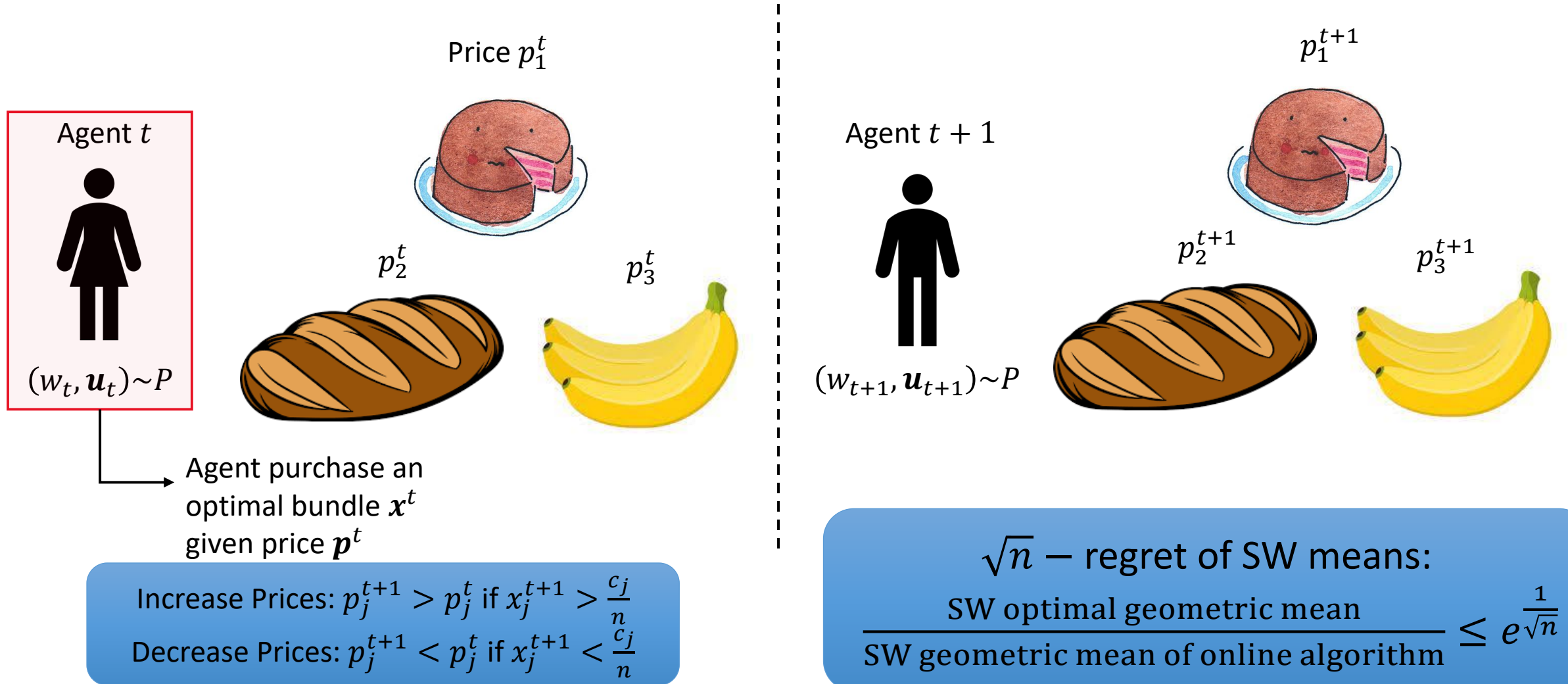
end

Step-size: $O\left(\frac{1}{\sqrt{n}}\right)$

Only requires knowledge of user consumption
(and not their budgets or utilities) to update prices

Theorem: Under i.i.d. budget and utility parameters with strictly positive support and when good capacities are $O(n)$, Algorithm 2 achieves an expected regret of $R_n(\boldsymbol{\pi}) \leq O(\sqrt{n})$ and expected constraint violation of $V_n(\boldsymbol{\pi}) \leq O(\sqrt{n})$, where n is the number of arriving users.

Again, the price of a good is increased if the arriving user purchase more than its market share of the good

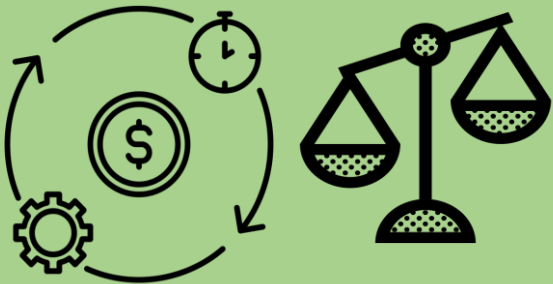


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- **Conclusion/Takeaways**

We study Fisher markets in the online and incomplete information setting and develop algorithms with sub-linear regret guarantees

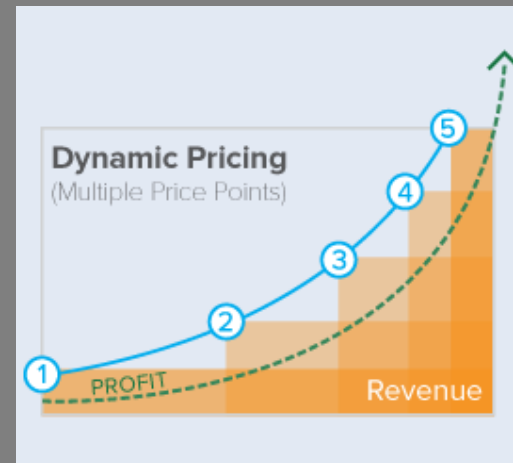
The weighted geometric average objective has both efficiency and fairness properties



Static equilibrium pricing approaches have performance limitations



We develop an adaptive expected equilibrium pricing algorithm with much improved performance

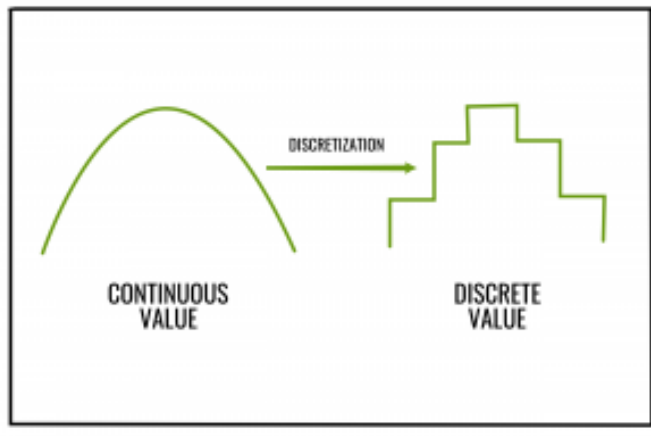


We develop a revealed preference algorithm with sub-linear regret and capacity violation

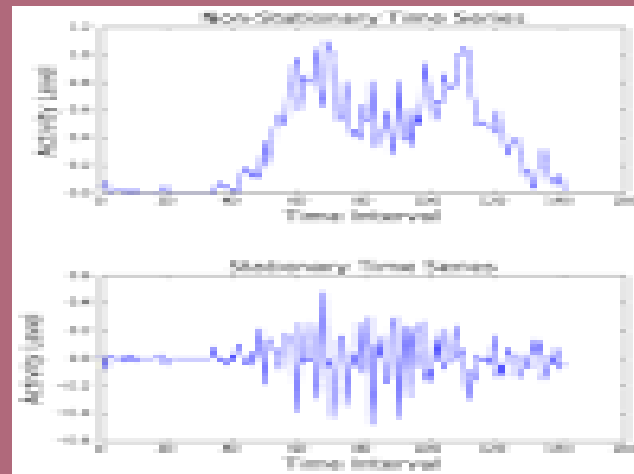


Future Work

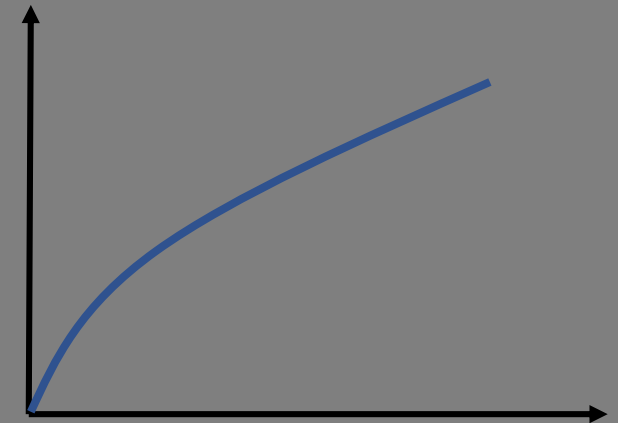
Loss in social objective under integral allocations



Extensions for Nonstationary Agent Data in the Market



Extension of online Fisher markets under general concave utility functions



Thank you