Newsvendor Optimization with Limited Distribution Information*

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Abstract

We report preliminary results on stochastic optimization with limited distributional information. Lack of complete distribution calls for stochastically robust models that, after exploiting available limited or partial information, offer risk-shielded solutions, i.e., solutions that are insensitive to all possible distributions of random variables. We focus on the well-known newsvendor problem in this study, where the distribution of the random demand is only specified by its mean and one of the following: its standard deviation or its support. We propose a stochastically robust model for the newsvendor problem. More specifically, our model tries to minimize the regret that is defined as the ratio of the expected cost based on limited information to that based on complete information, called Relative Expected Value of Distribution (REVD). We show how to derive an optimal solution to the REVD model. Numerical examples are provided to compare our model with other similar approaches. The goal is to establish a confidence ratio that the decision from our model is not worse, relatively, too much than the decision based on the true distribution which would be never known exactly in real world applications.

1 Introduction

We consider the newsvendor problem that can be described as follows. A newsvendor sells certain product whose demand is unknown but follows cer-

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tain probability distribution. The newsvendor needs to place an order of q units before observing the demand. There are three types of costs: the unit ordering cost c, the unit inventory holding cost p, and the unit stock-out cost t. The objective is to decide the optimal order quantity q so that the expected total cost is minimized. The problem can be formulated as

$$\min_{q} G_f(q) \equiv cq + tE_F(z-q)^+ + pE_F(q-z)^+, \tag{1}$$

where z is a random demand with a probability distribution f (pdf) and a cumulative distribution F(cdf). We assume w.o.l.g. that $c \leq t$, otherwise there is a trivial optimal ordering $q^* = 0$ for arbitrary distribution, since $G_f'(q) \geq 0$ for every $q > 0$. We also assume that $c > 0$.

When f is known, the optimal ordering quantity, denoted by q_f^*, can be easily obtained as it satisfies

$$\int_{q_f^*}^{\infty} dF(x) = \frac{p+c}{p+t}.$$

However, in many practical situation, the distribution f is not necessarily known. On the other hand, it is much easier to estimate points of the mean, the standard deviation, and the support of z based on historical data. Scarf [4] (see also Vairaktarakis [7] for more general settings) proposed a min-max approach to the newsvendor problem when only the mean μ and standard deviation σ of the demand are known. More specifically, he solves the following min-max problem

$$\min_{q\geq 0} \max_{f\in H(\mu,\sigma)} G_f(q),$$

where $H(\mu, \sigma)$ is the set of distributions with mean of μ and standard deviation of σ. The distribution-free solution Scarf introduced is

$$q_u^* \equiv \mu + \frac{\sigma}{2}\left(\sqrt{\frac{t-c}{p+c}} - \sqrt{\frac{p+c}{t-c}}\right),$$

where the solution q_u^* is often referred to as the "Scarf's rule". Scarf's results have been extended to more general setting, see, for example, Gallego and Moon [2].

In a Boeing Stochastic&Robust Optimization project report [1], for a class of stochastic optimization problems the authors established a relative regret bound on a solution optimal to a deterministic counterpart, that is, simply adapting the mean values of random variables in decision making. The tightness of the bound depends on the closeness in cost function gradients and may be efficiently checked when the deterministic solution is found. The goal of the study can be also used to measure the benefits of distribution information.

Scarf's approach is simple and offers a robust solution to the newsvendor model. However, as the model tries to minimizes the worst case cost, the solution obtained is quite pessimistic. Therefore, recently new robust models are proposed to the newsvendor problem by Yue, Chen, and Wang [5] and by Perakis and Roels [3]. Rather than minimizing the worst case cost (over all possible distribution), the approach proposed by [5, 3] minimizes a regret of the newsvendor, which is defined as follows. Assume that q is the order quantity where the decision is made without knowing the true distribution. Assume the distribution is f, then the cost associated with the decision q is $G_f(q)$, while the optimal cost is $G_f(q_f^*)$ by solving (1). The regret, also called the Absolute Expected Value of Distribution (AEVD), is then defined by the difference of $G_f(q)$ and $G_f(q_f^*)$, i.e.,

$$AEVD_f(q) \equiv G_f(q) - G_f(q_f^*).$$

Therefore, one would like to choose an optimal q such that

$$\max_{f \in H(I)} AEVD_f(q)$$

is minimized, where $H(I)$ is the set of all possible distributions.

In [5], $H(I)$ is chosen as $H(\mu, \sigma)$—all distributions with the given mean and standard deviation; and in [3], $H(I)$ is chosen as $H(\mu, [A, B])$—all distributions with the given mean and support $[A, B]$ of the random demand. Closed form solutions for these two cases are provided in both [5] and [3] while other kinds of partial distribution information (i.e. symmetry, unimodality) are also investigated in [3]. (The closed form solution for $H(\mu, [A, B])$ derived in [3] was incomplete, so that we shall give complete results and proofs in Appendix (6.5).)

It can be expected that the min-max regret approach is less conservative than Scarf's approach. However, the optimal objective value obtained in the AEVD model must be interpreted in context. For instance, for two distributions, one with a regret of \$100 and another one of \$200, which might be meaningless if \$100 is 50% of the corresponding minimal cost and \$200 is only 10% of the corresponding minimal cost.

Therefore, we propose a new robust model where the regret is defined as the ratio of $G_f(q)$ and $G_f(q_f^*)$, called Relative Expected Value of Distribution (REVD), i.e.,

$$REVD_f(q) \equiv \frac{G_f(q)}{G_f(q_f^*)}.$$

We show how to compute

$$\max_{f \in H(I)} REVD_f(q) \tag{2}$$

when $H(I)$ is $H(\mu, \sigma)$ or $H(\mu, [A, B])$. The analysis is slightly more complicated than that in the AEVD model because a fractional objective function is involved here.

Now, it is straightforward to interpret the objective value of our REVD model. For example, if the optimal $REVD$ is 1.1, it means that the chosen decision is at most 10% above the expected minimal cost when one knows the complete distribution of the demand. Furthermore, we develop techniques to find the optimal q such that

$$\max_{f \in H(I)} REVD_f(q)$$

is minimized.

The above three approaches were suggested under general settings of robust optimization in [6] by Kouvelis and Yu, corresponding to three different robustness criteria: Absolute Robustness, Robust Deviation and Relative Robustness. In a specific decision, some or all of the robustness criteria might be applied. The Scarf's rule (absolute robust criterion) tends to lead to decisions that are very conservative in nature and the main concern is how to hedge against the worst possible happening. The AEVD (robust deviation) and REVD (relative robustness) tend to be less conservative in their decision selection, and more in tune with a logic that attempts to exploit opportunities for improvement, in other words, minimize the regret. And, the AEVD model is to minimize the absolute regret, while the REVD model is to minimize the relative regret. For most of the cases, the AEVD and REVD model will tend to favor similar decisions, but the REVD should be used in the situations that either the performance of the optimal AEVD ordering is highly variable (i.e. can be very small or large); or the demand distribution fluctuates in a wide range (i.e. the standard deviation of the demand is very large). These situations will be illustrated by numerical simulations in Section 4.

We are also investigating the multiple-item and capacitated inventory problem based on the same models, where similar results are obtained and will be reported in a new report.

2 Min-Max REVD with mean and support

In this section, we compute (2) with $H(I) = H(\mu, [A, B])$ by using the duality of linear fractional programming. In particular, consider a primal problem

$$\begin{array}{lll} (P) & \max & \frac{\xi^T x}{d^T x} \\ & s.t. & Ax = b, \\ & & x \geq 0. \end{array}$$

The primal problem can be reformulated as a linear program by applying the following variable transformation: $t = \frac{1}{d^T x}$, $z = tx$:

$$\begin{array}{lll} (P') & \max & \xi^T z \\ & s.t. & Az - bt = 0, \\ & & d^T z = 1, \\ & & z \geq 0, \ t \geq 0. \end{array}$$

Therefore, the dual problem is

$$\begin{array}{lll} (D) & \min & \lambda \\ & s.t. & A^T y + d\lambda \geq \xi, \\ & & -b^T y \geq 0, \\ & & y, \lambda \text{ free.} \end{array}$$

Now, apply the above result to the min-max REVD:

$$\begin{array}{ll} & \min_q \ \max_{f \in H(\mu,[A,B])} \ REVD_f(q) \\ = & \min_q \ \max_{f \in H(\mu,[A,B])} \ \frac{G_f(q)}{G_f(q_f^*)} \\ = & \min_q \ \max_{f \in H(\mu,[A,B])} \ \frac{G_f(q)}{\min_{\hat{q}} \ G_f(\hat{q})} \\ = & \min_q \ \max_{f \in H(\mu,[A,B])} \ \max_{\hat{q}} \ \frac{G_f(q)}{G_f(\hat{q})} \\ = & \min_q \ \max_{\hat{q}} \ \max_{f \in H(\mu,[A,B])} \ \frac{G_f(q)}{G_f(\hat{q})} \\ = & \min_{A \leq q \leq B} \ \max_{A \leq \hat{q} \leq B} \ \max_{f \in H(\mu,[A,B])} \ \frac{G_f(q)}{G_f(\hat{q})}. \end{array}$$

Let's consider the inner optimization problem

$$h(q, \hat{q}) := \max_{f \in H(\mu,[A,B])} \frac{G_f(q)}{G_f(\hat{q})}, \ A \leq q \leq B \text{ and } A \leq \hat{q} \leq B,$$

which can be formulated as

$$\begin{array}{ll} \max & \frac{G_f(q)}{G_f(\hat{q})} \\ s.t. & \int_A^B dF(x) = 1, \\ & \int_A^B x dF(x) = \mu, \end{array}$$

where $F(x)$ is the cumulative distribution function of f.

Rewrite $G_f(q)$ and $G_f(\hat{q})$ as

$$\begin{array}{rcl} G_f(q) & = & \int_A^B cq \ dF(x) + \int_A^q p(q-x) dF(x) + \int_q^B t(x-q) dF(x) \\ & = & \int_A^q [p(q-x) + cq] dF(x) + \int_q^B [t(x-q) + cq] dF(x), \end{array}$$

and

$$G_f(\hat{q}) \ = \ \int_A^{\hat{q}} [p(\hat{q}-x) + c\hat{q}] dF(x) + \int_{\hat{q}}^B [t(x-\hat{q}) + c\hat{q}] dF(x).$$

Then, we can view it as an infinite fractional LP with

$$\xi^T = (p(q-x)+cq|_{x\le q},\ t(x-q)+cq|_{x\ge q}),$$

$$d^T = (p(\hat q-x)+c\hat q|_{x\le \hat q},\ t(x-\hat q)+c\hat q|_{x\ge \hat q}),$$

and the variable vector $f(x)|_{A\le x\le B}$, where $f(x)$ is the density of the distribution $F(x)$.

Hence, the dual of this infinite fractional LP problem is

$$\begin{array}{lll} \min & \lambda & \\ s.t. & y_0+xy_1+\lambda[p(\hat q-x)+c\hat q] \ \ge & [p(q-x)+cq]\cdot 1_{x\le q} \\ & & +[t(x-q)+cq]\cdot 1_{x\ge q}, \quad \text{for } A\le x\le \hat q \\ & y_0+xy_1+\lambda[t(x-\hat q)+c\hat q] \ \ge & [p(q-x)+cq]\cdot 1_{x\le q} \\ & & +[t(x-q)+cq]\cdot 1_{x\ge q}, \quad \text{for } \hat q\le x\le B \\ & y_0+\mu y_1\le 0. & \end{array}$$

It is equivalent to

$$\begin{array}{lll} \min & \lambda & \\ s.t. & -\mu y_1+xy_1+\lambda[p(\hat q-x)+c\hat q] \ \ge & [p(q-x)+cq]\cdot 1_{x\le q} \\ & & +[t(x-q)+cq]\cdot 1_{x\ge q}, \quad A\le x\le \hat q \\ & -\mu y_1+xy_1+\lambda[t(x-\hat q)+c\hat q] \ \ge & [p(q-x)+cq]\cdot 1_{x\le q} \\ & & +[t(x-q)+cq]\cdot 1_{x\ge q}, \quad \hat q\le x\le B. \end{array}$$

Now, from the dual problem we get the following results: (Note: Minimizing λ is equivalent to minimizing the function value of the left-hand-side at $x=\mu$.)

Lemma 1. For $(q,\hat q)\in[A,B]^2$, we have

$$h(q,\hat q)=\begin{cases} 1+\frac{(q-\hat q)(p+c)}{c\hat q+p\hat q-p\mu}, & \text{if } \hat q\ge\mu \text{ and } q\ge\hat q, \\ 1+\frac{\hat q-q}{\hat q-A}\frac{(t-c)(\mu-A)-(p+c)(\hat q-\mu)}{c\hat q+p\hat q-p\mu}, & \text{if } \hat q\ge\mu \text{ and } q\le\hat q, \\ 1+\frac{q-\hat q}{B-\hat q}\frac{(p+c)(B-\mu)-(t-c)(\mu-\hat q)}{c\hat q+t\mu-t\hat q}, & \text{if } \hat q\le\mu \text{ and } q\ge\hat q, \\ 1+\frac{(\hat q-q)(t-c)}{c\hat q+t\mu-t\hat q}, & \text{if } \hat q\le\mu \text{ and } q\le\hat q. \end{cases}$$

Notice that the result are symmetric under the following exchange: $p\leftrightarrow -t$, $t\leftrightarrow -p$ and $A\leftrightarrow B$.

Define $\text{REVD}_{max}(q)\equiv\max_{f\in H(\mu,[A,B])} REVD_f(q)$ as the largest REVD, and let q^e be the decision that minimizes it, that is

$$q^e = \arg\min \text{REVD}_{max}(q).$$

Then we have the following lemma:

Lemma 2.

$$REVD_{max}(q) = \begin{cases} h_+(q), & \text{if } q \le q^e, \\ h_-(q), & \text{if } q \ge q^e, \end{cases}$$

where

$$h_+(q) = \max_{\min\{B, \mu + \frac{t-c}{p+c}(\mu - A)\} \ge \hat{q} \ge \mu} \frac{\hat{q} - q}{\hat{q} - A} \frac{(t-c)(\mu - A) - (p+c)(\hat{q} - \mu)}{c\hat{q} + p\hat{q} - p\mu} + 1,$$

$$h_-(q) = \max_{\max\{A, \mu - \frac{p+c}{t-c}(B - \mu)\} \le \hat{q} \le \mu} \frac{q - \hat{q}}{B - \hat{q}} \frac{(p+c)(B - \mu) - (t-c)(\mu - \hat{q})}{c\hat{q} + t\mu - t\hat{q}} + 1.$$

We can further simplify it by introducing the following variable transformations:

$$\theta = \frac{q-\mu}{\mu},\ \alpha = \frac{t-c}{p},\ \beta = \frac{p+c}{p},\ a = \frac{\mu - A}{\mu},\ b = \frac{B-\mu}{\mu},\ x = \frac{\hat{q}-\mu}{\mu},\ y = \frac{\mu - \hat{q}}{\mu}.$$

Then, the above formula becomes

$$REVD_{max}(q) = \begin{cases} \max_{0 \le x \le \min\ (\frac{\alpha}{\beta} a, b)}\ g_+(\theta, x), & \text{if } q \le q^e, \\ \max_{0 \le y \le \min\ (a, \frac{\beta}{\alpha} b)}\ g_-(\theta, y), & \text{if } q \ge q^e, \end{cases}$$

where

$$g_+(\theta, x) = 1 + \frac{x - \theta}{x + a} \cdot \frac{\alpha a - \beta x}{\beta - 1 + \beta x},$$

$$g_-(\theta, y) = 1 + \frac{y + \theta}{y + b} \cdot \frac{\beta b - \alpha y}{\beta - 1 + \alpha y}.$$

Notice that $h_+(q)$ is a decreasing function while $h_-(q)$ is a increasing function, and $h_+(B) \le 1 \le h_-(B)$, $h_-(A) \le 1 \le h_+(A)$. It follows that $h_+(q^e) = h_-(q^e)$, which is equivalent to

$$\max_{0 \le x \le \min(\frac{\alpha}{\beta} a, b)} g_+(\theta, x) = \max_{0 \le y \le \min(a, \frac{\beta}{\alpha} b)} g_-(\theta, y).$$

Therefore, we can calculate q^e by solving the following optimization problem:

$$\max_{0 \le x \le \min(\frac{\alpha}{\beta} a, b), 0 \le y \le \min(a, \frac{\beta}{\alpha} b)} g_+(\theta, x) \quad s.t. \quad g_+(\theta, x) = g_-(\theta, y).$$

Eliminating θ from the constraint and then plug it into the objective function, we get the following optimization problem:

$$\max_{0 \le x \le \min(\frac{\alpha}{\beta} a, b), 0 \le y \le \min(a, \frac{\beta}{\alpha} b)} g_{a,b,\alpha,\beta}(x, y),$$

where

$$g_{a,b,\alpha,\beta}(x, y) = 1 + \frac{(x+y)(\alpha a - \beta x)(\beta b - \alpha y)}{(y+b)(\alpha a - \beta x)(\beta - 1 + \alpha y) + (x+a)(\beta b - \alpha y)(\beta - 1 + \beta x)}.$$

We now have

Theorem 1. Let (x^*, y^*) be the solution of the above optimization problem. Then, the min-max REVD and the associated decision q^e are given by

$$REVD_{max}(q^e) = g(x^*, y^*) \quad \text{and} \quad q^e = (1 + \theta^e)\mu,$$

where

$$\theta^e = \frac{x(y+b)(\alpha a - \beta x)(\beta - 1 + \alpha y) - y(x+a)(\beta b - \alpha y)(\beta - 1 + \beta x)}{(y+b)(\alpha a - \beta x)(\beta - 1 + \alpha y) + (x+a)(\beta b - \alpha y)(\beta - 1 + \beta x)}.$$

From Theorem 1, we can further show that the minimum of $REVD_{max}$ increases as a or b increases, which is also illustrated in the numerical examples given in the next section. Actually, we can see this from the model directly. Note that $a + b = \frac{B-A}{\mu}$ measures the size of the support, which is an indicator of the uncertainty. Hence, as a or b increases, the support gets larger, and the amount of uncertainty increases, therefore it's reasonable to expect larger minimum $REVD_{max}$.

To analyze the above optimization problem, we first look at a special case when $\alpha = \beta$ (i.e. $t - c = p + c$) and $a = b$ (i.e. $\mu = \frac{A+B}{2}$). In this case the objective function is symmetric w.r.t x and y. Since it is concave, the optimal is achieved at $x = y$. Therefore, the above optimization problem is deduced to the following univariate optimization problem:

$$\max_{0 \le x \le a} \phi_{a,\alpha}(x) \equiv \frac{x(a-x)}{(x+a)(1 - \frac{1}{\alpha} + x)} + 1.$$

Notice, $\phi'_{a,\alpha}(x) = -[(x+a)^2(x+1-\frac{1}{\alpha})^2]^{-2}[(2a+1-\frac{1}{\alpha})x^2 + 2a(1-\frac{1}{\alpha})x - a^2(1-\frac{1}{\alpha})]$. Therefore the optimal x^* that maximize $\phi_{a,\alpha}(x)$ is the root of the second order equation

$$(2a + 1 - \frac{1}{\alpha})x^2 + 2a(1 - \frac{1}{\alpha})x - a^2(1 - \frac{1}{\alpha}) = 0$$

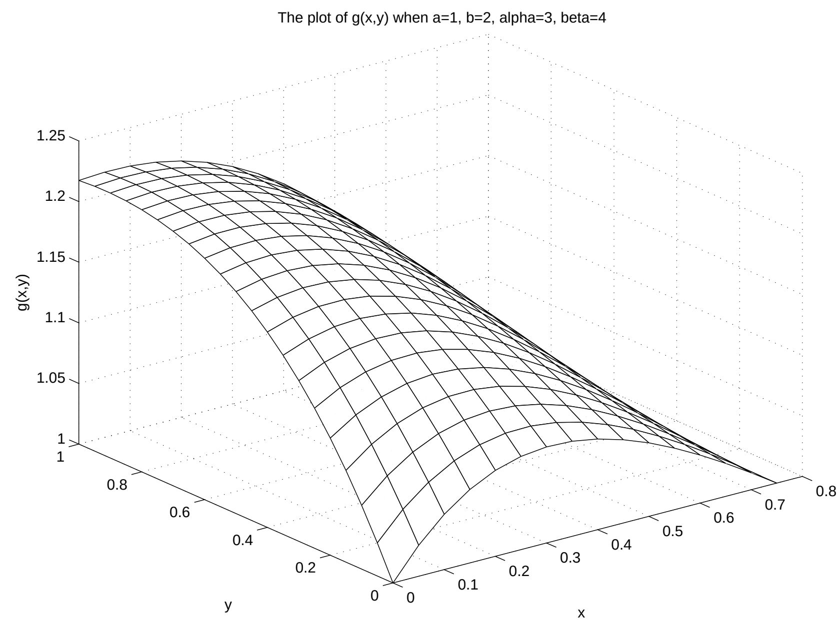
in the interval $[0, a]$. Hence

$$x^* = \frac{-a(\alpha - 1) + \sqrt{2a^2(\alpha - 1)(\alpha a + \alpha - 1)}}{2\alpha a + \alpha - 1},$$

and the minimum $REVD_{max}$ for this case is $\phi_{a,\alpha}(x^*)$.

To illustrate the variance of REVD, let's look at a few plots of $g(x, y)$ for several sets of a, b, α and β. The plot of $g(x, y)$ for the case of $a = b = 1$, $\alpha = \beta = 2$ is given as below:

For this case, the optimal $x^* = 0.2899$, $y^* = 0.2899$, with the min-max REVD 1.202. That is, our decision is at most 20% above the expected minimal cost when one knows the complete distribution of the demand. The plot of $g(x, y)$ for the case of $a = 1$, $b = 2$, $\alpha = 3$ and $\beta = 4$ is given as below:



For this case, we can find the optimal $x^* = 0.062$, $y^* = 0.910$, with the min-max REVD 1.220. The plot of $g(x, y)$ for the case of $a = 1$, $b = 2$, $\alpha = 3$

and $\beta = 8$ is given as below:

For this case, we can find the optimal $x^* = 0$, $y^* = 1$, with the Min-max REVD 1.215.

3 Min-Max REVD with mean and standard deviation

We use the same notation as in last section to minimizes the maximum REVD for distributions with given mean μ and standard deviation σ. (Thus, we may have different expression for $g(x, y)$ in different sections.)

Let $H(\mu, \sigma)$ be the set of all distributions with given mean μ and standard deviation σ. Then all the two-point pdf's in $H(\mu, \sigma)$ can be represented by $\{T(\gamma)| - t < \gamma < p\}$, where $T(\gamma)$ is a two-point pdf that assigns weights $\omega_1(\gamma) = \frac{t+\gamma}{p+t}$ and $\omega_2(\gamma) = \frac{p-\gamma}{p+t}$ to points

$$q_1(\gamma) = \mu - \sigma(\frac{p - \gamma}{t + \gamma})^{\frac{1}{2}} \text{ and } q_2(\gamma) = \mu + \sigma(\frac{t + \gamma}{p - \gamma})^{\frac{1}{2}},$$

respectively.

If $f = T(\gamma)$, the objective function $G_f(q)$ after some algebraic manipulation, can be simplified as follows:

$$G_{T(\gamma)}(q) = \begin{cases} cq - t(q - \mu), & \text{if } q \leq q_1(\gamma), \\ cq + \sigma[(p - \gamma)(t + \gamma)]^{\frac{1}{2}} + \gamma(q - \mu), & \text{if } q_1(\gamma) \leq q \leq q_2(\gamma), \\ cq + p(q - \mu), & \text{if } q \geq q_2(\gamma). \end{cases}$$

We have

Lemma 3. (Similar to Theorem 3 in [5]) For any q and $f \in H(\mu,\sigma)$, there exists a two-point pdf $T(\gamma) \in H(\mu,\sigma)$ with $-t < \gamma < p$ such that

$$REVD_f(q) \le REVD_{T(\gamma)}(q).$$

In view of Lemma 3, for any given decision q, we only have to search among two-point pdf's $T(\gamma)$ to find its largest REVD, i.e., to find a parameter $-t \le \gamma \le p$ that maximizes $REVD_{T(\gamma)}(q)$:

$$\text{REVD}_{max}(q) = \max_{-t\le\gamma\le p}\{\text{REVD}_{T(\gamma)}(q)\}.$$

From the geometric interpretation we can define

$$\begin{aligned} V_+(q,\gamma) &= \text{REVD}_{T(\gamma)}(q)|_{-c\le\gamma\le p, q_1(\gamma)\le q\le q_2(\gamma)} \\ &= \tfrac{G_{T(\gamma)}(q)}{G_{T(\gamma)}(q_1(\gamma))}|_{q_1(\gamma)\le q\le q_2(\gamma)} \\ &= \tfrac{cq+\sigma[(p-\gamma)(t+\gamma)]^{\frac{1}{2}}+\gamma(q-\mu)}{cq_1(\gamma)+\sigma[(p-\gamma)(t+\gamma)]^{\frac{1}{2}}-\gamma(\sigma(\frac{p-\gamma}{t+\gamma})^{\frac{1}{2}})}, \end{aligned}$$

and

$$\begin{aligned} V_-(q,\gamma) &= \text{REVD}_{T(\gamma)}(q)|_{-t\le\gamma\le -c, q_1(\gamma)\le q\le q_2(\gamma)} \\ &= \tfrac{G_{T(\gamma)}(q)}{G_{T(\gamma)}(q_2(\gamma))}|_{q_1(\gamma)\le q\le q_2(\gamma)} \\ &= \tfrac{cq+\sigma[(p-\gamma)(t+\gamma)]^{\frac{1}{2}}+\gamma(q-\mu)}{cq_2(\gamma)+\sigma[(p-\gamma)(t+\gamma)]^{\frac{1}{2}}+\gamma(\sigma(\frac{t+\gamma}{p-\gamma})^{\frac{1}{2}})}. \end{aligned}$$

Then, we derive

Lemma 4.

$$\max_{-t\le\gamma\le p}\{\text{REVD}_{T(\gamma)}(q)\} = \max\{\max_{-c\le\gamma_1\le p} V_+(q,\gamma_1), \max_{-t\le\gamma_2\le -c} V_-(q,\gamma_2)\}$$

Thus, combining with the result in Lemma 3, we have

$$\text{REVD}_{max}(q) = \max\{\max_{-c\le\gamma_1\le p} V_+(q,\gamma_1), \max_{-t\le\gamma_2\le -c} V_-(q,\gamma_2)\}.$$

Let q^e be the decision that minimizes the largest REVD, that is

$$q^e = \arg\min \text{REVD}_{max}(q).$$

Since $V_+(q,\gamma_1) \le 1$ for all $q \le q_1(\gamma_1)$, $-c \le \gamma_1 \le p$ and $V_-(q,\gamma_2) \le 1$ for all $q \ge q_2(\gamma_2)$,$-t \le \gamma_2 \le -c$. Then $\max_{-c\le\gamma_1\le p} V_+(q,\gamma_1) \le 1$ for all $q \le q_1(-c)$ and $\max_{-t\le\gamma_2\le -c} V_-(q,\gamma_1) \le 1$ for all $q \ge q_2(-c)$, because $\min_{-c\le\gamma_1\le p} q_1(\gamma_1) = q_1(-c)$ and $\max_{-t\le\gamma_2\le -c} q_2(\gamma_2) = q_2(-c)$.

Notice that $V_+(q,\gamma_1)$ and $\max_{-c\le\gamma_1\le p} V_+(q,\gamma_1)$ are both increasing functions of q, while $V_-(q,\gamma_2)$ and $\max_{-t\le\gamma_2\le -c} V_-(q,\gamma_2)$ are both decreasing functions of q. It follows that

$$q_1(-c) \le q_e \le q_2(-c)$$

and

$$\max_{-c\le\gamma_1\le p} V_+(q^e,\gamma_1) = \max_{-t\le\gamma_2\le -c} V_-(q^e,\gamma_2).$$

Therefore, we can simplify the formula for $REVD_{max}(q)$ as follows:

$$REVD_{max}(q) = \begin{cases} \max_{-c\le\gamma_1\le p} V_+(q,\gamma_1), & \text{if } q \ge q^e, \\ \max_{-t\le\gamma_2\le -c} V_-(q,\gamma_2), & \text{if } q \le q^e. \end{cases}$$

We can further simplify it by introducing the following variable transformations:

$$\theta = \frac{q-\mu}{\sigma},\ \alpha = \frac{t-c}{p},\ \beta = \frac{p+c}{p},\ \lambda = \frac{\mu}{\sigma},\ x = \frac{\gamma_1+c}{p},\ y = -\frac{\gamma_2+c}{p}.$$

Then, the above formula becomes

$$REVD_{max}(q) = \begin{cases} \max_{0\le x\le\beta} g_+(\theta,x), & \text{if } q \ge q^e, \\ \max_{0\le y\le\alpha} g_-(\theta,y), & \text{if } q \le q^e, \end{cases}$$

where

$$g_+(\theta,x) = V_+(q,\gamma_1) = 1 + \frac{1+\theta\sqrt{\frac{\alpha+x}{\beta-x}}}{\alpha+(\beta-1)\lambda\sqrt{\frac{\alpha+x}{\beta-x}}}x,$$

$$g_-(\theta,y) = V_-(q,\gamma_2) = 1 + \frac{1-\theta\sqrt{\frac{\beta+y}{\alpha-y}}}{\beta+(\beta-1)\lambda\sqrt{\frac{\beta+y}{\alpha-y}}}y.$$

The above formula also illustrates how to calculate the maximum REVD for any given decision q. Now, let us calculate q^e by the following optimization procedure:

$$\max_{0\le x\le\beta,0\le y\le\alpha} g_+(\theta,x)\ s.t.\ g_+(\theta,x) = g_-(\theta,y).$$

Eliminating θ from the constraint and then plug it into the objective function, we get the following optimization problem:

$$\max_{0\le x\le\beta,0\le y\le\alpha} g_{\alpha,\beta,\lambda}(x,y),$$

where

$$g_{\alpha,\beta,\lambda}(x,y) = 1 + \frac{(\sqrt{\frac{\alpha+x}{\beta-x}}+\sqrt{\frac{\beta+y}{\alpha-y}})xy}{\sqrt{\frac{\alpha+x}{\beta-x}}[\beta+(\beta-1)\lambda\sqrt{\frac{\beta+y}{\alpha-y}}]x+\sqrt{\frac{\beta+y}{\alpha-y}}[\alpha+(\beta-1)\lambda\sqrt{\frac{\alpha+x}{\beta-x}}]y}.$$

We now have

Theorem 2. Let (x^*, y^*) be the solution of the above optimization problem. Then, the min-max REVD and the associated decision q^e are given by

$$REVD_{max}(q^e) = g(x^*, y^*) \quad \text{and} \quad q^e = \mu + \theta^e \sigma,$$

where

$$\theta^e = \frac{-[\beta + (\beta-1)\lambda\sqrt{\frac{\beta+y^*}{\alpha-y^*}}]x^* + [\alpha + (\beta-1)\lambda\sqrt{\frac{\alpha+x^*}{\beta-x^*}}]y^*}{\sqrt{\frac{\alpha+x^*}{\beta-x^*}}[\beta + (\beta-1)\lambda\sqrt{\frac{\beta+y^*}{\alpha-y^*}}]x^* + \sqrt{\frac{\beta+y^*}{\alpha-y^*}}[\alpha + (\beta-1)\lambda\sqrt{\frac{\alpha+x^*}{\beta-x^*}}]y^*}.$$

From Theorem 2, we can further show that the minimum $REVD_{max}$ decreases as $\lambda = \frac{\mu}{\sigma}$ increases, which is also illustrated in the numerical examples given in the next section. Notice $\frac{\mu}{\sigma}$ is a measure of the uncertainty, hence we can see that as $\frac{\mu}{\sigma}$ increases, the amount of uncertainty decreases accordingly, therefore it's reasonable to expect smaller minimum $REVD_{max}$.

To analyze the above optimization problem, we first look at a special case when $\alpha = \beta$, i.e. $t - c = p + c$. Notice, in this case the objective function is symmetric w.r.t x and y, since it is concave, then the optimal is assumed when $x = y$. Therefore, the above optimization problem is deduced to the following univariate optimization problem:

$$\max_{0 \le x \le \alpha} \phi_{\alpha,\lambda}(x) \equiv \frac{x}{\alpha + (\alpha-1)\lambda\sqrt{\frac{\alpha+x}{\alpha-x}}} + 1.$$

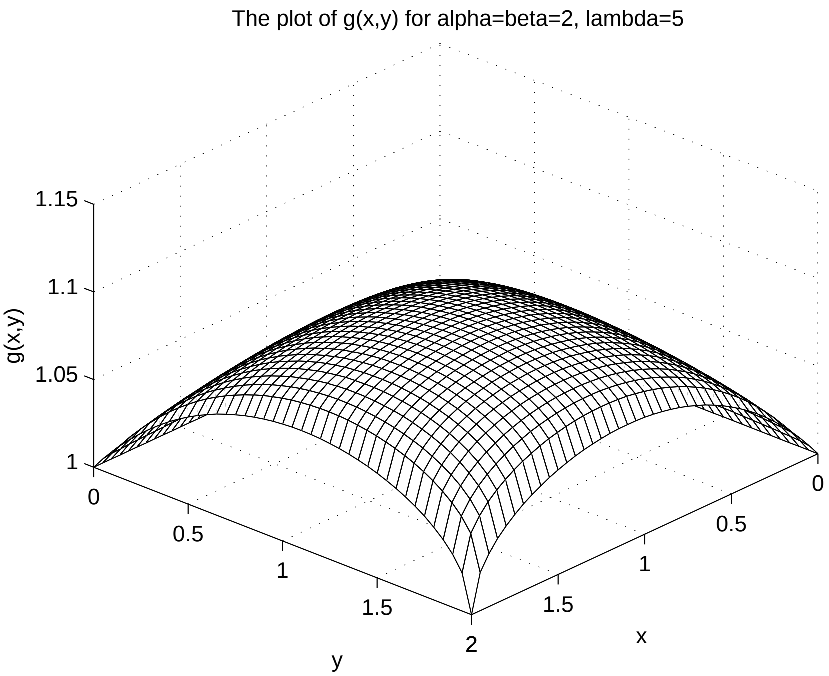
One can find some upper bounds for $REVD_{max}(q^e)$, one trivial upper bound is $1 + \frac{1}{1+(1-\frac{1}{\alpha})\lambda}$ for $\alpha = \beta$ case; and $1 + \frac{\max\{\alpha^2,\beta^2\}}{\alpha\beta+(\beta-1)\lambda\sqrt{\alpha\beta}}$ for the general case.

Notice, $\phi'_{\alpha,\lambda}(x) = (\alpha + (\alpha-1)\lambda\sqrt{\frac{\alpha+x}{\alpha-x}})^{-2}[\alpha - (\alpha-1)\lambda(x^2 + \alpha x - \alpha^2)(\alpha + x)^{-1/2}(\alpha - x)^{-3/2}]$. Therefore the optimal x^* that maximize $\phi_{\alpha,\lambda}(x)$ is the root of the following equation

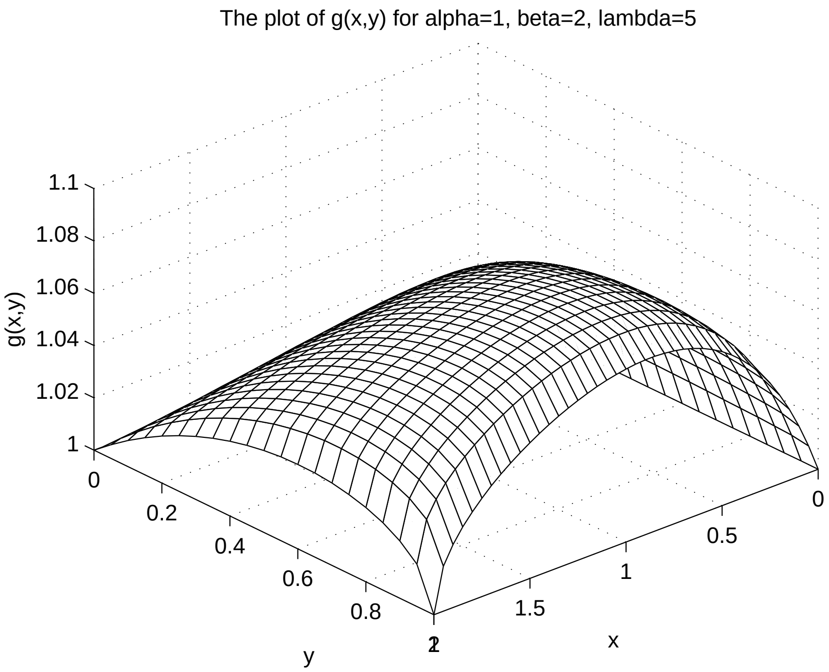
$$\alpha - (\alpha-1)\lambda(x^2 + \alpha x - \alpha^2)(\alpha + x)^{-1/2}(\alpha - x)^{-3/2}] = 0$$

in the interval $[0, \alpha]$, and the minimum $REVD_{max}$ for this case is $\phi_{\alpha,\lambda}(x^*)$.

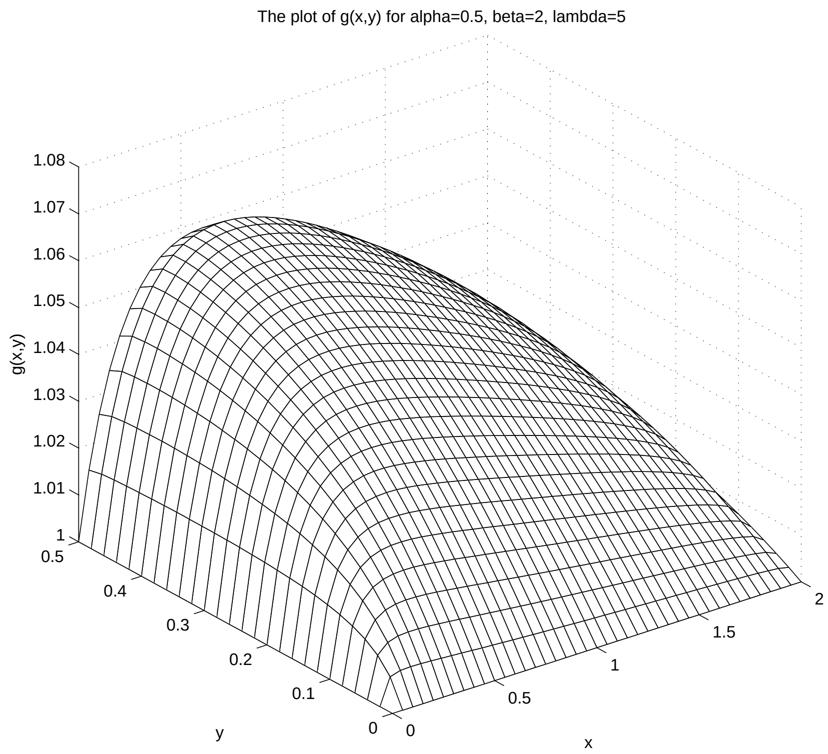
Again, to illustrate the variance of REVD, let's look at a few plots of $g(x, y)$ for several sets of a, b, α and β. The plot of $g(x, y)$ for the case of $\alpha = \beta = 2$ and $\lambda = 5$ is given as below:



For this case, we can find the optimal $x^* = 1.324$, $y^* = 1.324$, with the min-max REVD 1.101. That is, our decision is at most 10% above the expected minimal cost when one knows the complete distribution of the demand. The plot of $g(x, y)$ for the case of $\alpha = 1$, $\beta = 2$ and $\lambda = 5$ is given as below:



For this case, we can find the optimal $x^* = 0.963$, $y^* = 0.828$, with the min-max REVD 1.077. The plot of $g(x, y)$ for the case of $\alpha = 0.5$, $\beta = 2$ and $\lambda = 5$ is given as below:



For this case, we can find the optimal $x^* = 0.627$, $y^* = 0.433$, with the min-max REVD 1.060.

4 Numerical examples

Here we present some numerical examples to confirm our model and theoretical findings.

4.1 Given mean μ and standard deviation σ

Consider the following newsvendor example used in Gallego and Moon [2]:

$$p = 10.10, \; t = 15.20, \; \mu = 900, \; \sigma = 122.$$

For different value of the fixed ordering cost c, we use the following ordering quantities: the simple mean μ, the Scarf's rule q_u^*, the min-max AEVD ordering q^d, the min-max REVD q_c^r, the optimal order quantities under the normal distribution, Gamma distribution, Log-normal distribution and uniform distribution respectively. Furthermore, for each order quantity, we calculate the maximum AEVD, the maximum REVD and the cost range. Notice, [5] did not take the ordering cost c into account, hence we enhance their result in Appendix (6.6) when the ordering cost exists, so that we can compare their result with that from our model. The cost range can be computed from the tight upper bound $G_u(q)$ and lower bound $G_l(q)$ of $G_f(q)$ given in [5]:

$$G_u(q) \equiv cq + \frac{p-t}{2}(q-\mu) + \frac{t+p}{2}[\sigma^2 + (q-\mu)^2]^{1/2},$$

and

$$G_l(q) \equiv cq + t(\mu - q)^+ + p(q-\mu)^+.$$

For example, we run the MATLAB routine $[output] = compare(p, t, \mu, \sigma, c)$ for the case of $c = 1$ as follows: (We round the order quantities into integers.)

```
>> [output]=compare(10.10, 15.20, 900, 122, 1)

output =
  1.0e+002 *
   9.00000000000000   5.59421544044897   0.01363044836708
   9.15000000000000   4.85642471804169   0.01328808067625
   9.12000000000000   4.62729638619057   0.01308715890476
   9.10000000000000   4.76069763139554   0.01296496449780
   9.19000000000000   5.16847035268217   0.01356899231343
   9.13000000000000   4.70319441055238   0.01315321455813
   9.11000000000000   4.68096004455819   0.01302201391452
   9.26000000000000   5.73156192371308   0.01409742369965
```

Table 1: Ordering cost $c = 0.2$

$c = 0.2$	Quantity	Max AEVD	Max REVD	Cost Range
μ	900	$612.25	1.849	[$180, $1723]
q_u^*	923	$495.31	2.316	[$417, $1696]
q^d	918	$459.80	2.030	[$365, $1698]
q_c^r	911	$516.45	1.645	[$293, $1704]
$Normal$	929	$541.31	2.659	[$479, $1698]
$Gamma$	923	$495.31	2.316	[$417, $1696]
$Log-normal$	921	$480.29	2.202	[$396, $1697]
$Uniform$	939	$620.87	3.232	[$582, $1709]

Table 2: Ordering cost $c = 1$

$c = 1$	Quantity	Max AEVD	Max REVD	Cost Range
μ	900	\$559.42	1.363	[\$900, \$2443]
q_u^*	915	\$485.64	1.329	[\$1067, \$2432]
q^d	912	\$462.73	1.309	[\$1033, \$2432]
q_c^r	910	\$476.07	1.296	[\$1011, \$2433]
$Normal$	919	\$516.85	1.357	[\$1111, \$2432]
$Gamma$	913	\$470.32	1.315	[\$1044, \$2432]
$Log-normal$	911	\$468.10	1.302	[\$1022, \$2433]
$Uniform$	926	\$573.16	1.410	[\$1189, \$2438]

Table 3: ordering cost $c = 5$

$c = 5$	Quantity	Max AEVD	Max REVD	Cost Range
μ	900	\$619.02	1.120	[\$4500, \$6043]
q_u^*	876	\$496.27	1.097	[\$4745, \$6014]
q^d	881	\$459.15	1.090	[\$4694, \$6015]
q_c^r	882	\$465.67	1.088	[\$4684, \$6016]
$Normal$	870	\$542.10	1.107	[\$4806, \$6016]
$Gamma$	865	\$581.29	1.115	[\$4857, \$6020]
$Log-normal$	863	\$597.20	1.119	[\$4877, \$6022]
$Uniform$	859	\$629.41	1.126	[\$4918, \$6028]

Table 4: Ordering cost $c = 10$

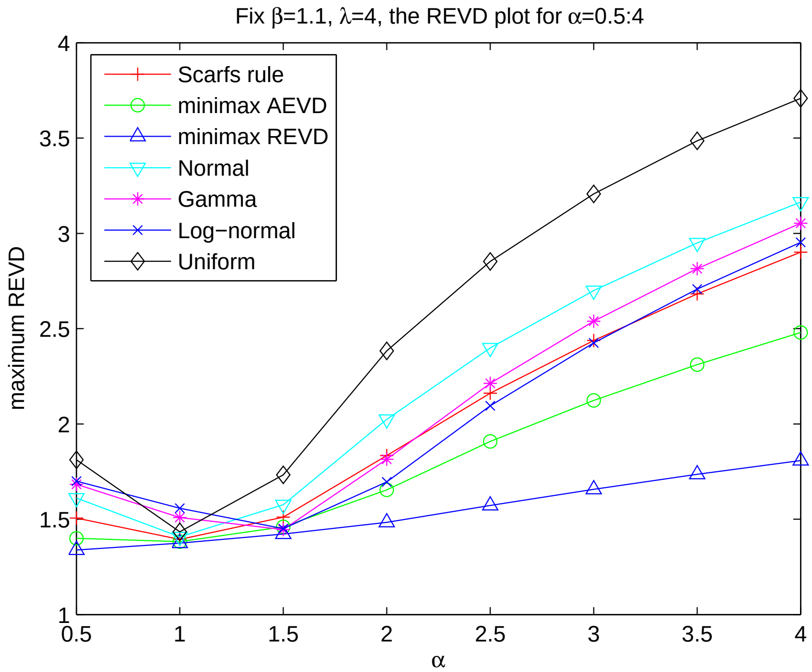
$c = 10$	Quantity	Max AEVD	Max REVD	Cost Range
μ	900	\$1006.86	1.107	[\$9000, \$10543]
q_u^*	811	\$501.74	1.054	[\$9463, \$10247]
q^d	831	\$407.46	1.042	[\$9359, \$10259]
q_c^r	831	\$407.46	1.042	[\$9359, \$10259]
$Normal$	800	\$555.50	1.060	[\$9520, \$10250]
$Gamma$	798	\$565.33	1.061	[\$9530, \$10252]
$Log-normal$	798	\$565.33	1.061	[\$9530, \$10252]
$Uniform$	776	\$674.46	1.073	[\$9645, \$10277]

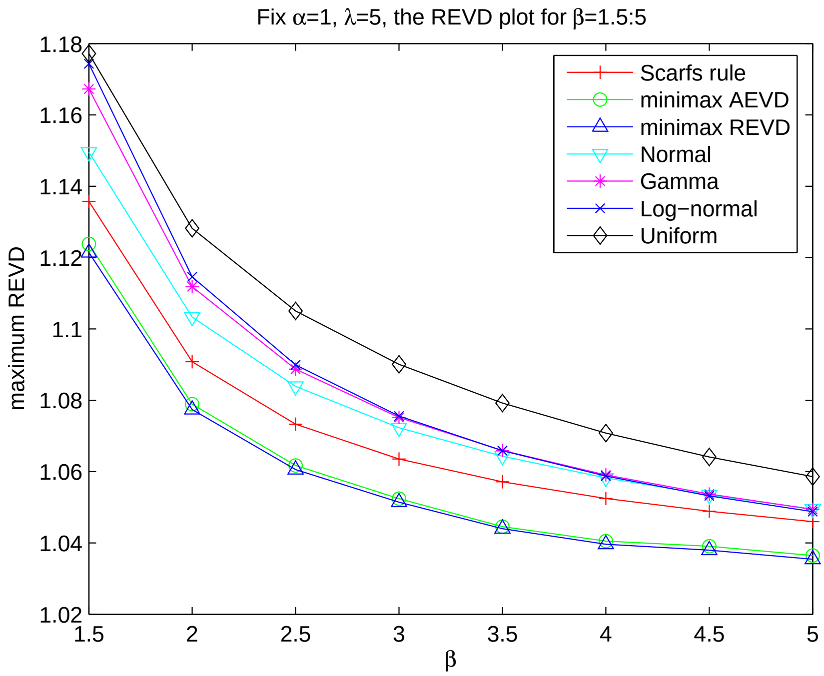
Table 5: ordering cost $c = 15$

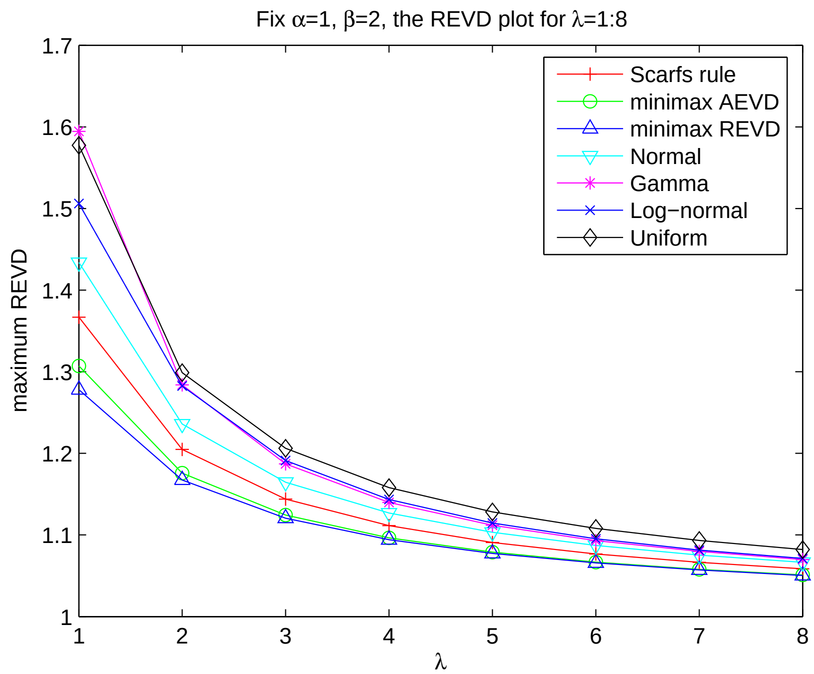
$c = 15$	Quantity	Max AEVD	Max REVD	Cost Range
μ	900	\$1519.09	1.112	[\$13500, \$15043]
q_u^*	222	\$135.60	1.010	[\$13636, \$13773]
q^d	378	\$104.34	1.008	[\$13604, \$13782]
q_c^r	780	\$613.93	1.0018	[\$13524, \$14171]
$Normal$	606	\$253.26	1.0044	[\$13559, \$13866]
$Gamma$	632	\$283.49	1.0040	[\$13554, \$13888]
$Log-normal$	644	\$299.10	1.0038	[\$13551, \$13900]
$Uniform$	692	\$375.30	1.0031	[\$13541, \$13961]

Notice, the maximum REVD only depends on α, β and λ. Hence, we can fix two of them, and change the value of the other parameter to see the corresponding changes of maximum REVD.

The results are as follows:







From the numerical results, we can observe the followings.

1. For the example in Table 1, the order quantity from the AEVD model has a large maximum REVD 2.030, i.e. the worst case cost is 103% more than the minimal cost. In contrast, the order quantity from the REVD model has maximum REVD 1.645, whose worst case cost is only 64.5% more than the minimal cost. And the regrets from not ordering

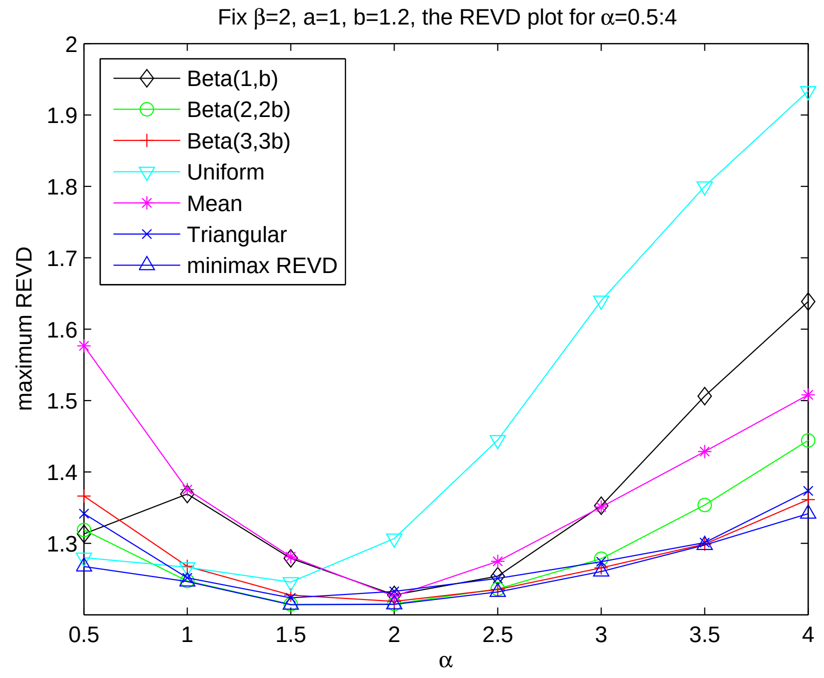
optimal for these two ordering quantities are 459.80 and 516.45, which is almost in the same scale. That's why our REVD model is preferable to AEVD model in those situations.

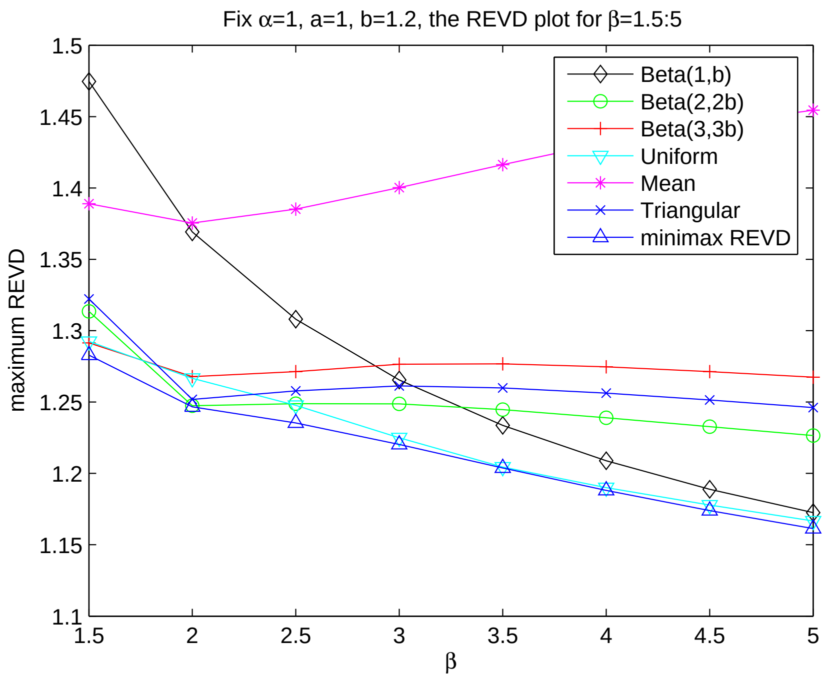
2. In the first plot for the case of $\beta = 1 + \frac{c}{p} = 1.1$ and $\lambda = \frac{\mu}{\sigma} = 4$, $\alpha = \frac{t-c}{p}$ varying from 0.5 to 4, we can see that the maximum REVD quantities corresponding to the optimal orderings under AEVD model and REVD model are quite different. Typically, the above two quantities are quite close, but differ largely when λ is small (i.e. more uncertainty) and/or $\beta = 1 + \frac{c}{p}$ is small (i.e. the inventory holding cost is dominant).

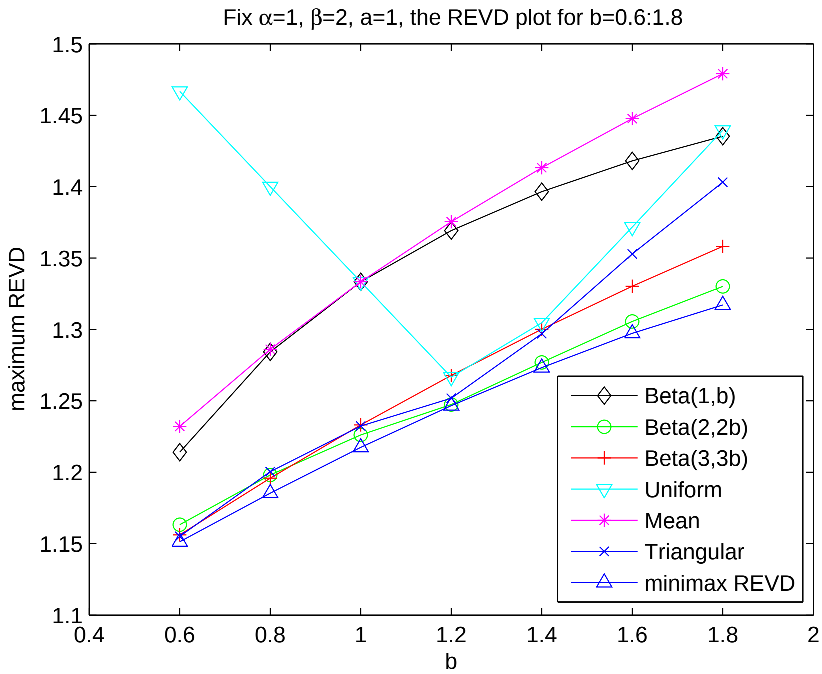
3. We have considered four distributions: normal, Gamma, Log-normal, Uniform. Although none of them dominates the others, it is quite clear that uniform distribution may not be a good choice for the newsvendor in making decisions.

4.2 Given mean μ and support $[A, B]$

Notice, the maximum REVD only depends on α, β, a and b. Hence, we can fix three of them, and change the value of the other parameter to see the corresponding changes of the maximum REVD. In particular, if we fix $A = 0$, $B = 1$ and $p = 1$, then $a = 1$. We can change α, β or b, and compute the maximum REVD corresponding to the optimal ordering under different distributions, say, Beta(1, b), Beta(2, 2b), Beta(3, 3b), Triangular distribution and Uniform distribution. Compare them with the results for maximum REVD of min-max REVD ordering and mean ordering, we get the following:


Fix β=2, a=1, b=1.2, the REVD plot for α=0.5:4
Beta(1,b)
Beta(2,2b)
Beta(3,3b)
Uniform
Mean
Triangular
minimax REVD
maximum REVD
α
2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
0.5
1
1.5
2
2.5
3
3.5
4


Fix α=1, a=1, b=1.2, the REVD plot for β=1.5:5
Beta(1,b)
Beta(2,2b)
Beta(3,3b)
Uniform
Mean
Triangular
minimax REVD
maximum REVD
β
1.5
1.45
1.4
1.35
1.3
1.25
1.2
1.15
1.1
1.5
2
2.5
3
3.5
4
4.5
5



From the numerical results, we can observe the followings.

1. The triangle and some beta distributions seem to shield risk quite well for the worse case, so that they might be good benchmark distributions for inventory decision making when the demand mean and support are known.

2. Again, it is quite clear that uniform distribution may not be a good choice for the newsvendor in making decisions.

Remarks Note that in these numerical examples we did not include the ordering quantity that minimizes the maximum AEVD for the following two reasons: First, we did not derive the method to compute the ordering quantity that minimizes the maximum AEVD with mean μ and support $[A, B]$. In fact, the method provided in [5] to compute the min-max AEVD ordering is only for the case of given mean μ and standard deviation σ. And, [3] and Appendix 6.5 in our report deal with the case of given μ and support $[A, B]$, but what we computed is RVAI (Robust Value of Additional Information), which is the difference between the profit and optimal profit, instead of the difference between the costs in AEVD. Although these two are equivalent somehow, but we still need to take some time to convert them from one to another.

Secondly, neither [5] nor [3] takes the ordering cost c into consideration, hence at the time when we derive Theorem 3 in Appendix 6.5, ordering cost was not taken into account either. But if we need to compare the AEVD with our REVD model, 'c' needs to be taken into account. We plan to do so soon.

5 Conclusions and Future Work

Our findings have important implications to general stochastic optimization. For example, in the case of knowing the mean and support, the triangle distribution could be used as a "robust" assumption in decision models and sampling methods. For knowing the mean and standard deviation, the normal distribution seems save for a class of problems. We plan to investigate these distributions further.

The problem discussed above has only one random variable. As we mentioned earlier, we are now investigating the multiple-item, dependent random variable, and capacitated inventory problem under the same model.

More importantly, we plan to apply the model and approach to general stochastic optimization problems, where we also like to derive theoretical bounds on REVD for certain classes of problems.

References

[1] Manar El-Chammas, Anthony So, Wei Wu, and Yinyu Ye: *Boeing Report: Bounds for Stochastic Optimization*, March, 2006.

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[3] George Perakis and Guillaume Roels: *Regret in the Newsvendor Model with Partial Information*, 2006

[4] Scarf, H. (1958): *A min-max solution of an inventory problem*, Studies in The Mathematical Theory of Inventory and Production, 201-209, Stanford University Press, California, 1958.

[5] Jinfeng Yue, Bintong Chen and Min-Chiang Wang: *Expected Value of Distribution Information for the Newsvendor Problem*, Operations Research, to appear, 2006.

[6] Panos Kouvelis and Gang Yu: *Robust Discrete Optimization and Its Application*, KLUWER ACADEMIC PUBLISHERS, 1997.

[7] George L. Vairaktarakis: *Robust multi-item newsboy models with a budget constraint*, Int. J. Production Economics 66 (2000) 213-226.

6 Appendix

6.1 Proof of Lemma 1.

Proof: (i) $\hat{q} \geq \mu$ and $q \geq \hat{q}$
$\Rightarrow$

$$(y_1 - \lambda p)x + \lambda(c\hat{q} + p\hat{q}) - \mu y_1 = p(q - x) + cq \quad \text{for } A \leq x \leq \hat{q},$$

and

$$(y_1 + \lambda t)x + \lambda(c\hat{q} - t\hat{q}) - \mu y_1 \geq t(x - q) + cq \quad \text{for } \hat{q} \leq x \leq B.$$

(Note: coincide with the first segment.)
Therefore,

$$\lambda_{\min} = 1 + \frac{(q - \hat{q})(p + c)}{c\hat{q} + p\hat{q} - p\mu}$$
$$y_1 = \frac{(q - \hat{q})(p + c)p}{c\hat{q} + p\hat{q} - p\mu}$$

(ii) $\hat{q} \geq \mu$ and $q \leq \hat{q}$
$\Rightarrow$

$$(y_1 - \lambda p)x + \lambda(c\hat{q} + p\hat{q}) - \mu y_1 \geq p(q - x) + cq \quad \text{for } A \leq x \leq \hat{q},$$

and

$$(y_1 + \lambda t)x + \lambda(c\hat{q} - t\hat{q}) - \mu y_1 \geq t(x - q) + cq \quad \text{for } \hat{q} \leq x \leq B.$$

$$(y_1 - \lambda p)x + \lambda(c\hat{q} + p\hat{q}) - \mu y_1 = p(q - x) + cq \quad \text{for } x = A$$

and

$$(y_1 - \lambda p)x + \lambda(c\hat{q} + p\hat{q}) - \mu y_1 = t(x - q) + cq \quad \text{for } x = \hat{q}.$$

(Note: intersect with two point at $x = A$ and $x = \hat{q}$).
Therefore,

$$\lambda_{\min} = 1 + \frac{\hat{q} - q}{\hat{q} - A}\frac{(t - c)(\mu - A) - (p + c)(\hat{q} - \mu)}{c\hat{q} + p\hat{q} - p\mu}$$
$$y_1 = \frac{\hat{q} - q}{\hat{q} - A}\frac{(p + c)t\hat{q} - (t - c)pA}{c\hat{q} + p\hat{q} - p\mu}$$

(iii) $\hat{q} \leq \mu$ and $q \geq \hat{q}$
$\Rightarrow$

$$(y_1 - \lambda p)x + \lambda(c\hat{q} + p\hat{q}) - \mu y_1 \geq p(q - x) + cq \quad \text{for } A \leq x \leq \hat{q},$$

and

$$(y_1 + \lambda t)x + \lambda(c\hat{q} - t\hat{q}) - \mu y_1 \geq t(x - q) + cq \quad \text{for } \hat{q} \leq x \leq B.$$

$$(y_1 + \lambda t)x + \lambda(c\hat{q} - t\hat{q}) - \mu y_1 = p(q - x) + cq \quad \text{for } x = \hat{q}$$

and

$$(y_1 + \lambda t)x + \lambda(c\hat{q} - t\hat{q}) - \mu y_1 = t(x - q) + cq \quad \text{for } x = B.$$

(Note: intersect with two point at $x = \hat{q}$ and $x = B$).
Therefore,

$$\lambda_{\min} = 1 + \frac{q - \hat{q}}{B - \hat{q}} \frac{(p + c)(B - \mu) - (t - c)(\mu - \hat{q})}{c\hat{q} + t\mu - t\hat{q}}$$

$$y_1 = \frac{q - \hat{q}}{B - \hat{q}} \frac{(t - c)p\hat{q} - (p + c)tB}{c\hat{q} + t\mu - t\hat{q}}$$

(vi) $\hat{q} \leq \mu$ and $q \leq \hat{q}$
$\Rightarrow$

$$(y_1 - \lambda p)x + \lambda(c\hat{q} + p\hat{q}) - uy_1 \geq p(q - x) + cq \quad \text{for } A \leq x \leq \hat{q},$$

and

$$(y_1 + \lambda t)x + \lambda(c\hat{q} - t\hat{q}) - uy_1 = t(x - q) + cq \quad \text{for } \hat{q} \leq x \leq B.$$

(Note: coincide with the second segment.)
Therefore,

$$\lambda_{\min} = 1 + \frac{(\hat{q} - q)(t - c)}{c\hat{q} + t\mu - t\hat{q}}$$

$$y_1 = -\frac{(\hat{q} - q)(t - c)t}{c\hat{q} + t\mu - t\hat{q}}$$

The above four cases together completes the proof.

6.2 Proof of Lemma 2.

Notice

$$\begin{aligned} REVD_{max}(q) &= \max\nolimits_{f \in H(\mu,[A,B])} REVD_f(q) \\ &= \max\nolimits_{A \leq \hat{q} \leq B} h(q, \hat{q}) \\ &= \max\{\max\nolimits_{\mu \leq \hat{q} \leq B} h(q, \hat{q}), \max\nolimits_{A \leq \hat{q} \leq \mu} h(q, \hat{q})\} \\ &= \max\{\max\nolimits_{\max(\mu,q) \leq \hat{q} \leq B} h(q, \hat{q}), \max\nolimits_{A \leq \hat{q} \leq \min(\mu,q)} h(q, \hat{q})\} \end{aligned}$$

the last equality is because of the following facts:

$$h(q,\hat{q}) = 1 + \frac{(q-\hat{q})(p+c)}{c\hat{q}+p\hat{q}-p\mu} \leq h(q,\mu),\ \text{when } \mu \leq \hat{q} \leq q;$$

$$h(q,\hat{q}) = 1 + \frac{(\hat{q}-q)(t-c)}{c\hat{q}+t\mu-t\hat{q}} \leq h(q,\mu),\ \text{when } q \leq \hat{q} \leq \mu.$$

Then,

$$\begin{aligned}
h_+(q) &:= \max_{\hat{q}\geq\max(\mu,q)} h(q,\hat{q}) \\
&= \max_{\hat{q}\geq\max(\mu,q)} \tfrac{\hat{q}-q}{\hat{q}-A}\tfrac{(t-c)(\mu-A)-(p+c)(\hat{q}-\mu)}{c\hat{q}+p\hat{q}-p\mu} + 1 \\
&= \max_{\min\{B,\mu+\frac{t-c}{p+c}(\mu-A)\}\geq\hat{q}\geq\max(\mu,q)} \tfrac{\hat{q}-q}{\hat{q}-A}\tfrac{(t-c)(\mu-A)-(p+c)(\hat{q}-\mu)}{c\hat{q}+p\hat{q}-p\mu} + 1 \\
&= \max_{\min\{B,\mu+\frac{t-c}{p+c}(\mu-A)\}\geq\hat{q}\geq\mu} \tfrac{\hat{q}-q}{\hat{q}-A}\tfrac{(t-c)(\mu-A)-(p+c)(\hat{q}-\mu)}{c\hat{q}+p\hat{q}-p\mu} + 1
\end{aligned}$$

the last equality is because

$$h(q,\hat{q}) = \frac{\hat{q}-q}{\hat{q}-A}\frac{(t-c)(\mu-A)-(p+c)(\hat{q}-\mu)}{c\hat{q}+p\hat{q}-p\mu} + 1 \leq 1,$$

when

$$\mu \leq \hat{q} \leq \min\{B, \mu + \frac{t-c}{p+c}(\mu-A), q\}.$$

By the same way, we have

$$\begin{aligned}
h_-(q) &:= \max_{\hat{q}\leq\min(\mu,q)} h(q,\hat{q}) \\
&= \max_{\max\{A,\mu-\frac{p+c}{t-c}(B-\mu)\}\leq\hat{q}\leq\mu} \tfrac{q-\hat{q}}{B-\hat{q}}\tfrac{(p+c)(B-\mu)-(t-c)(\mu-\hat{q})}{c\hat{q}+t\mu-t\hat{q}} + 1
\end{aligned}$$

Notice that $h_+(q)$ is decreasing function while $h_-(q)$ is increasing function, and $h_+(B) \leq 1 \leq h_-(B)$, $h_-(A) \leq 1 \leq h_+(A)$. It follows that

$$h_+(q^e) = h_-(q^e).$$

Therefore, we can get the formula for $REVD_{max}(q)$ as follows:

$$REVD_{max}(q) = \begin{cases} h_+(q), & \text{if } q \leq q^e, \\ h_-(q), & \text{if } q \geq q^e. \end{cases}$$

6.3 Proof of Lemma 3.

We have two ways to prove it.

Method 1. Apply the result of Theorem 3 and remark (7) in paper [5]. Let $T(\gamma)$ be the two-point pdf chosen in Theorem 3 such that $q_1(\gamma) = q_f^*$. Then $AEVD_f(q) \leq AEVD_{T(\gamma)}(q)$ and $G_{T(\gamma)}(q_f^*) \leq G_f(q_f^*)$ (by (7) in [5]). (Note: The proof for $AEVD_f(q) \leq AEVD_{T(\gamma)}(q)$ is slightly different from the proof in [5], because the definition of $G_f(q)$ is different.)

Then

$$\frac{AEVD_f(q)}{G_f(q_f^*)} \leq \frac{AEVD_{T(\gamma)}(q)}{G_{T(\gamma)}(q_f^*)},$$

$\Rightarrow$

$$\frac{G_f(q)}{G_f(q_f^*)} = \frac{AEVD_f(q)}{G_f(q_f^*)} + 1 \leq \frac{AEVD_{T(\gamma)}(q)}{G_{T(\gamma)}(q_f^*)} + 1 = \frac{G_{T(\gamma)}(q)}{G_{T(\gamma)}(q_f^*)},$$

$\Rightarrow$

$$REVD_f(q) = \frac{G_f(q)}{G_f(q_f^*)} \leq \frac{G_{T(\gamma)}(q)}{G_{T(\gamma)}(q_f^*)} \leq \frac{G_{T(\gamma)}(q)}{G_{T(\gamma)}(q_{T(\gamma)}^*)} = REVD_{T(\gamma)}(q),$$

the last inequality is because $G_{T(\gamma)}(q_{T(\gamma)}^*) \leq G_{T(\gamma)}(q_f^*)$.

Method 2: Use the LP duality to prove just like what we did in section 2. (see the proof of Lemma 1 or [3] Theorem 7)

Method 2: Use the LP duality to prove. (see [3] Theorem 7)

Let $r = REVD_{max}(q)$, then consider the following LP problem:

$$\begin{array}{rl} \max & G_f(q) - rG_f(\hat{q}) \\ s.t. & \int_\Omega x^i dF(x) = q_i, \quad \text{for } i = 0, 1, 2, \end{array}$$

where $q_0 = 1$, $q_1 = \mu$ and $q_2 = \mu^2 + \sigma^2$.

Its dual is

$$\begin{array}{rrl} \min & y_0 + y_1\mu + y_2(\sigma^2 + \mu^2) & \\ s.t. & y_0 + y_1 x + y_2 x^2 \geq & [p(q - x) + cq] \cdot 1_{x \leq q} + [t(x - q) + cq] \cdot 1_{x \geq q} \\ & & -r\{[p(\hat{q} - x) + c\hat{q}] \cdot 1_{x \leq \hat{q}} + [t(x - \hat{q}) + c\hat{q}] \cdot 1_{x \geq \hat{q}}\}, \quad \forall x. \end{array}$$

From the geometrical interpretation of this dual problem , we can see that there are at most two intersection points (i.e. two active constraints of the dual), which means the optimal distribution is at most a two-point distribution.

6.4 Proof of Lemma 4.

Consider the following two cases:

(1) When $-c \leq \gamma \leq p$, then $q_{T(\gamma)}^* = q_1(\gamma)$. We have

$$REVD_{T(\gamma)}(q) = \frac{G_{T(\gamma)}(q)}{G_{T(\gamma)}(q_1(\gamma))}$$

(a) If $q_1(\gamma) \leq q \leq q_2(\gamma)$, then

$$REVD_{T(\gamma)}(q) = V_+(q, \gamma).$$

(b) If $q \geq q_2(\gamma)$, then

$$REVD_{T(\gamma)}(q) = \frac{cq + p(q-\mu)}{cq_1(\gamma) + \sigma[(p-\gamma)(t+\gamma)]^{\frac{1}{2}} - \gamma(\sigma(\frac{p-\gamma}{t+\gamma})^{\frac{1}{2}})} \leq \frac{cq + p(q-\mu)}{c\mu} = V_+(q,p).$$

(c) If $q \leq q_1(\gamma)$, then

$$REVD_{T(\gamma)}(q) = \frac{cq - t(q-\mu)}{cq_2(\gamma) + \sigma[(p-\gamma)(t+\gamma)]^{\frac{1}{2}} - \gamma(\sigma(\frac{p-\gamma}{t+\gamma})^{\frac{1}{2}})} \leq \frac{cq - t(q-\mu)}{c\mu} = V_-(q,-t).$$

On the other hand, from the geometric point of view, since $G_{T(\gamma)}(q)$ is a convex function, we have

$$REVD_{T(\gamma)}(q) \geq V_+(q,\gamma).$$

It follows that

$$V_+(q,\gamma) \leq REVD_{T(\gamma)}(q) \leq \max\{V_+(q,\gamma), V_+(q,p), V_-(q,-t)\}.$$

(2) When $-t \leq \gamma \leq -c$, by the same way we have

$$V_-(q,\gamma) \leq REVD_{T(\gamma)}(q) \leq \max\{V_-(q,\gamma), V_+(q,p), V_-(q,-t)\}.$$

Combining these two, we have

$$\max_{-t\leq\gamma\leq p}\{\mathrm{REVD}_{T(\gamma)}(q)\} = \max\{\max_{-c\leq\gamma_1\leq p} V_+(q,\gamma_1), \max_{-t\leq\gamma_2\leq -c} V_-(q,\gamma_2)\}.$$

6.5 The ordering that minimizes the RVAI for distributions with given mean μ and support [A,B]

In [3], the min-max regret problem is formulated as

$$\min_{Q\geq 0}\max_{F\in D(q)}\{[\max_{\hat{Q}\geq 0}\Pi(\hat{Q},F)] - \Pi(Q,F)\},$$

where $\Pi(Q,F) = pE_F[\min\{Q,D\}] - cQ$ is the expected profit. Notice, here we use the same notation as in [3]: Q is the order quantity before the selling season, $\hat{Q}$ is the order quantity after observing the demand distribution, c is the unit order cost and p is the unit selling price. We use these notations throughout all this subsection.

Definition 1 The Robust Value of Additional Information (RVAI) corresponds to the maximum profit loss from knowing only partial information on the demand distribution. Mathematically, the RVAI is the optimal value of the above problem.

By inverting the order of maximization, it can be formulated as follows:

$$\begin{aligned}
& \min_{Q\geq 0} RVAI \\
= \ & \min_{Q\geq 0} \max_{F\in D(q)}\{[\max_{\hat{Q}\geq 0} \Pi(\hat{Q},F)] - \Pi(Q,F)\} \\
= \ & \min_{Q\geq 0} \max_{\hat{Q}\geq 0}\{\max_{F\in D(q)}[\Pi(\hat{Q},F) - \Pi(Q,F)]\} \\
= \ & \min_{Q\geq 0} \max_{\hat{Q}\geq 0} p\{\max_{F\in D(q)} \int_0^\infty (\min\{x,\hat{Q}\} - \min\{x,Q\})dF(x)\} + c(Q-\hat{Q})
\end{aligned}$$

Theorem 3. (The original one, Theorem 2 given in [3], was imcomplete) If the demand distribution is nonnegative, with support [A, B] and mean μ, the min-max regret order quantity is the following:
$Q^* =$

$$\begin{cases}
A + \frac{p-c}{p}\frac{\mu-A}{B-\mu}(B-A), & \text{if } \frac{p}{c} \leq \min\{1+\frac{1}{2}\frac{B-\mu}{\mu-A}, 1+\frac{B-\mu}{B-A}\}, \\
B - \frac{(2p-c)^2}{4p(p-c)}(B-\mu), & \text{if } 1+\frac{1}{2}\frac{B-\mu}{\mu-A} \leq \frac{p}{c} \leq 1+\frac{1}{2}\sqrt{\frac{B-\mu}{\mu-A}}, \\
A + (\sqrt{\frac{c}{p}} - \sqrt{\frac{c}{p} - \frac{\mu-A}{B-A}})^2\frac{(B-A)^2}{\mu-A}, & \text{if } 1+\frac{B-\mu}{B-A} \leq \frac{p}{c} \leq 1+(\frac{\mu-A}{B-\mu}+\frac{1}{2}\frac{B-A}{B-\mu}\sqrt{\frac{\mu-A}{B-\mu}})^{-1}, \\
Q_0, & \text{if } \max\{1+(\frac{\mu-A}{B-\mu}+\frac{1}{2}\frac{B-A}{B-\mu}\sqrt{\frac{\mu-A}{B-\mu}})^{-1}, 1+\frac{1}{2}\frac{B-\mu}{\mu-A}\} \\
 & \leq \frac{p}{c} \leq \min\{1+2\sqrt{\frac{B-\mu}{\mu-A}}, \frac{B-A}{\mu-A}+\frac{1}{2}\frac{B-A}{\mu-A}\sqrt{\frac{B-\mu}{\mu-A}}\}, \\
B - (\sqrt{\frac{p-c}{p}} - \sqrt{\frac{p-c}{p} - \frac{B-\mu}{B-A}})^2\frac{(B-A)^2}{B-\mu}, & \text{if } \frac{B-A}{\mu-A}+\frac{1}{2}\frac{B-A}{\mu-A}\sqrt{\frac{B-\mu}{\mu-A}} \leq \frac{p}{c} \leq 1+\frac{B-A}{\mu-A}, \\
A + \frac{(p+c)^2}{4pc}(\mu-A), & \text{if } 1+2\sqrt{\frac{B-\mu}{\mu-A}} \leq \frac{p}{c} \leq 1+2\frac{B-\mu}{\mu-A}, \\
B - \frac{c}{p}\frac{B-\mu}{\mu-A}(B-A), & \text{if } \frac{p}{c} \geq \max\{1+\frac{B-A}{\mu-A}, 1+2\frac{B-\mu}{\mu-A}\}.
\end{cases}$$

and the RVAI amounts to
RVAI=

$$\begin{cases}
(1-\frac{p-c}{p}\frac{B-A}{B-\mu})(p-c)(\mu-A), & \text{if } \frac{p}{c} \leq \min\{1+\frac{1}{2}\frac{B-\mu}{\mu-A}, 1+\frac{B-\mu}{B-A}\}, \\
\frac{c^2}{4p}(B-\mu), & \text{if } 1+\frac{1}{2}\frac{B-\mu}{\mu-A} \leq \frac{p}{c} \leq 1+\frac{1}{2}\sqrt{\frac{B-\mu}{\mu-A}}, \\
(1-\sqrt{1-\frac{p}{c}\frac{\mu-A}{B-A}})^2(\frac{c}{p}\frac{B-A}{\mu-A}-1)c(B-A), & \text{if } 1+\frac{B-\mu}{B-A} \leq \frac{p}{c} \leq 1+(\frac{\mu-A}{B-\mu}+\frac{1}{2}\frac{B-A}{B-\mu}\sqrt{\frac{\mu-A}{B-\mu}})^{-1}, \\
(\sqrt{p(B-\mu)} - \sqrt{(p-c)(B-Q_0)})^2, & \text{if } \max\{1+(\frac{\mu-A}{B-\mu}+\frac{1}{2}\frac{B-A}{B-\mu}\sqrt{\frac{\mu-A}{B-\mu}})^{-1}, 1+\frac{1}{2}\frac{B-\mu}{\mu-A}\} \\
 & \leq \frac{p}{c} \leq \min\{1+2\sqrt{\frac{B-\mu}{\mu-A}}, \frac{B-A}{\mu-A}+\frac{1}{2}\frac{B-A}{\mu-A}\sqrt{\frac{B-\mu}{\mu-A}}\}, \\
(1-\sqrt{1-\frac{p}{p-c}\frac{B-\mu}{B-A}})^2(\frac{p-c}{p}\frac{B-A}{B-\mu}-1) & \\
\quad \cdot(p-c)(B-A), & \text{if } \frac{B-A}{\mu-A}+\frac{1}{2}\frac{B-A}{\mu-A}\sqrt{\frac{B-\mu}{\mu-A}} \leq \frac{p}{c} \leq 1+\frac{B-A}{\mu-A}, \\
\frac{(p-c)^2}{4p}(\mu-A), & \text{if } 1+2\sqrt{\frac{B-\mu}{\mu-A}} \leq \frac{p}{c} \leq 1+2\frac{B-\mu}{\mu-A}, \\
(1-\frac{c}{p}\frac{B-A}{\mu-A})c(B-\mu), & \text{if } \frac{p}{c} \geq \max\{1+\frac{B-A}{\mu-A}, 1+2\frac{B-\mu}{\mu-A}\}.
\end{cases}$$

where $Q_0 = \Gamma_2 \cap \tilde{\Gamma}_2$ is the intersection point of Γ_2 and $\tilde{\Gamma}_2$ in the monotonic interval, i.e. the unique solution of
$\sqrt{p(\mu-A)} - \sqrt{c(Q-A)} = \sqrt{p(B-\mu)} - \sqrt{(p-c)(B-Q)}$, such that Q is in the monotonic interval of both Γ_2 and $\tilde{\Gamma}_2$.

Proof: Let

$$f(Q,\hat{Q}) = p\{\max_{F\in D(q)}\int_0^\infty (\min\{x,\hat{Q}\} - \min\{x,Q\})dF(x)\} + c(Q-\hat{Q}),$$

then

$$\begin{aligned} RVAI &= \min_{Q\geq 0}\max_{\hat{Q}\geq 0} f(Q,\hat{Q}) \\ &= \min_{Q\geq 0}\max\{\max_{\hat{Q}\geq Q} f(Q,\hat{Q}), \max_{\hat{Q}\leq Q} f(Q,\hat{Q})\} \\ &= \min_{Q\geq 0}\max\{f_+(Q), f_-(Q)\} \end{aligned}$$

where $f_+(Q) = \max_{\hat{Q}\geq Q} f(Q,\hat{Q})$, $f_-(Q) = \max_{\hat{Q}\leq Q} f(Q,\hat{Q})$.

The inner maximization problem of $f(Q,\hat{Q})$ over all distributions in $D(q)$, can be formulated as a semi-infinite linear optimization problem. By strong duality, the primal problem is equivalent to the following dual problem:

$$\begin{array}{lc} \min_{y_0,y_1} & y_0 + y_1\mu \\ s.t. & y_0 + y_1 x \geq \min\{x,\hat{Q}\} - \min\{x,Q\}, \quad \forall A\leq x\leq B. \end{array}$$

(a) For the case of $\hat{Q}\geq Q$. Using geometric interpretation, we can get the same result as in [3]:

$$f_+(Q) = \begin{cases} (p-c)(\mu-Q), & \text{if } Q\leq A+\frac{c}{p}(\mu-A), \\ (\sqrt{p(\mu-A)}-\sqrt{c(Q-A)})^2, & \text{if } A+\frac{c}{p}(\mu-A)\leq Q\leq A+\frac{c}{p}\frac{(B-A)^2}{\mu-A}, \\ (p\frac{\mu-A}{B-A}-c)(B-Q), & \text{if } Q\geq A+\frac{c}{p}\frac{(B-A)^2}{\mu-A}. \end{cases}$$

(b) For $\hat{Q}\leq Q$, we can get the following results by the same way:

$$f_-(Q) = \begin{cases} c(Q-\mu), & \text{if } Q\geq B-\frac{p-c}{p}(B-\mu), \\ (\sqrt{p(B-\mu)}-\sqrt{(p-c)(B-Q)})^2, & \text{if } B-\frac{p-c}{p}\frac{(B-A)^2}{B-\mu}\leq Q\leq B-\frac{p-c}{p}(B-\mu), \\ (p\frac{B-\mu}{B-A}-(p-c))(Q-A), & \text{if } Q\leq B-\frac{p-c}{p}\frac{(B-A)^2}{B-\mu}. \end{cases}$$

Let $\Gamma_1,\Gamma_2,\Gamma_3$ denote the three segments of $f_+(Q)$, $\tilde{\Gamma}_1,\tilde{\Gamma}_2,\tilde{\Gamma}_3$ denote the three segments of $f_-(Q)$ respectively. (the order is the same as the order listed in (a) (b) above.)

Notice the optimal order Q^* satisfies $f_+(Q^*) = f_-(Q^*)$, hence to minimize $\max\{f_+(Q), f_-(Q)\}$, we need to find the intersection point of $f_+(Q)$ and $f_-(Q)$. (take into account the monotonity of $f_+(Q)$ and $f_-(Q)$.) (Notice $\Gamma_1\cap\tilde{\Gamma}_1 = \varnothing$)

We consider the following two cases:

Case 1: $\frac{p}{c} \geq \frac{B-A}{\mu-A}$, then we need to find the intersection point between $\Gamma_1, \Gamma_2, \Gamma_3$ and $\tilde{\Gamma}_1, \tilde{\Gamma}_2$. (Actually part of $\tilde{\Gamma}_2$). We get

$$\begin{cases} \Gamma_3 \cap \tilde{\Gamma}_1, & \text{if } \frac{p}{c} \geq \max\{1+\frac{B-A}{\mu-A}, 1+2\frac{B-\mu}{\mu-A}\}, \\ \Gamma_3 \cap \tilde{\Gamma}_2, & \text{if } \frac{B-A}{\mu-A}+\frac{1}{2}\frac{B-A}{\mu-A}\sqrt{\frac{B-\mu}{\mu-A}} \leq \frac{p}{c} \leq 1+\frac{B-A}{\mu-A}, \\ \Gamma_2 \cap \tilde{\Gamma}_2, & \text{if } 1+\frac{1}{2}\sqrt{\frac{B-\mu}{\mu-A}} \leq \frac{p}{c} \leq \min\{1+2\sqrt{\frac{B-\mu}{\mu-A}}, \frac{B-A}{\mu-A}+\frac{1}{2}\frac{B-A}{\mu-A}\sqrt{\frac{B-\mu}{\mu-A}}\}, \\ \Gamma_2 \cap \tilde{\Gamma}_1, & \text{if } 1+2\sqrt{\frac{B-\mu}{\mu-A}} \leq \frac{p}{c} \leq 1+2\frac{B-\mu}{\mu-A}\}, \\ \Gamma_1 \cap \tilde{\Gamma}_2, & \text{if } \frac{p}{c} \leq 1+\frac{1}{2}\sqrt{\frac{B-\mu}{\mu-A}}. \end{cases}$$

Case 2: $\frac{p}{c} \leq \frac{B-A}{\mu-A} \Leftrightarrow \frac{p}{p-c} \geq \frac{B-A}{B-\mu}$, then we need to find the intersection point between Γ_1, Γ_2 and $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3$. (Actually part of Γ_2). (It's symmetric to case 1 under the following exchange: $p \leftrightarrow p$, $c \leftrightarrow (p-c)$, $A \leftrightarrow B$ and $\Gamma_i \leftrightarrow \tilde{\Gamma}_i$.) Then, we get

$$\begin{cases} \tilde{\Gamma}_3 \cap \Gamma_1, & \text{if } \frac{p}{p-c} \geq \max\{1+\frac{B-A}{B-\mu}, 1+2\frac{\mu-A}{B-\mu}\}, \\ \tilde{\Gamma}_3 \cap \Gamma_2, & \text{if } \frac{B-A}{B-\mu}+\frac{1}{2}\frac{B-A}{B-\mu}\sqrt{\frac{\mu-A}{B-\mu}} \leq \frac{p}{p-c} \leq 1+\frac{B-A}{B-\mu}, \\ \tilde{\Gamma}_2 \cap \Gamma_2, & \text{if } 1+\frac{1}{2}\sqrt{\frac{\mu-A}{B-\mu}} \leq \frac{p}{p-c} \leq \min\{1+2\sqrt{\frac{\mu-A}{B-\mu}}, \frac{B-A}{B-\mu}+\frac{1}{2}\frac{B-A}{B-\mu}\sqrt{\frac{\mu-A}{B-\mu}}\}, \\ \tilde{\Gamma}_2 \cap \Gamma_1, & \text{if } 1+2\sqrt{\frac{\mu-A}{B-\mu}} \leq \frac{p}{p-c} \leq 1+2\frac{\mu-A}{B-\mu}\}, \\ \tilde{\Gamma}_1 \cap \Gamma_2, & \text{if } \frac{p}{p-c} \leq 1+\frac{1}{2}\sqrt{\frac{\mu-A}{B-\mu}}. \end{cases}$$

Combing the above two cases, we get the results as follows:

(1) $\frac{B-\mu}{\mu-A} \leq 1$

$$Q^* = \begin{cases} A+\frac{p-c}{p}\frac{\mu-A}{B-\mu}(B-A), & \text{if } \frac{p}{c} \leq 1+\frac{1}{2}\frac{B-\mu}{\mu-A}, \\ B-\frac{(2p-c)^2}{4p(p-c)}(B-\mu), & \text{if } 1+\frac{1}{2}\frac{B-\mu}{\mu-A} \leq \frac{p}{c} \leq 1+\frac{1}{2}\sqrt{\frac{B-\mu}{\mu-A}}, \\ Q_0, & \text{if } 1+\frac{1}{2}\sqrt{\frac{B-\mu}{\mu-A}} \leq \frac{p}{c} \leq \frac{B-A}{\mu-A}+\frac{1}{2}\frac{B-A}{\mu-A}\sqrt{\frac{B-\mu}{\mu-A}}, \\ B-(\sqrt{\frac{p-c}{p}}-\sqrt{\frac{p-c}{p}-\frac{B-\mu}{B-A}})^2\frac{(B-A)^2}{B-\mu}, & \text{if } \frac{B-A}{\mu-A}+\frac{1}{2}\frac{B-A}{\mu-A}\sqrt{\frac{B-\mu}{\mu-A}} \leq \frac{p}{c} \leq 1+\frac{B-A}{\mu-A}, \\ B-\frac{c}{p}\frac{B-\mu}{\mu-A}(B-A), & \text{if } \frac{p}{c} \geq 1+\frac{B-A}{\mu-A}. \end{cases}$$

(2) $\frac{B-\mu}{\mu-A} \geq 1$

$$Q^* = \begin{cases} A + \frac{p-c}{p}\frac{\mu-A}{B-\mu}(B-A), & \text{if } \frac{p}{c} \le 1 + \frac{B-\mu}{B-A}, \\ A + (\sqrt{\frac{c}{p}} - \sqrt{\frac{c}{p} - \frac{\mu-A}{B-A}})^2 \frac{(B-A)^2}{\mu-A}, & \text{if } 1 + \frac{B-\mu}{B-A} \le \frac{p}{c} \le 1 + (\frac{\mu-A}{B-\mu} + \frac{1}{2}\frac{B-A}{B-\mu}\sqrt{\frac{\mu-A}{B-\mu}})^{-1}, \\ Q_0, & \text{if } 1 + (\frac{\mu-A}{B-\mu} + \frac{1}{2}\frac{B-A}{B-\mu}\sqrt{\frac{\mu-A}{B-\mu}})^{-1} \le \frac{p}{c} \le 1 + 2\sqrt{\frac{B-\mu}{\mu-A}}, \\ A + \frac{(p+c)^2}{4pc}(\mu-A), & \text{if } 1 + 2\sqrt{\frac{B-\mu}{\mu-A}} \le \frac{p}{c} \le 1 + 2\frac{B-\mu}{\mu-A}, \\ B - \frac{c}{p}\frac{B-\mu}{\mu-A}(B-A), & \text{if } \frac{p}{c} \ge 1 + 2\frac{B-\mu}{\mu-A}. \end{cases}$$

Therefore, we can write them together as follows:
$Q^* =$

$$\begin{cases} A + \frac{p-c}{p}\frac{\mu-A}{B-\mu}(B-A), & \text{if } \frac{p}{c} \le \min\{1 + \frac{1}{2}\frac{B-\mu}{\mu-A}, 1 + \frac{B-\mu}{B-A}\}, \\ B - \frac{(2p-c)^2}{4p(p-c)}(B-\mu), & \text{if } 1 + \frac{1}{2}\frac{B-\mu}{\mu-A} \le \frac{p}{c} \le 1 + \frac{1}{2}\sqrt{\frac{B-\mu}{\mu-A}}, \\ A + (\sqrt{\frac{c}{p}} - \sqrt{\frac{c}{p} - \frac{\mu-A}{B-A}})^2 \frac{(B-A)^2}{\mu-A}, & \text{if } 1 + \frac{B-\mu}{B-A} \le \frac{p}{c} \le 1 + (\frac{\mu-A}{B-\mu} + \frac{1}{2}\frac{B-A}{B-\mu}\sqrt{\frac{\mu-A}{B-\mu}})^{-1}, \\ Q_0, & \text{if } \max\{1 + (\frac{\mu-A}{B-\mu} + \frac{1}{2}\frac{B-A}{B-\mu}\sqrt{\frac{\mu-A}{B-\mu}})^{-1}, 1 + \frac{1}{2}\frac{B-\mu}{\mu-A}\} \\ & \le \frac{p}{c} \le \min\{1 + 2\sqrt{\frac{B-\mu}{\mu-A}}, \frac{B-A}{\mu-A} + \frac{1}{2}\frac{B-A}{\mu-A}\sqrt{\frac{B-\mu}{\mu-A}}\}, \\ B - (\sqrt{\frac{p-c}{p}} - \sqrt{\frac{p-c}{p} - \frac{B-\mu}{B-A}})^2 \frac{(B-A)^2}{B-\mu}, & \text{if } \frac{B-A}{\mu-A} + \frac{1}{2}\frac{B-A}{\mu-A}\sqrt{\frac{B-\mu}{\mu-A}} \le \frac{p}{c} \le 1 + \frac{B-A}{\mu-A}, \\ A + \frac{(p+c)^2}{4pc}(\mu-A), & \text{if } 1 + 2\sqrt{\frac{B-\mu}{\mu-A}} \le \frac{p}{c} \le 1 + 2\frac{B-\mu}{\mu-A}, \\ B - \frac{c}{p}\frac{B-\mu}{\mu-A}(B-A), & \text{if } \frac{p}{c} \ge \max\{1 + \frac{B-A}{\mu-A}, 1 + 2\frac{B-\mu}{\mu-A}\}. \end{cases}$$

and the RVAI amounts to
RVAI=

$$\begin{cases} (1 - \frac{p-c}{p}\frac{B-A}{B-\mu})(p-c)(\mu-A), & \text{if } \frac{p}{c} \le \min\{1 + \frac{1}{2}\frac{B-\mu}{\mu-A}, 1 + \frac{B-\mu}{B-A}\}, \\ \frac{c^2}{4p}(B-\mu), & \text{if } 1 + \frac{1}{2}\frac{B-\mu}{\mu-A} \le \frac{p}{c} \le 1 + \frac{1}{2}\sqrt{\frac{B-\mu}{\mu-A}}, \\ (1 - \sqrt{1 - \frac{p}{c}\frac{\mu-A}{B-A}})^2(\frac{c}{p}\frac{B-A}{\mu-A} - 1)c(B-A), & \text{if } 1 + \frac{B-\mu}{B-A} \le \frac{p}{c} \le 1 + (\frac{\mu-A}{B-\mu} + \frac{1}{2}\frac{B-A}{B-\mu}\sqrt{\frac{\mu-A}{B-\mu}})^{-1}, \\ (\sqrt{p(B-\mu)} - \sqrt{(p-c)(B-Q_0)})^2, & \text{if } \max\{1 + (\frac{\mu-A}{B-\mu} + \frac{1}{2}\frac{B-A}{B-\mu}\sqrt{\frac{\mu-A}{B-\mu}})^{-1}, 1 + \frac{1}{2}\frac{B-\mu}{\mu-A}\} \\ & \le \frac{p}{c} \le \min\{1 + 2\sqrt{\frac{B-\mu}{\mu-A}}, \frac{B-A}{\mu-A} + \frac{1}{2}\frac{B-A}{\mu-A}\sqrt{\frac{B-\mu}{\mu-A}}\}, \\ (1 - \sqrt{1 - \frac{p}{p-c}\frac{B-\mu}{B-A}})^2(\frac{p-c}{p}\frac{B-A}{B-\mu} - 1) & \\ \quad \cdot(p-c)(B-A), & \text{if } \frac{B-A}{\mu-A} + \frac{1}{2}\frac{B-A}{\mu-A}\sqrt{\frac{B-\mu}{\mu-A}} \le \frac{p}{c} \le 1 + \frac{B-A}{\mu-A}, \\ \frac{(p-c)^2}{4p}(\mu-A), & \text{if } 1 + 2\sqrt{\frac{B-\mu}{\mu-A}} \le \frac{p}{c} \le 1 + 2\frac{B-\mu}{\mu-A}, \\ (1 - \frac{c}{p}\frac{B-A}{\mu-A})c(B-\mu), & \text{if } \frac{p}{c} \ge \max\{1 + \frac{B-A}{\mu-A}, 1 + 2\frac{B-\mu}{\mu-A}\}. \end{cases}$$

6.6 The ordering that minimizes the maximum AEVD for distributions with given mean μ and standard deviation σ

Let $\mathcal{F}$ be the set of all pdf's with a given mean μ and a given standard deviation σ. Then all the two-point pdf's in $\mathcal{F}$ can be represented by $\{T(\gamma)|-t < \gamma < p\}$, where $T(\gamma)$ is a two-point pdf that assigns weights $\omega_1(\gamma) = \frac{t+\gamma}{p+t}$ and $\omega_2(\gamma) = \frac{p-\gamma}{p+t}$ to points

$$q_1(\gamma) = \mu - \sigma(\frac{p-\gamma}{t+\gamma})^{\frac{1}{2}} \text{ and } q_2(\gamma) = \mu + \sigma(\frac{t+\gamma}{p-\gamma})^{\frac{1}{2}},$$

respectively.

If $f = T(\gamma)$, the objective function $G_f(q)$ after some algebraic calculations, can be simplified as follows:

$$G_{T(\gamma)}(q) = \begin{cases} cq - t(q-\mu), & \text{if } q \leq q_1(\gamma), \\ cq + \sigma[(p-\gamma)(t+\gamma)]^{\frac{1}{2}} + \gamma(q-\mu), & \text{if } q_1(\gamma) \leq q \leq q_2(\gamma), \\ cq + p(q-\mu), & \text{if } q \geq q_2(\gamma). \end{cases}$$

Lemma 5. (Similar to Theorem 3 in [5]) For any q and $f \in \mathcal{F}$, there exists a two-point pdf $T(\gamma) \in \mathcal{F}$ with $-t < \gamma < p$ such that

$$AEVD_f(q) \leq AEVD_{T(\gamma)}(q).$$

Proof: We have two ways to prove it.

Method 1. Similar to the proof of Theorem 3 in paper [5]. (Note: The proof for $AEVD_f(q) \leq AEVD_{T(\gamma)}(q)$ is slightly different from the proof in [5], because the definition of $G_f(q)$ is different.)

Method 2: Use duality to prove it, which is pretty much the same as the proof for Lemma 3.

In view of Lemma 5, for any given decision q, we only have to search among two-point pdf's $T(\gamma)$ to find its largest AEVD, i.e., to find a parameter $-t \leq \gamma \leq p$ that maximizes $REVD_{T(\gamma)}(q)$:

$$\text{AEVD}_{max}(q) = \max_{-t\leq\gamma\leq p}\{\text{AEVD}_{T(\gamma)}(q)\}.$$

From the geometric interpretation we can define

$$\begin{aligned} V_+(q,\gamma) &= \text{AEVD}_{T(\gamma)}(q)|_{-c\leq\gamma_1\leq p, q_1(\gamma)\leq q\leq q_2(\gamma)} \\ &= [G_{T(\gamma)}(q) - G_{T(\gamma)}(q_1(\gamma))]|_{q_1(\gamma)\leq q\leq q_2(\gamma)} \\ &= (q-\mu+\sigma\sqrt{\tfrac{p-\gamma}{t+\gamma}})(\gamma+c) \end{aligned}$$

$$\begin{aligned} V_-(q,\gamma) &= \text{AEVD}_{T(\gamma)}(q)|_{-t\le\gamma\le -c, q_1(\gamma)\le q\le q_2(\gamma)} \\ &= [G_{T(\gamma)}(q) - G_{T(\gamma)}(q_2(\gamma))]|_{q_1(\gamma)\le q\le q_2(\gamma)} \\ &= (q-\mu-\sigma\sqrt{\tfrac{t+\gamma}{p-\gamma}})(\gamma+c) \end{aligned}$$

Lemma 6.

$$\max_{-t\le\gamma\le p}\{\text{AEVD}_{T(\gamma)}(q)\} = \max\{\max_{-c\le\gamma_1\le p} V_+(q,\gamma_1), \max_{-t\le\gamma_2\le -c} V_-(q,\gamma_2)\}$$

Proof: We omitted the proof because it is only slightly different from the proof in [5]. The only difference is that we take into account the ordering cost 'c'.

Then, combining with the result in Lemma 5, we have

$$\text{AEVD}_{max}(q) = \max\{\max_{-c\le\gamma_1\le p} V_+(q,\gamma_1), \max_{-t\le\gamma_2\le -c} V_-(q,\gamma_2)\}.$$

Let q^e be the decision that minimizes the largest AEVD, that is

$$q^e = \text{arg min } \text{AEVD}_{max}(q).$$

Since $V_+(q,\gamma_1) \le 0$ for all $q \le q_1(\gamma_1)$, $-c \le \gamma_1 \le p$ and $V_-(q,\gamma_2) \le 0$ for all $q \ge q_2(\gamma_2)$,$-t \le \gamma_2 \le -c$. Then $\max_{-c\le\gamma_1\le p} V_+(q,\gamma_1) \le 0$ for all $q \le q_1(-c)$ and $\max_{-t\le\gamma_2\le -c} V_-(q,\gamma_1) \le 0$ for all $q \ge q_2(-c)$, because $\min_{-c\le\gamma_1\le p} q_1(\gamma_1) = q_1(-c)$ and $\max_{-t\le\gamma_2\le -c} q_2(\gamma_2) = q_2(-c)$.

Notice that $V_+(q,\gamma_1)$ and $\max_{-c\le\gamma_1\le p} V_+(q,\gamma_1)$ are both increasing functions of q, while $V_-(q,\gamma_2)$ and $\max_{-t\le\gamma_2\le -c} V_-(q,\gamma_2)$ are both decreasing functions of q. It follows that

$$q_1(-c) \le q_e \le q_2(-c)$$

and

$$\max_{-c\le\gamma_1\le p} V_+(q^e,\gamma_1) = \max_{-t\le\gamma_2\le -c} V_-(q^e,\gamma_2).$$

Therefore, we can simplify the formula for $AEVD_{max}(q)$ as follows:

$$AEVD_{max}(q) = \begin{cases} \max_{-c\le\gamma_1\le p} V_+(q,\gamma_1), & \text{if } q \ge q^e, \\ \max_{-t\le\gamma_2\le -c} V_-(q,\gamma_2), & \text{if } q \le q^e. \end{cases}$$

We can further simplify it by introducing the following variable transformations:

$$\theta = \frac{q-\mu}{\sigma},\ \alpha = \frac{t-c}{p},\ \beta = \frac{p+c}{p},\ x = \frac{\gamma_1+c}{p},\ y = -\frac{\gamma_2+c}{p}.$$

Then, the above formula becomes

$$AEVD_{max}(q) = (p\sigma) \begin{cases} \max_{0 \le x \le \beta} g_+(\theta, x), & \text{if } q \ge q^e, \\ \max_{0 \le y \le \alpha} g_-(\theta, y), & \text{if } q \le q^e, \end{cases}$$

where

$$g_+(\theta, x) = V_+(q, \gamma_1) = [\theta + \sqrt{\frac{\beta - x}{\alpha + x}}]x,$$

$$g_-(\theta, y) = V_-(q, \gamma_2) = [-\theta + \sqrt{\frac{\alpha - y}{\beta + y}}]y.$$

The above formula illustrated how to calculate the maximum AEVD for any given decision q. Now, let us calculate q^e by the following optimization procedure:

$$\max_{0 \le x \le \beta, 0 \le y \le \alpha} g_+(\theta, x) \ s.t. \ g_+(\theta, x) = g_-(\theta, x).$$

Eliminating θ from the constraint and then plug it into the objective function, we get the following non-constrained optimization problem:

$$\max_{0 \le x \le \beta, 0 \le y \le \alpha} g_{\alpha,\beta}(x, y),$$

where

$$g_{\alpha,\beta}(x, y) = \frac{xy}{x + y}(\sqrt{\frac{\beta - x}{\alpha + x}} + \sqrt{\frac{\alpha - y}{\beta + y}}).$$

Theorem 4. Let (x^*, y^*) be the solution of the above optimization problem. Then, the min-max AEVD and the associated decision q^e are given by

$$AEVD_{max}(q^e) = (p\sigma)g(x^*, y^*) \quad \text{and} \quad q^e = \mu + \theta^e \sigma,$$

where

$$\theta^e = \frac{1}{x^* + y^*}(-\sqrt{\frac{\beta - x^*}{\alpha + x^*}}x^* + \sqrt{\frac{\alpha - y^*}{\beta + y^*}}y^*).$$