Semidefinite Programming and Universal Rigidity

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Outline

Introduction to Semidefinite Programming

SDP Solution Rank Theorems

Sensor Network Localization and Graph Realization

SDP Relaxation and Localizability

Semidefinite Programming Problem (SDP)

Find a real $n \times n$ symmetric matrix $X \in \mathcal{S}^n$ that satisfies

$$A_i \bullet X = b_i, \quad \forall i = 1, \ldots, m, \quad X \succeq \mathbf{0}$$

where $A_1, \ldots, A_m$ are given real symmetric matrices with real scalars $(b_1, \ldots, b_m)$,

$$A \bullet X = \sum_{i,j} a_{ij} x_{ij},$$

and $\succeq$ represents the matrix inequality of positive semi-definiteness.

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$$x_1 + x_2 + x_3 = 1,$$
$$\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq \mathbf{0}.$$

Semidefinite Programming Optimization Problem

$$
\begin{array}{llll}
(SDP) & \inf & C \bullet X & \\
& \text{subject to} & A_i \bullet X = b_i & i = 1, \ldots, m, \\
& & X \succeq \mathbf{0}. &
\end{array}
$$

where C is a given (objective) symmetric matrix. The dual problem to (SDP) can be written as:

$$
\begin{array}{lll}
(SDD) & \sup & \mathbf{b} \cdot \mathbf{y} \\
& \text{subject to} & S = C - \sum_i^m y_i A_i \succeq \mathbf{0},
\end{array}
$$

where $\mathbf{y} \in \mathbf{R}^m$.

Duality Theorems for SDPs

Theorem

(Weak duality theorem in SDP) Let $\mathcal{F}_p$ and $\mathcal{F}_d$, the feasible sets for the primal and dual, be non-empty. Then,

$$C \bullet X \geq \mathbf{b} \cdot \mathbf{y}, \quad \forall \quad X \in \mathcal{F}_p, \ (\mathbf{y}, S) \in \mathcal{F}_d.$$

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Theorem

(Strong duality theorem in SDP) Let $\mathcal{F}_p$ and $\mathcal{F}_d$ be non-empty and at least one of them has an interior. Then, X is optimal for (SDP) and $(\mathbf{y}, S)$ optimal for (SDD) if and only if the following conditions hold:

i) $X \in \mathcal{F}_p$ *and* $(\mathbf{y}, S) \in \mathcal{F}_d$;

ii) $C \bullet X = \mathbf{b} \cdot \mathbf{y}$ *or* $X \bullet S = 0$ *or* $XS = \mathbf{0}$.

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If there is S^* such that $\text{rank}(S^*) \geq n - d$, then the rank of any X^* is bounded above by d.

SDP Solution Uniqueness: Zhu 2010

Theorem

Let X^* be a max-rank solution of (SDP) and let $X^* = P^T P$ where $P \in \mathbf{R}^{r \times n}$. Then, X^* is the unique solution for (SDP) if and only if the null space of the linear space spanned by $PA_iP^T,\ i = 1, \ldots, m$, is $\{\mathbf{0}\}$.

Corollary

If all (SDP) solutions share the same rank, then (SDP) has a unique solution.

SDP Computational Complexity and Solution Rank

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- Barvinok (1995) showed that if the problem is solvable, then there exists a solution X whose rank r satisfies $r(r+1) \leq 2m$, and it is essentially tight.
- Given any solution, a such low-rank solution can be found in polynomial time based on Carathéodory's theorem (Pataki 1999, Alfakih/Wolkowicz 1999).

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- We are interested in finding a fixed low–rank (say d) solution to the SDP problem.
- However, there are some issues:
  - Such a solution may not exist!
  - Even if it does, one may not be able to find it efficiently.
- Suppose we consider an approximate low-rank solution to

$$A_i \bullet X = b_i, \quad \forall i = 1, \ldots, m, \quad X \succeq \mathbf{0},$$

where $A_i \succeq \mathbf{0}$ for all i.

Approximate Low-Rank SDP Solution

More precisely, find an $\hat{X} \succeq 0$ with rank at most d that satisfies the SDP constraints approximately:

$$\beta(m,n,d) \cdot b_i \le A_i \bullet \hat{X} \le \alpha(m,n,d) \cdot b_i \quad \forall\ i = 1, \ldots, m.$$

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Clearly, the closer are both to 1, the better the approximation quality.

Approximation Result, So, Y and Zhang 2007

Theorem

For any $d \geq 1$*, there exists an* $\hat{X} \succeq \mathbf{0}$ *with* $rank(\hat{X}) \leq d$ *such that*

$$\alpha(m,n,d) = \begin{cases} 1 + \dfrac{18\ln(2m)}{d} & \text{for } 1 \leq d \leq 18\ln(2m) \\ 1 + \sqrt{\dfrac{18\ln(2m)}{d}} & \text{for } d > 18\ln(2m) \end{cases},$$

$$\beta(m,n,d) = \begin{cases} \dfrac{1}{e(2m)^{2/d}} & \text{for } 1 \leq d \leq 4\ln(2m) \\ \max\left\{\dfrac{1}{e(2m)^{2/d}}, 1 - \sqrt{\dfrac{4\ln(2m)}{d}}\right\} & \text{for } d > 4\ln(2m) \end{cases}$$

Some Remarks

- There is always a low-rank approximate SDP solution with respect to a bounded distortion. As the allowable rank increases, the distortion bounds become smaller and smaller.

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- Moreover, there exists an efficient randomized algorithm for finding such an $\hat{X}$.
- The distortion factors are independent of n.
- The factors are sharp; but they can be improved if we only consider one–sided inequalities.
- This result contains as special cases several well-known results in the literature.

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SDP Relaxation and Localizability

Sensor Network Localization and Graph Realization

Given a graph $G = (V, E)$ and sets of non–negative weights, say $\{d_{ij} : (i, j) \in E\}$ on edges, the goal is to compute a realization of G in the Euclidean space $\mathbf{R}^d$ for a given low dimension d. That is,

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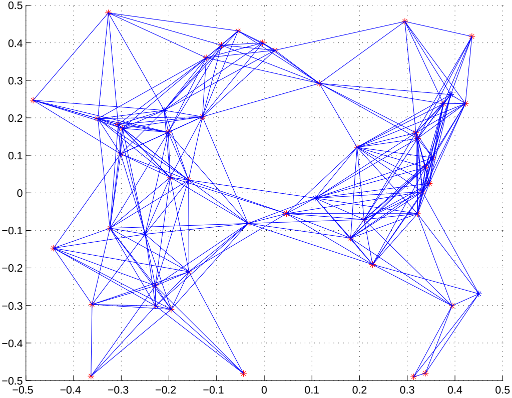
- the Euclidean distance between a pair of adjacent vertexes (i, j) equals to (or bounded by) the prescribed weight $d_{ij} \in E$.

The positions of a few vertexes are known and they are called anchors for SNL.

50-node 2-D Sensor Localization



Quadratic Equality and Inequality Systems

Given graph $G = (V, E)$ and $d_{ij} \in E$, find $\mathbf{x}_j \in \mathbf{R}^d$ such that

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Or given $\mathbf{a}_k \in \mathbf{R}^d$, $d_{ij} \in N_x$, and $\hat{d}_{kj} \in N_a$, find $\mathbf{x}_i \in \mathbf{R}^d$ such that

$$\begin{array}{lll} \|\mathbf{x}_i - \mathbf{x}_j\|^2 & (\leq) = (\geq) & d_{ij}^2, \ \forall\ (i,j) \in N_x,\ i < j, \\ \|\mathbf{a}_k - \mathbf{x}_j\|^2 & (\leq) = (\geq) & \hat{d}_{kj}^2, \ \forall\ (k,j) \in N_a; \end{array}$$

that is, edge (ij) (or (kj)) connects sensors i and j (or anchor k and sensor j) with the Euclidean length equal to d_{ij} (or $\hat{d}_{kj}$).

Key Questions Related to SNL

Let the "bar system" be:

$$\begin{aligned} \|\mathbf{x}_i - \mathbf{x}_j\|^2 &= d_{ij}^2, \ \forall\ (i,j) \in N_x,\ i < j, \\ \|\mathbf{a}_k - \mathbf{x}_j\|^2 &= \hat{d}_{kj}^2, \ \forall\ (k,j) \in N_a, \end{aligned}$$

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- Does the system have a localization or configuration for all $\mathbf{x}_j$'s?
- Is the localization/configuration unique, and it can be (approximately) certified?
- Is the system partially localizable with an (approximate) certification?

Outline

Matrix Representation

Find $Y = X^T X$, where X is $d \times n$, such that

$$\begin{array}{rcl} (\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)^T \bullet \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} & = & d_{ij}^2, \ \forall\ i, j \in N_x,\ \ i < j, \\ (\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \bullet \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} & = & \hat{d}_{kj}^2, \ \forall\ k, j \in N_a. \end{array}$$

where $\mathbf{e}_j$ is the vector of all zeros except 1 at the jth position.

SDP Relaxation of SNL, Biswas and Y 2004

Relax $Y = X^T X$ to $Y \succeq X^T X$;

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Relax $Y = X^T X$ to $Y \succeq X^T X$;or equivalently

$$Z := \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succeq \mathbf{0}.$$

Then, we face a standard SDP (feasibility) problem.

$$\begin{array}{rcl} (\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)^T \bullet Z & = & d_{ij}^2, \ \forall\ i,j \in N_x,\ i < j, \\ (\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \bullet Z & = & \hat{d}_{kj}^2, \ \forall\ k,j \in N_a, \\ Z & \succeq & \mathbf{0}. \end{array}$$

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- A solution matrix Z has rank at least d.
- It's d if and only if $Y = X^T X$ and it produces one localization to the original SNL problem.
- If there exists a dual solution with rank n, then the rank of any primal solution Z must be d.

The Dual of the SDP Relaxation

$$
\begin{array}{ll}
\text{minimize} & I \bullet V + \sum_{i<j \in N_x} w_{ij} d_{ij}^2 + \sum_{k,j \in N_a} \hat{w}_{kj} \hat{d}_{kj}^2 \\
\text{subject to} & \begin{pmatrix} V & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \sum_{i<j \in N_x} w_{ij} (\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)^T \\
& + \sum_{k,j \in N_a} w_{kj} (\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T = S \succeq 0,
\end{array}
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where variable matrix $V \in \mathcal{S}^d$, variable w_{ij} is the (stress) weight on edge between $\mathbf{x}_i$ and $\mathbf{x}_j$, and $\hat{w}_{kj}$ is the (stress) weight on edge between $\mathbf{a}_k$ and $\mathbf{x}_j$.

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- The max-rank of any optimal dual stress matrix S is n.
- It's n if the problem admits strict complementarity.

Unique Localizability and Uniquely Localizable Problem I

A sensor network is uniquely-localizable (UL) if there exists a unique localization in $\mathbf{R}^2$ and there is no nontrivial localizations, $x_j \in \mathbf{R}^h,\ j = 1, ..., n$, where $h > 2$, such that

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = d_{ij}^2,\ \forall\ i,j \in N_x,\ i < j,$$
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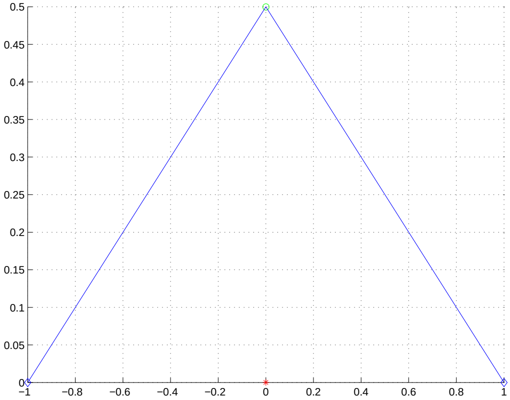
It basically says that the problem cannot be localized in a higher dimension space where anchor points are simply augmented to $(\mathbf{a}_k; \mathbf{0}) \in \mathbf{R}^h$, $k = 1, ..., m$.

One sensor-Two anchors: Not localizable



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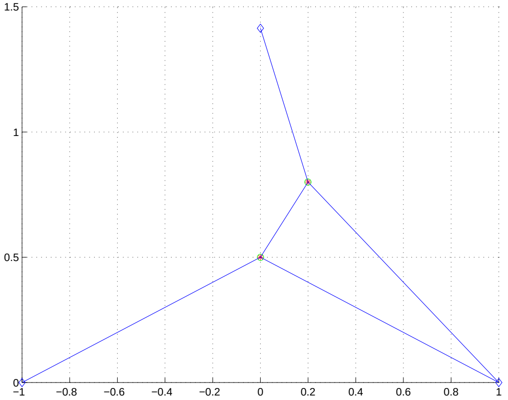
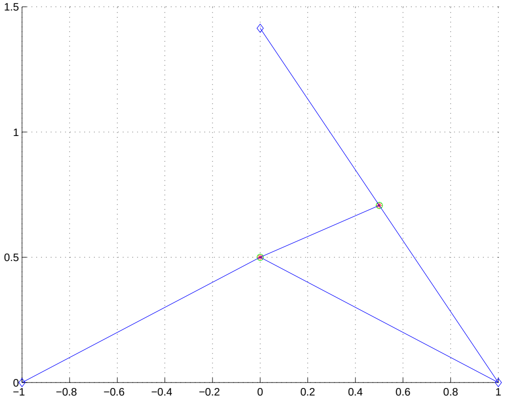
- It's also necessary for every sensor to have at least $d+1$ edges.
- For a UL instance, if there exists a dual stress matrix with rank n, then it is called strongly-localizable (SL).
- UL is stronger than the notion of *Global Rigidity*, and it is most similar to *Universal Rigidity*.

Two sensor-Three anchors: Strongly Localizable



Two sensor-Three anchors: Localizable but not Strongly



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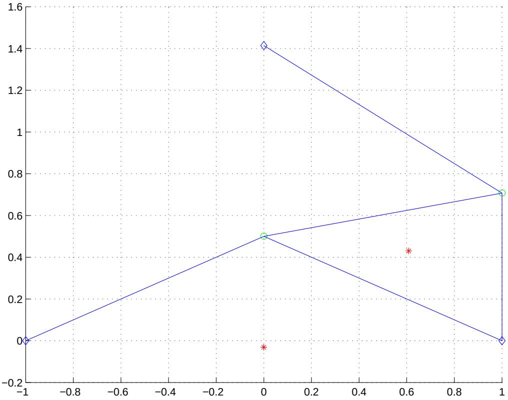

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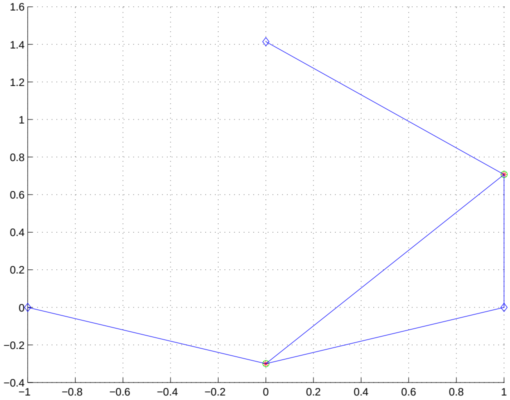

An Equivalence Theorem, So and Y 2005

Theorem

The following statements are equivalent:

1. *The sensor network is uniquely-localizable;*
2. *The max-rank solution of the SDP relaxation has rank d;*
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Moreover, the localization of a uniquely localizable instance can be computed approximately in a time polynomial in n, d, *and the accuracy* $\log(1/\epsilon)$.

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The existence of a rank n stress matrix (or strong localizability) is sufficient but not necessary for UL.

Localize All Localizable Points, So and Y 2005

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If an SNL problem (graph) contains a subproblem (subgraph) that is strongly localizable, then the solution submatrix corresponding to the subproblem in the SDP has rank d. Thus, the SDP relaxation computes a solution that localize all possibly localizable sensor points.

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Diagonals of the "co-variance" matrix

$$\hat{Y} - \hat{X}^T \hat{X},$$

can be a certification for uniquely localizable sensor points. That is, $\hat{Y}_{jj} - \|\hat{\mathbf{x}}_j\|^2 = 0$ if any only if the jth sensor point has a unique localization.

Uniquely-Localizable Graphs

Theorem

Let the SNL problem have at least $d+1$ anchors and they, together with all sensors, are in general positions.

- *If G is complete or every edge length is specified, then the sensor network is uniquely-localizable (Schoenberg 1942).*

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- *If G is complete or every edge length is specified, then the sensor network is uniquely-localizable (Schoenberg 1942).*
- *The $(d + 1)$-lateration graph is uniquely-localizable (So 2006 and Zhu, So and Y 2009). Moreover, it is near the sparsest graph (with only $(d + 1)n$ edges) that is uniquely-localizable.*

$(d+1)$-Lateration Graph

A $(d+1)$-lateration ordering for a graph G is an ordering of the vertexes $1, \cdots, d+1, d+2, \cdots, n$ such that the first $d+1$ vertexes form the complete graph, and every vertex $j > d+1$ has $d+1$ edges connected to its preceding vertexes on the sequence.

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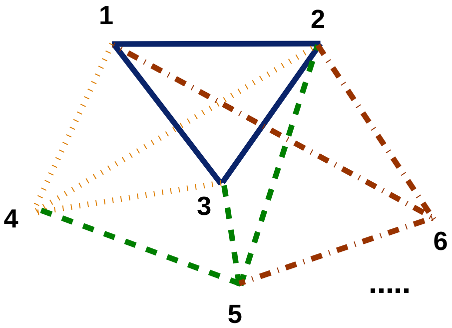

Universal Rigidity Theorems in Generic Position

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- A framework problem that contains a spanning $(d + 1)$-lateration graph is universally rigid if and only if there exists a max-rank PSD stress matrix (Alfakih, Taheri and Y 2010).
- Given configuration/position matrix P and the lateration order, such a PSD stress matrix can be computed exactly in strongly polynomial time.

Proof Sketch I

Let the extended position matrix

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Recall that symmetric matrix S is a stress matrix if and only if

$$\text{orthogonality:} \quad AS = \mathbf{0}, \tag{1}$$

and

$$\text{purity:} \quad S_{ij} = 0, \ \forall (i,j) \notin E. \tag{2}$$

Proof Sketch II

- We start a PSD matrix satisfies orthogonality condition (1), say

$$S^0 = I - A^T(AA^T)^{-1}A,$$

where the columns of A are ordered according to the lateration order, and we call it a "prestress" matrix.

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- In at most n steps, we reach a “prestress” matrix that finally satisfies purity condition (2).

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- Sufficient and necessary conditions based on the max-rank condition of primal SDP matrix solutions?
- Sufficient and necessary conditions of UL for points in general positions?