

Mathematical Optimization in Machine Learning/Decision-Making

ZHIJIANG LAB, HANGZHOU

JUNE 30, 2023

Yinyu Ye

Stanford University and CUHKSZ (Sabbatical Leave)

Ask ChatGPT about an Optimization Problem?

Knapsack on ChatGPT:



假设有5件物品和一个最多能承重5kg的背包。第1件物品的重量是2kg，其价值为18元；第2件物品的重量是1kg，其价值为10元；第3件物品的重量是2.5kg，其价值为20元；第4件物品的重量是3.5kg，其价值为28元；第5件物品的重量是1.5kg，其价值为15元。请问将哪些物品装入背包里，物品的价值总和最大？请使用整数规划求解该问题。



这是一个经典的背包问题，可以使用整数规划方法求解。

我们可以定义0/1变量 x_i ，表示是否将第 i 件物品放入背包中。同时，我们还需要定义一个非负整数变量 z ，表示物品的总价值。

然后，我们可以将问题转化为以下的线性规划问题：

$$\text{最大化 } z = 18x_1 + 10x_2 + 20x_3 + 28x_4 + 15x_5$$

满足约束条件：

$$2x_1 + x_2 + 2.5x_3 + 3.5x_4 + 1.5x_5 \leq 5$$

$$x_i \in \{0,1\}, i=1,2,3,4,5$$

由于 x_i 是整数变量，这是一个整数规划问题。可以使用整数规划求解器求解。

求解结果表明，将第1件、第3件、第5件物品装入背包中，可以使价值最大，为 $18+20+15=53$ 元。



ChatGPT 在不调用求解器的前提下，能否求解大规模优化问题？



在不调用求解器的前提下，对于大规模的优化问题，通常也比较难直接求解得到最优解，因为现实中的大多数优化问题都非常复杂，存在着多个目标、多个约束、非线性、离散等多种问题，很容易遇到组合爆炸的问题。



ChatGPT:

- * ChatGPT tried to formulate problem and attempted to give an answer, but could not guarantee the correctness.
- * ChatGPT highly recommended using reliable optimization solvers and consulting specialists

Linear Programming and LP Giants won Nobel Prize...

$$\begin{aligned} & \max \quad \sum \pi_j x_j \\ \text{s.t.} \quad & \sum_j a_j x_j \leq b, \\ & 0 \leq x_j \leq 1 \quad \forall j = 1, \dots, n \end{aligned}$$



Today's Sharing

1. Online Linear Programming Algorithms and Applications

2. Accelerated Second-Order Methods for Nonlinear Optimization and Applications

3. Zero-Order Optimization Algorithms and Applications

4. Pre-Trained Mixed Integer Linear Programming Solvers and Applications

Topic 1. Online Linear Programming an Online Auction Example

- There is a fixed selling period or number of buyers; and there is a fixed inventory of goods
- Customers come and require a bundle of goods and make a bid
- Decision: To sell or not to sell to each individual customer on the fly?
- Objective: Maximize the revenue.

Bid #	\$100	\$30	Inventory
Decision	x1	x2				
Pants	1	0	100
Shoes	1	0				50
T-Shirts	0	1				500
Jackets	0	0				200
Hats	1	1	1000

Online Linear Programming Model and Theory

- OLP theory and practice (Agrawal et al. 2010, 14, Li&Y 2022)

- OLP:

$$\begin{aligned} \max \quad & \sum \pi_j x_j \\ \text{s.t.} \quad & \sum_j a_j x_j \leq b, \\ & 0 \leq x_j \leq 1 \quad \forall j = 1, \dots, n \end{aligned}$$

- Variables together with their data points arrive sequentially and decision makers need decide x_j on the fly, that is, before knowing the “future” data points
- Learning-while-Doing vs Learning-First and Deciding-Second (collect and learn all relevant data, then solve for all \mathbf{x})
- Offline LP’s objective value is a upper bond for the online version
- Is there an optimal online decision algorithm/mechanism⁶

Price Mechanism for Online Auction

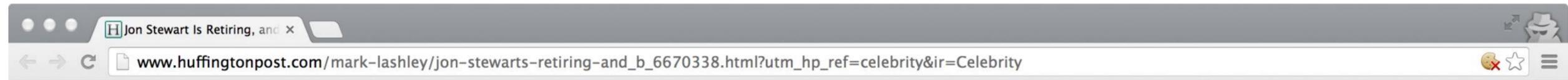
- Learn and compute itemized optimal prices
- Use the prices to price each bid internally
- Accept if it is a over bid, and reject otherwise
- There is an Optimal Online Algorithm to achieve the best you could do!

Bid #	\$100	\$30	Inventory	Price?
Decision	x1	x2					
Pants	1	0	100	45
Shoes	1	0				50	45
T-Shirts	0	1				500	10
Jackets	0	0				200	55
Hats	1	1	1000	15

$$\max \sum \pi_j x_j \quad \text{s.t.} \quad \sum_j a_j x_j \leq \mathbf{b}, \quad x_j \geq 0 \quad \forall j = 1, \dots, J$$

- The decision variable x_j represents the **total-times of pulling** the j -th arm.
- We have developed a two-phase algorithm
 - **Phase I**: Distinguish the optimal **super-basic** variables/arms from the optimal **non-basic** variables/arms with as fewer number of plays as possible
 - **Phase II**: Use the arms in the optimal face to exhaust the resource through an adaptive procedure and achieve **fairness**
- The algorithm achieves a problem dependent regret that bears a **logarithmic** dependence on the horizon T . Also, it identifies a number of LP-related parameters as the **bottleneck or condition-numbers** for the problem
 - Minimum non-zero **reduced cost**
 - Minimum **singular-values** of the optimal basis matrix.
- **First algorithm** to achieve the $O(\log T)$ regret/gap bound [Li, Sun & Y 2021 ICML] (<https://proceedings.mlr.press/v139/li21s.html>)

App. I: Online Matching for Display Advertising



Mark Lashley Assistant Professor, La Salle University

Jon Stewart Is Retiring, and it's Going to Be (Kind of) Okay

Posted: 02/13/2015 3:21 pm EST | Updated: 02/13/2015 3:59 pm EST



195 12 5 0 14

Like Share Tweet Pin it Comment

When the news broke Tuesday night that longtime *Daily Show* host Jon Stewart would be leaving his post in the coming months, the level of trauma on the internet was palpable. Some expected topics arose, within hours -- minutes, even -- of the announcement trickling out. Why would Stewart leave now? What's his plan? Who should replace him? Could the next *Daily Show* host be a woman? (Of course). Is this an elaborate ruse for Stewart to take over the *NBC Nightly News*? (Of course not).

The public conversation over the past two days has been so Stewart-centric that the retirement news effectively pushed NBC anchor Brian Williams's suspension off of social media's front pages. Part of that is the shock; we knew the other shoe was about to drop with (on?) Williams, but Stewart's departure was known only to Comedy Central brass before it was revealed to his studio audience. Part of it is how meme-worthy the parallels between the two hosts truly are ("fake newsman speaks truth, real newsman spins lies," some post on your Twitter timeline probably read). Breaking at

ADVERTISEMENT

AdChoices

Whatever your lifestyle, discover banking solutions built around it.

Visit the Citi Benefits Hub

citi

SUGGESTED FOR YOU



Incredible Seal Vs Octopus Battle Caught On Camera



阿里巴巴在2019年云栖大会上提到在智能履行决策上使用OLP的算法

2018 杭州·云栖大会 Alibaba Group

智能履行决策

商家

杭州-上海 杭州-广州 杭州-北京 杭州-武汉 ...

YTO ZTO YUNDA

菜鸟智能发货引擎

时效	服务	成本	单量平衡	...
线路容量	网点容量	局部优化	全局优化	...

最优快递

智能决策 ML & Optimization

商家的履行是带有全局约束的序列执行决策

- Online assignment problem
- Control based method
- Online linear programming

Ref: Agrawal, Shipra, Zizhuo Wang, and Yinyu Ye. "A dynamic near-optimal algorithm for online linear programming." *Operations Research* 62.4 (2014): 876-890.

决策变量

$$C_{ij} = c1 * \text{成本} + c2 * \text{服务} + c3 * \text{时效}$$
$$\max_x \sum_{i=1}^n \sum_{j=1}^m C_{ij} x_{ij}$$

将订单 I 匹配给快递公司 j 与否

$$\text{s.t.} \sum_{j=1}^m x_{ij} \leq 1$$

商家发货CP总单量比例约束

$$\sum_{i=1}^n x_{ij} * a_j \leq u_j$$

全局约束值, 比如总成本

$$\sum_{i=1}^n \sum_{j=1}^m x_{ij} b_{k,ij} \leq B_k$$

阿里巴巴团队在2020年CIKM会议论文Online Electronic Coupon Allocation based on Real-Time User Intent Detection上提到他们设计的发红包的机制也使用了OLP的方法 [2]

Spending Money Wisely: Online Electronic Coupon Allocation based on Real-Time User Intent Detection

Liangwei Li*
Liucheng Sun*
leon.llw@alibaba-inc.com
liucheng.slc@alibaba-inc.com
Alibaba Group
Hangzhou, Zhejiang

Chengfu Huo
chengfu.huocf@alibaba-inc.com
Alibaba Group
Hangzhou, Zhejiang

Chenwei Weng
wengchenwei.pt@alibaba-inc.com
Alibaba Group
Hangzhou, Zhejiang

Weijun Ren
afei@alibaba-inc.com
Alibaba Group
Hangzhou, Zhejiang

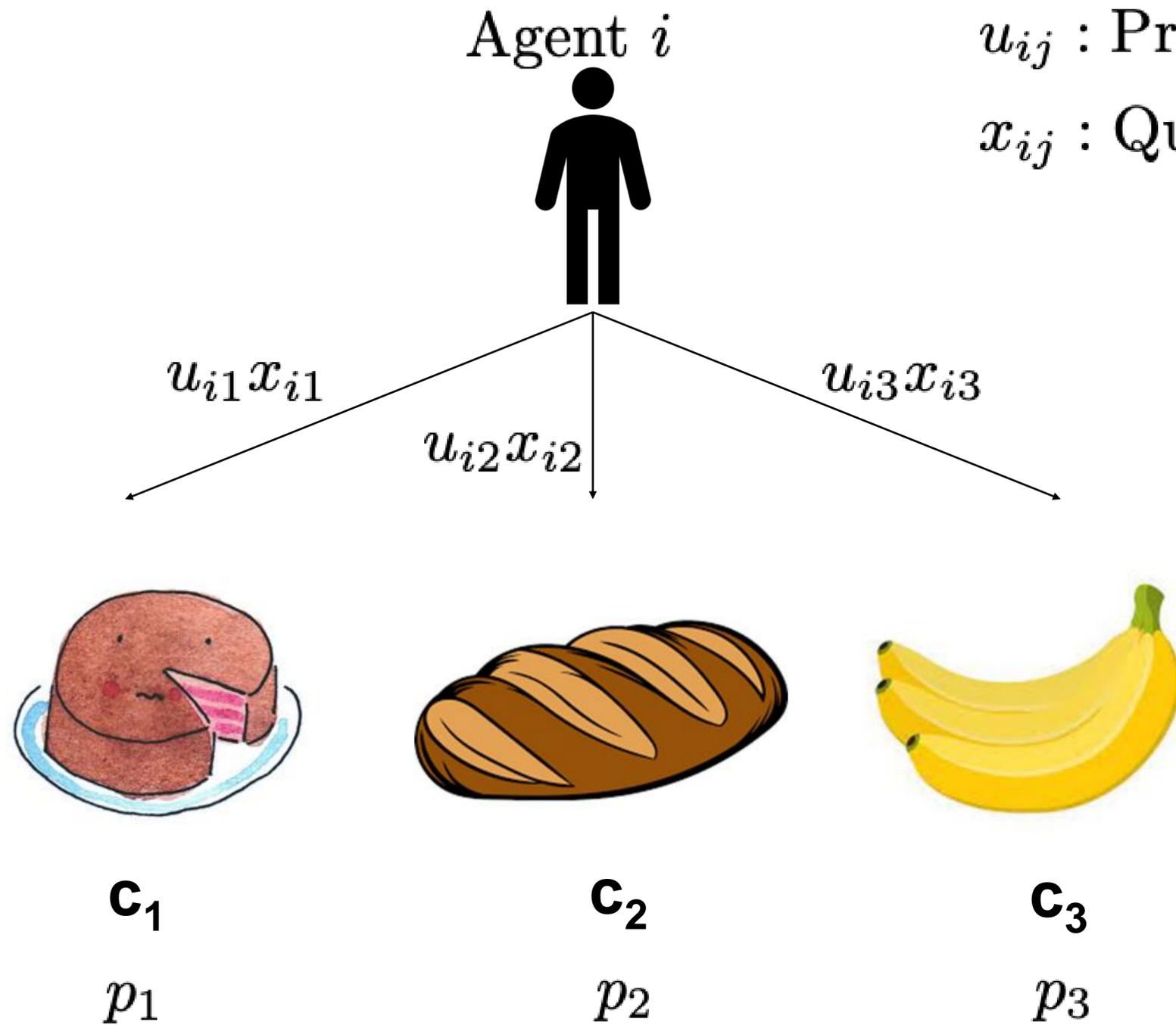
$$\begin{aligned} & \max \sum_{i=1}^M \sum_{j=1}^N v_{ij} x_{ij} \\ & \text{s.t. } \sum_{i=1}^M \sum_{j=1}^N c_j x_{ij} \leq B, \\ & \sum_j x_{ij} \leq 1, \quad \forall i \\ & x_{ij} \geq 0, \quad \forall i, j \end{aligned} \quad (5)$$

3.3 MCKP-Allocation

We adopt the primal-dual framework proposed by [2] to solve the problem defined in Equation 5. Let α and β_j be the associated dual variables respectively. After obtaining the dual variables, we can solve the problem in an online fashion. Precisely, according to the principle of the primal-dual framework, we have the following allocation rule:

$$x_{ij} = \begin{cases} 1, & \text{where } j = \arg \max_i (v_{ij} - \alpha c_j) \\ 0, & \text{otherwise} \end{cases} \quad (9)$$

APP II: A key resource allocation model to achieve efficient allocation is the Fisher Market



u_{ij} : Preference of Agent i for one unit of good j

x_{ij} : Quantity of good j purchased by person i

p_j : Price of Good j

w_i : Budget of Agent i

Individual Optimization Problem:

$$\max_{\mathbf{x}_i} \sum_j u_{ij} x_{ij}$$

$$\text{s.t. } \mathbf{p}^T \mathbf{x}_i \leq w_i$$

$$\mathbf{x}_i \geq \mathbf{0}$$

Do Prices exist to clear the market?

(Prices are posted and known to all agents so that they have freedom to choose)³

$M = \text{Total Number of Goods}$

The prices can be derived from a centralized optimization problem with a budget-weighted social objective (Eisenberg-Gale)

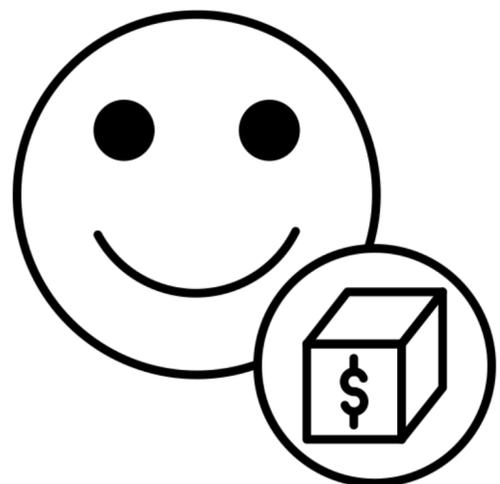
Individual Optimization Problem:

$$\begin{aligned} \max_{\mathbf{x}_i} \quad & \sum_j u_{ij} x_{ij} \\ \text{s.t.} \quad & \mathbf{p}^T \mathbf{x}_i \leq w_i \\ & \mathbf{x}_i \geq \mathbf{0} \end{aligned}$$

Social Optimization Problem:

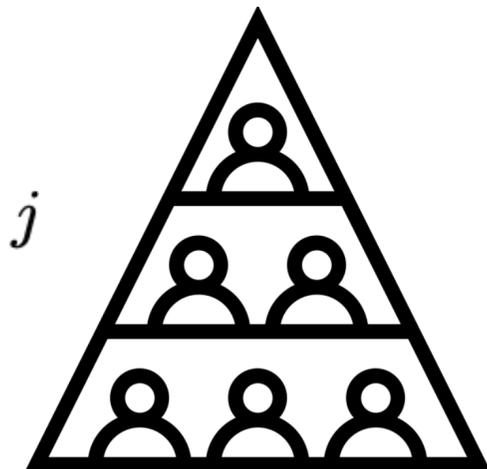
$$\begin{aligned} \max_{\mathbf{x}_i, \forall i \in [N]} \quad & \sum_i w_i \log \left(\sum_j u_{ij} x_{ij} \right) \\ \text{s.t.} \quad & \sum_i x_{ij} \leq c_j, \forall j \in [M] \\ & x_{ij} \geq 0, \forall i, j \end{aligned}$$

Capacity Constraints

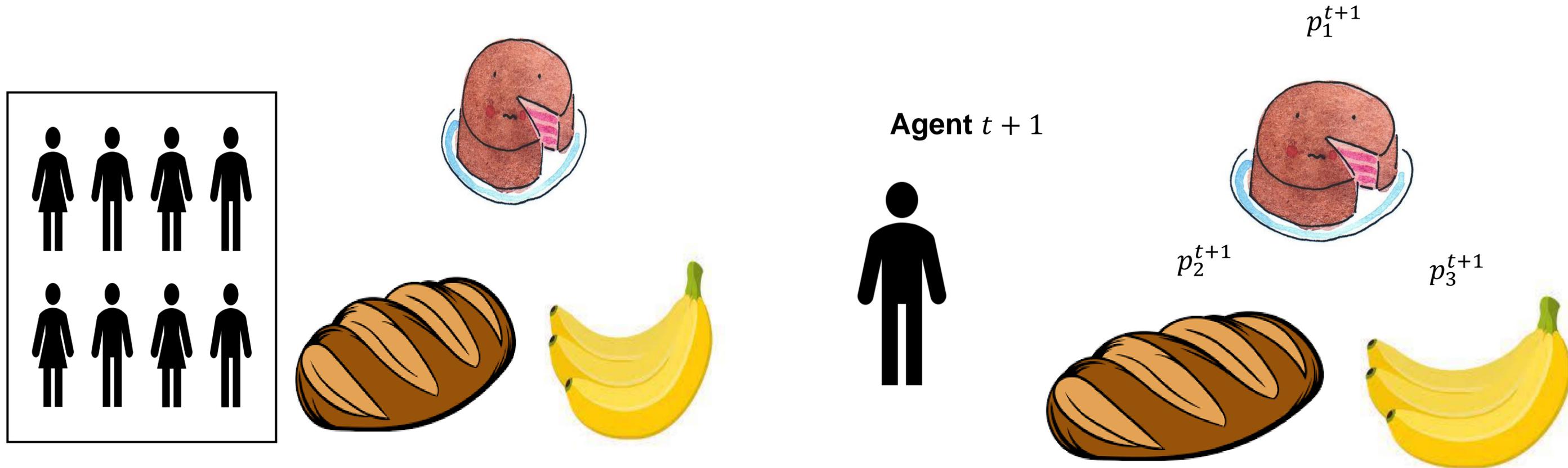


p_j : Price of Good j = Dual Variable of Constraint j

C_j can be decision variables subject to other resource constraints



Online Market Pricing: how to update posted-prices to minimize regret of the Eisenberg/Gale social welfare while achieving market clearness



Static Fisher Market Price Equilibrium
Theory and Limit

Now Agents Arrive Online

The price at time $t + 1$ is updated and reposted
based on observed consumption x^t at time t

Jalota and Y

<https://arxiv.org/abs/2205.00825>

We develop a revealed preference algorithm with sub-linear regret and constraint violation guarantees

Algorithm 2: Revealed Preference Algorithm for Online Fisher Markets

Input : Number of users n , Vector of good capacities per user $\mathbf{d} = \frac{\mathbf{c}}{n}$

Initialize $\mathbf{p}^1 > \mathbf{0}$;

for $t = 1, 2, \dots, n$ **do**

Phase I ;

 User purchases an optimal bundle of goods \mathbf{x}_t given the price \mathbf{p}^t ;

Phase II (Price Update): ;

$\mathbf{p}^{t+1} \leftarrow \mathbf{p}^t - \gamma_t (\mathbf{d} - \mathbf{x}_t)$;

Difference between market share of each agent and goods purchased

end

Step-size: $O\left(\frac{1}{\sqrt{n}}\right)$

Only requires knowledge of user consumption (and not their budgets or utilities) to update prices

Theorem: Under i.i.d. budget and utility parameters with strictly positive support and when good capacities are $O(n)$, Algorithm 2 achieves an expected regret of $R_n(\boldsymbol{\pi}) \leq O(\sqrt{n})$ and expected constraint violation of $V_n(\boldsymbol{\pi}) \leq O(\sqrt{n})$, where n is the number of arriving users.

Today's Sharing

1. Online Linear Programming Algorithms and Applications

2. Accelerated Second-Order Methods for Nonlinear Optimization and Applications

3. Zero-Order Optimization Algorithms and Applications

4. Pre-Trained Mixed Integer Linear Programming Solvers and Applications

Topic 2. Accelerated Second-Order Methods for Unconstrained Optimization and Applications

$$\min f(x), x \in X \text{ in } \mathbb{R}^n,$$

- where f is nonconvex and twice-differentiable,

$$g_k = \nabla f(x_k), H_k = \nabla^2 f(x_k)$$

- Goal: find x_k such that:

$$\|g_k\| \leq \epsilon \quad (\text{primary, first-order condition})$$

$$\lambda_{\min}(H_k) \geq -\sqrt{\epsilon} \quad (\text{secondary, second-order condition})$$

- First-order methods (FOM) typically need $\mathbf{O}(\mathbf{n}^2\epsilon^{-2})$ arithmetic operations
- Second-order methods (SOM) typically need $\mathbf{O}(\mathbf{n}^3\epsilon^{-1.5})$ arithmetic operations
- New? Yes, HSODM and DR-SOM: a single-loop method with $\mathbf{O}(\mathbf{n}^2\epsilon^{-1.75})$ operations
(Zhang et al. <https://arxiv.org/abs/2211.08212>)

An Integrated Descent Direction Using the SDP Homogeneous Model I (Zhang et al. SHUFE, 2022)

- Recall the fixed-radius trust-region method minimizes the Taylor quadratic model

$$\min_{d \in \mathbb{R}^n} m_k(d) := g_k^T d + \frac{1}{2} d^T H_k d \quad \text{s.t. } \|d\| \leq \Delta_k. \quad \longrightarrow \quad \psi_k(\xi_0, t; \delta) := \frac{1}{2} \begin{bmatrix} \xi_0 \\ t \end{bmatrix}^T \begin{bmatrix} H_k & g_k \\ g_k^T & -\delta \end{bmatrix} \begin{bmatrix} \xi_0 \\ t \end{bmatrix} = \frac{t^2}{2} \begin{bmatrix} \xi_0/t \\ 1 \end{bmatrix}^T \begin{bmatrix} H_k & g_k \\ g_k^T & -\delta \end{bmatrix} \begin{bmatrix} \xi_0/t \\ 1 \end{bmatrix}$$

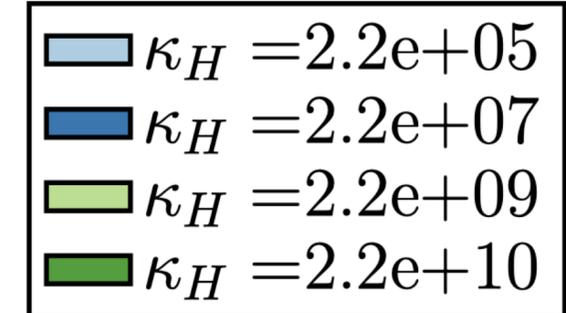
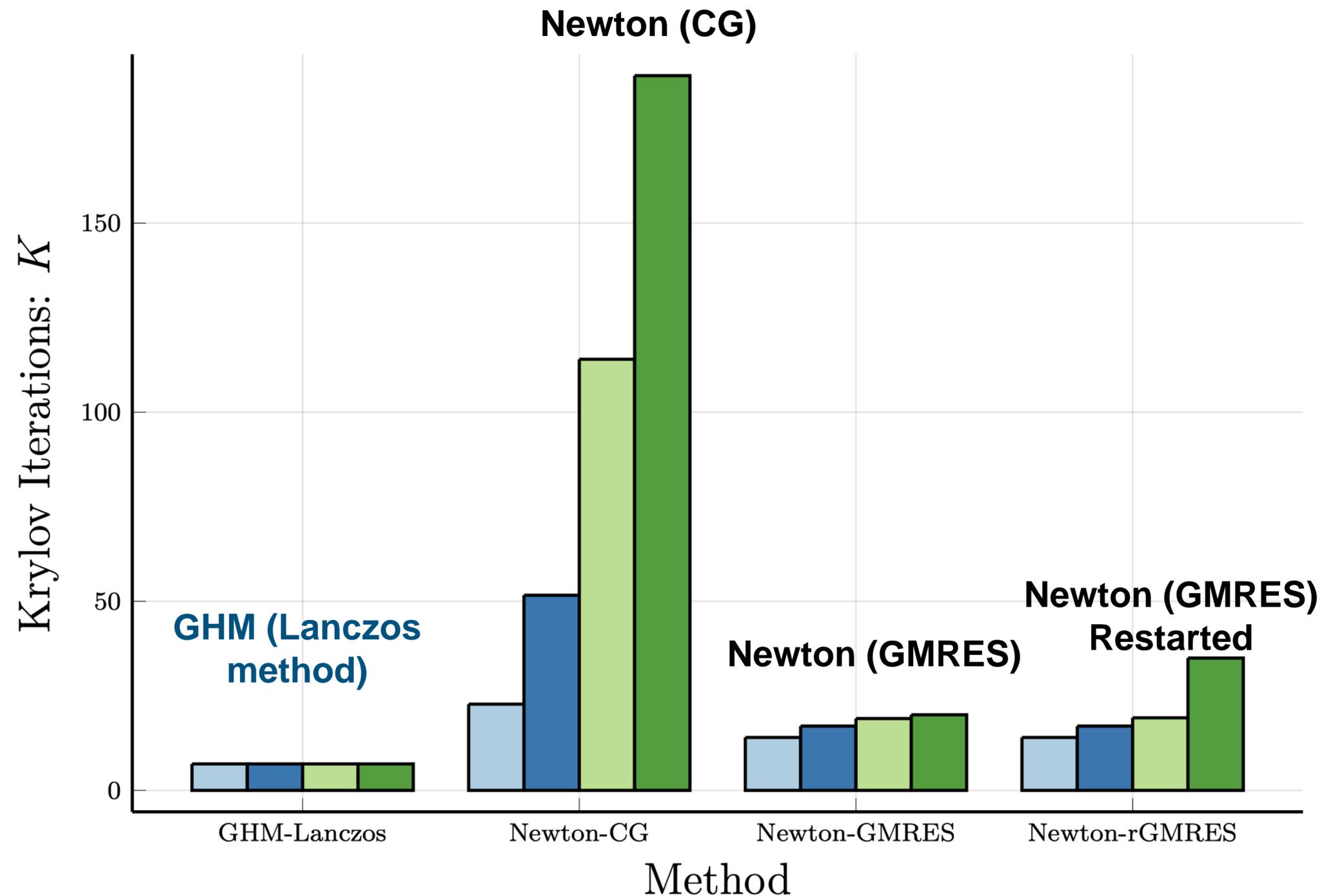
where $\Delta_k = \epsilon^{1/2}/M$ is the trust radius

- $-g_k$ is the first-order steepest descent direction but ignores Hessian; the most-left eigenvector of H_k -would be a descent direction for the second order term
- Could we construct a direction integrating both?

Answer: with a suitable δ_k and use it as the direction to go – a single loop algorithm to solve the original problem and replace the Newton step.

$O(n^2 \epsilon^{-1.75})$ vs $O(n^3)$ operations

Advantage of the Homogeneous Direction over Newton's



- Consider Hilbert matrix
- Computing a **Newton step** by iterative methods versus **Homogeneous Direction** from $H + \delta I$
- $\delta \downarrow$ Condition number $\kappa_H \uparrow$

App. III: HSODM for Policy Optimization in RL I

- Consider policy optimization of linearized objective in reinforcement learning

$$\max_{\theta \in \mathbb{R}^d} L(\theta) := L(\pi_\theta),$$

$$\theta_{k+1} = \theta_k + \alpha_k \cdot M_k \nabla \eta(\theta_k),$$

- M_k is usually a preconditioning matrix.

- The Natural Policy Gradient (NPG) method (Kakade, 2001) uses the Fisher information matrix where M_k is the inverse of

$$F_k(\theta) = \mathbb{E}_{\rho_{\theta_k}, \pi_{\theta_k}} \left[\nabla \log \pi_{\theta_k}(s, a) \nabla \log \pi_{\theta_k}(s, a)^T \right]$$

- Based on KL divergence, TRPO (Schulman et al. 2015) uses KL divergence in the constraint:

$$\max_{\theta} \nabla L_{\theta_k}(\theta_k)^T (\theta - \theta_k)$$

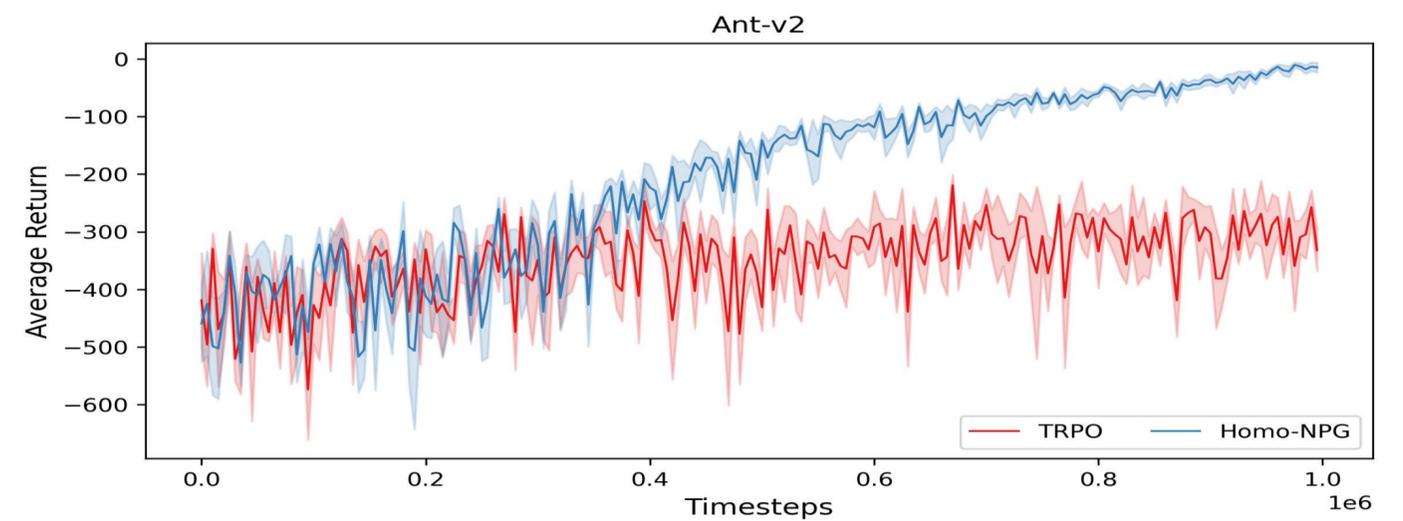
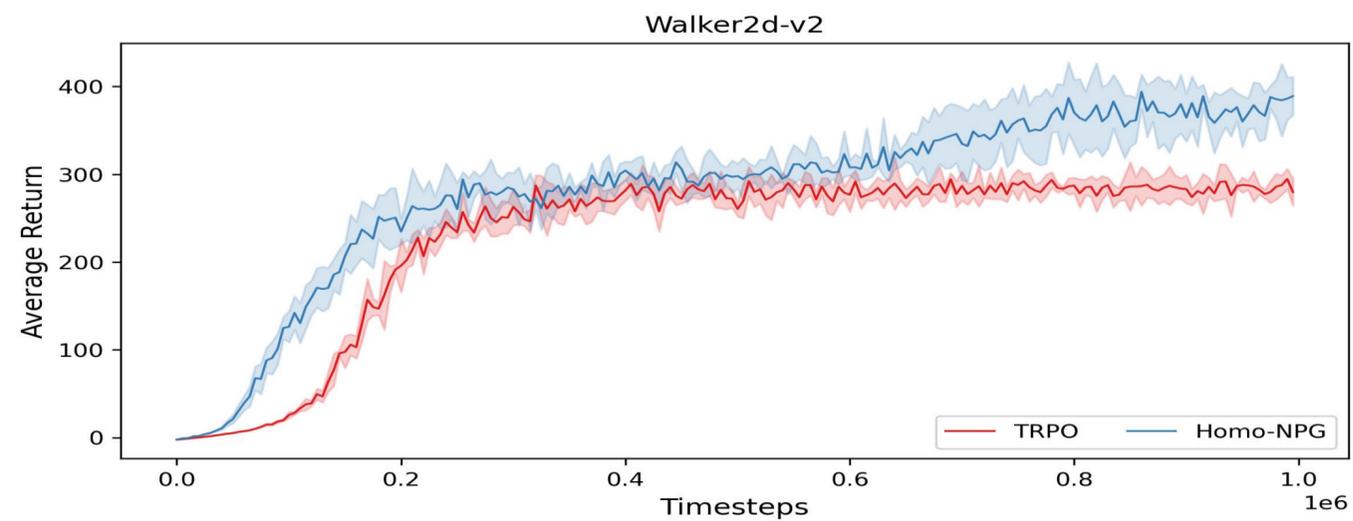
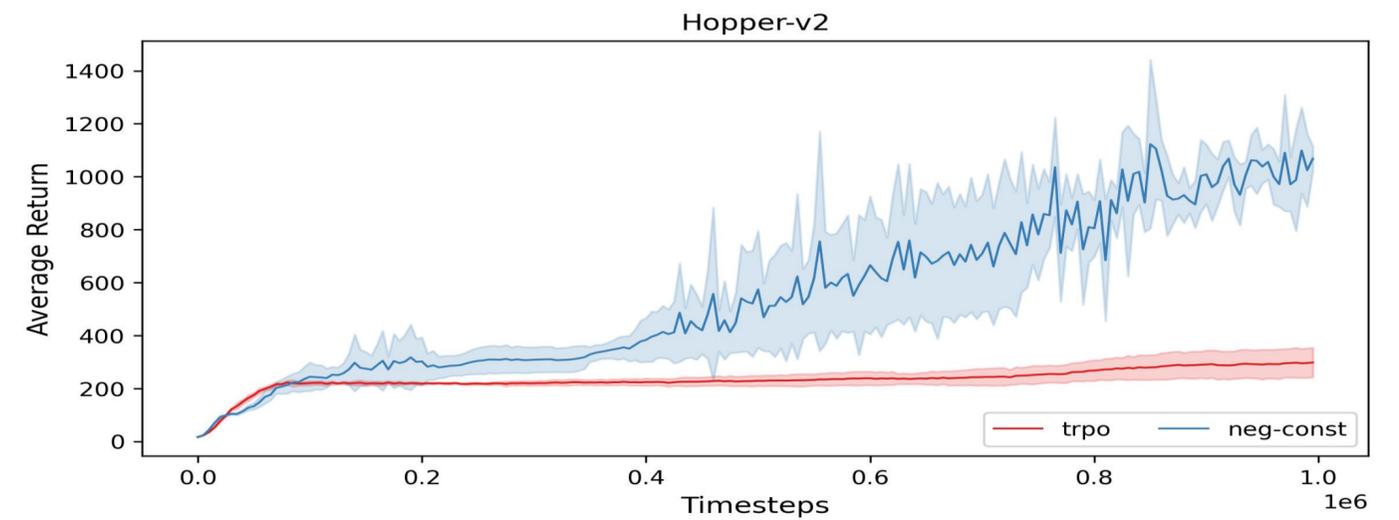
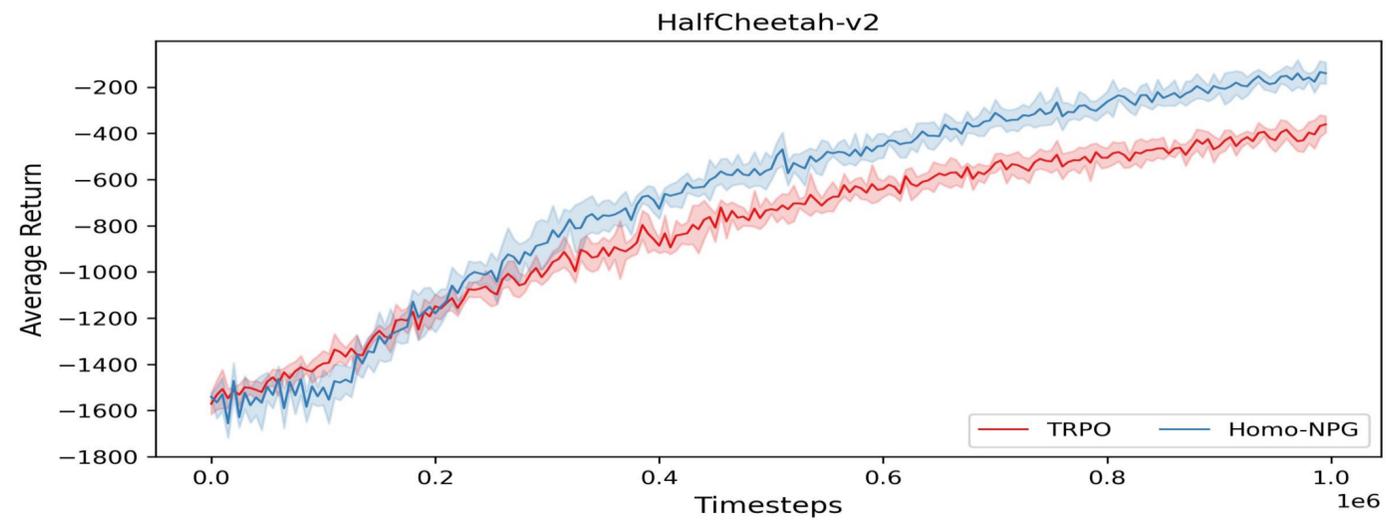
$$\text{s.t. } \mathbb{E}_{s \sim \rho_{\theta_k}} [D_{KL}(\pi_{\theta_k}(\cdot | s); \pi_{\theta}(\cdot | s))] \leq \delta.$$



**Homogeneous NPG:
Apply HSODM!**

HSODM for Policy Optimization in RL II

- A comparison of Homogeneous NPG and Trust-region Policy Optimization (Schultz, 2015)



- **Homogeneous model provides significant improvements over TRPO (public open-source solver)**

App. IV: HSODM for Sensor Network Localization I

- Consider Sensor Network Location (SNL)

$$N_x = \{(i, j) : \|x_i - x_j\| = d_{ij} \leq r_d\}, N_a = \{(i, k) : \|x_i - a_k\| = d_{ik} \leq r_d\}$$

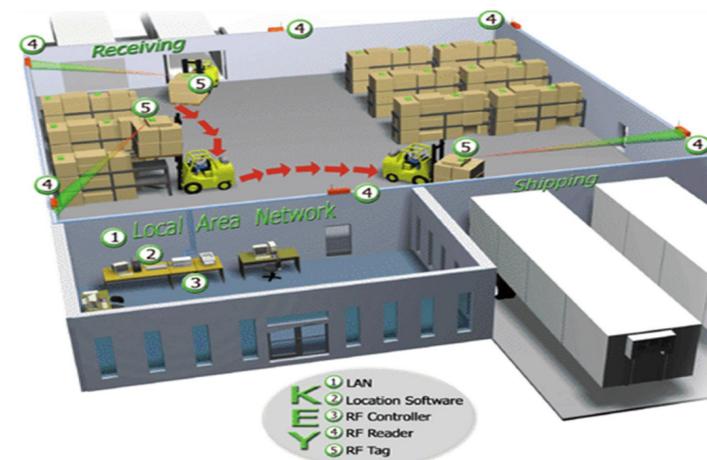
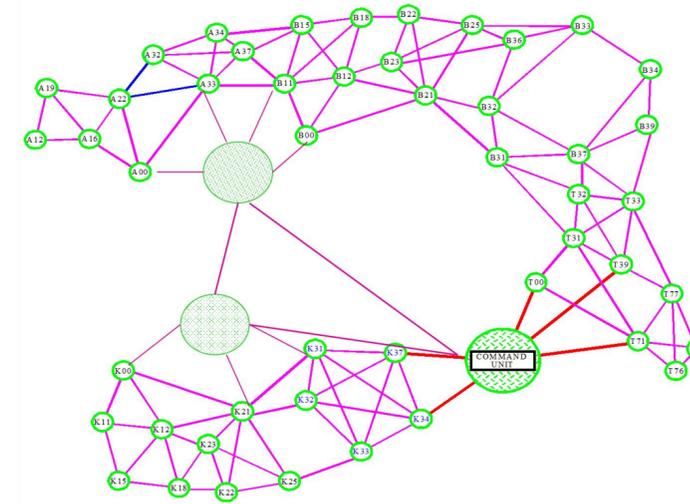
where r_d is a fixed parameter known as the radio range. The SNL problem considers the following QCQP feasibility problem,

$$\|x_i - x_j\|^2 = d_{ij}^2, \forall (i, j) \in N_x$$

$$\|x_i - a_k\|^2 = \bar{d}_{ik}^2, \forall (i, k) \in N_a$$

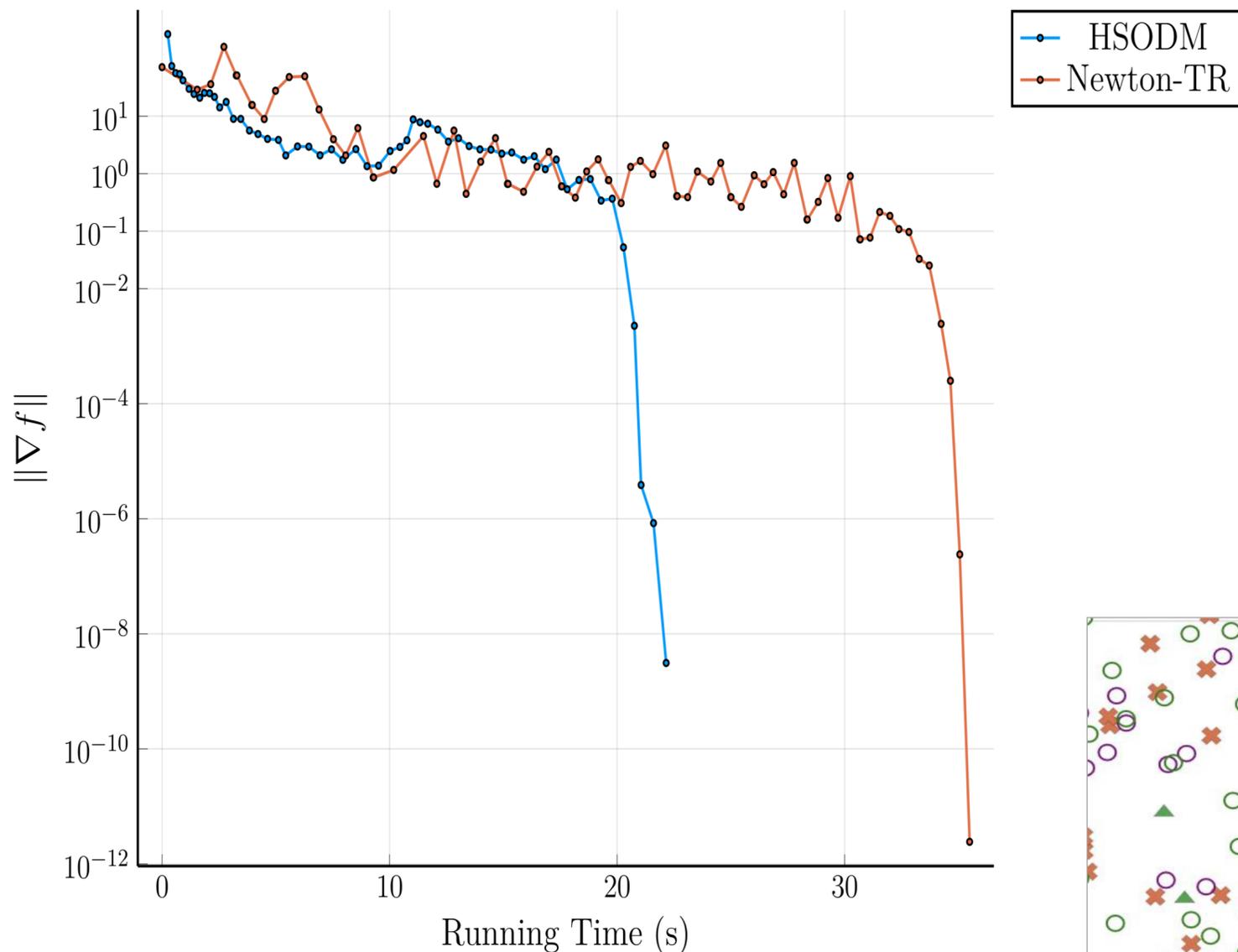
- We can solve SNL by the nonconvex nonlinear least square (NLS) problem

$$\min_X \sum_{(i,j) \in N_x} (\|x_i - x_j\|^2 - d_{ij}^2)^2 + \sum_{(k,j) \in N_a} (\|a_k - x_j\|^2 - \bar{d}_{kj}^2)^2.$$

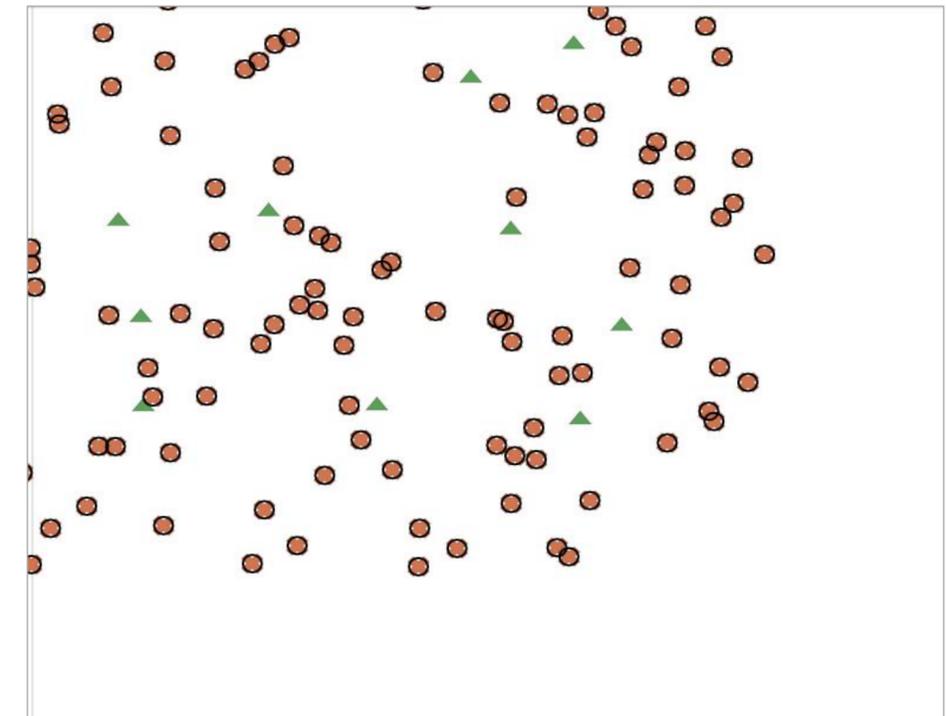
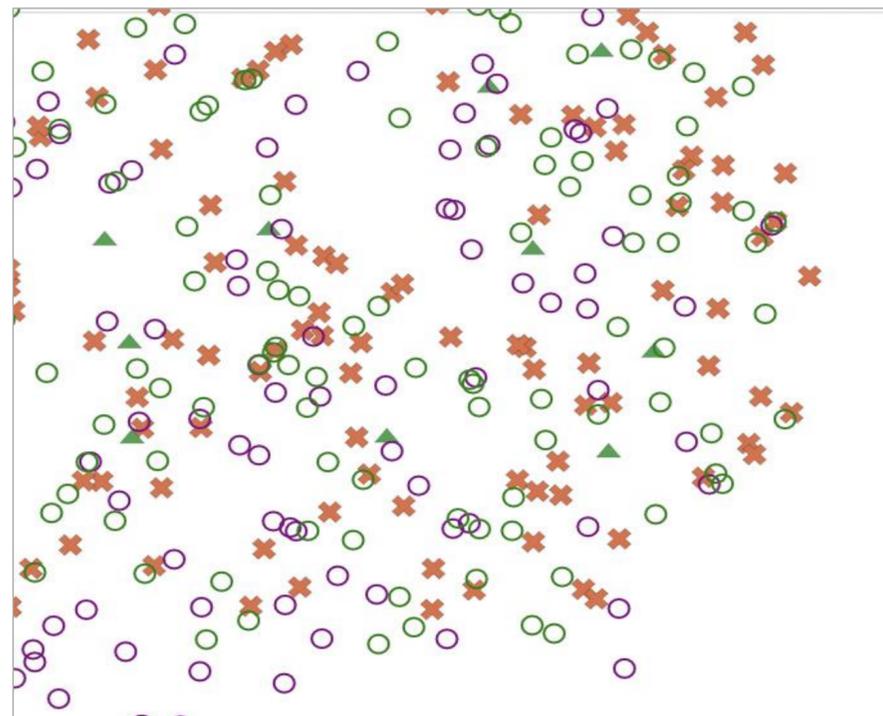


HSODM for Sensor Network Localization II

SNL, $n := 200$, $m := 20$



- Compare HSODM (with HVP), and Newton-TR Method.
- HSODM is faster due to the eigenvalue procedure
- The solution quality is much better than the FOMs



Dimension Reduced Second-Order Method (DRSOM)

- Motivation from Multi-Directional FOM and Subspace Method, such as CG and ADAM, DRSOM applies the trust-region method in low dimensional subspace.
 - This results in a low-dimensional quadratic sub-minimization problem:
 - Typically, DRSOM adopts two directions $d = -\alpha^1 \nabla f(x_k) + \alpha^2 d_k$
- where $g_k = \nabla f(x_k)$, $H_k = \nabla^2 f(x^k)$, $d_k = x_k - x_{k-1}$
- Then we solve a 2-d quadratic minimization problem to decide the two step-sizes:

$$\min m_k^\alpha(\alpha) := f(x_k) + (c_k)^T \alpha + \frac{1}{2} \alpha^T Q_k \alpha$$

$$G_k = \begin{bmatrix} \|\alpha\|_{G_k} \leq \Delta_k \\ g_k^T g_k & -g_k^T d_k \\ -g_k^T d_k & d_k^T d_k \end{bmatrix}, Q_k = \begin{bmatrix} g_k^T H_k g_k & -g_k^T H_k d_k \\ -g_k^T H_k d_k & d_k^T H_k d_k \end{bmatrix}, c_k = \begin{bmatrix} -\|g_k\|^2 \\ g_k^T d_k \end{bmatrix}$$

Constructing the two-dimensional quadratic model is the Key

How to cheaply obtain Q_k , that is, compute

$H_k g_k$ and $H_k d_k$?

- Use Hessian if readily available !

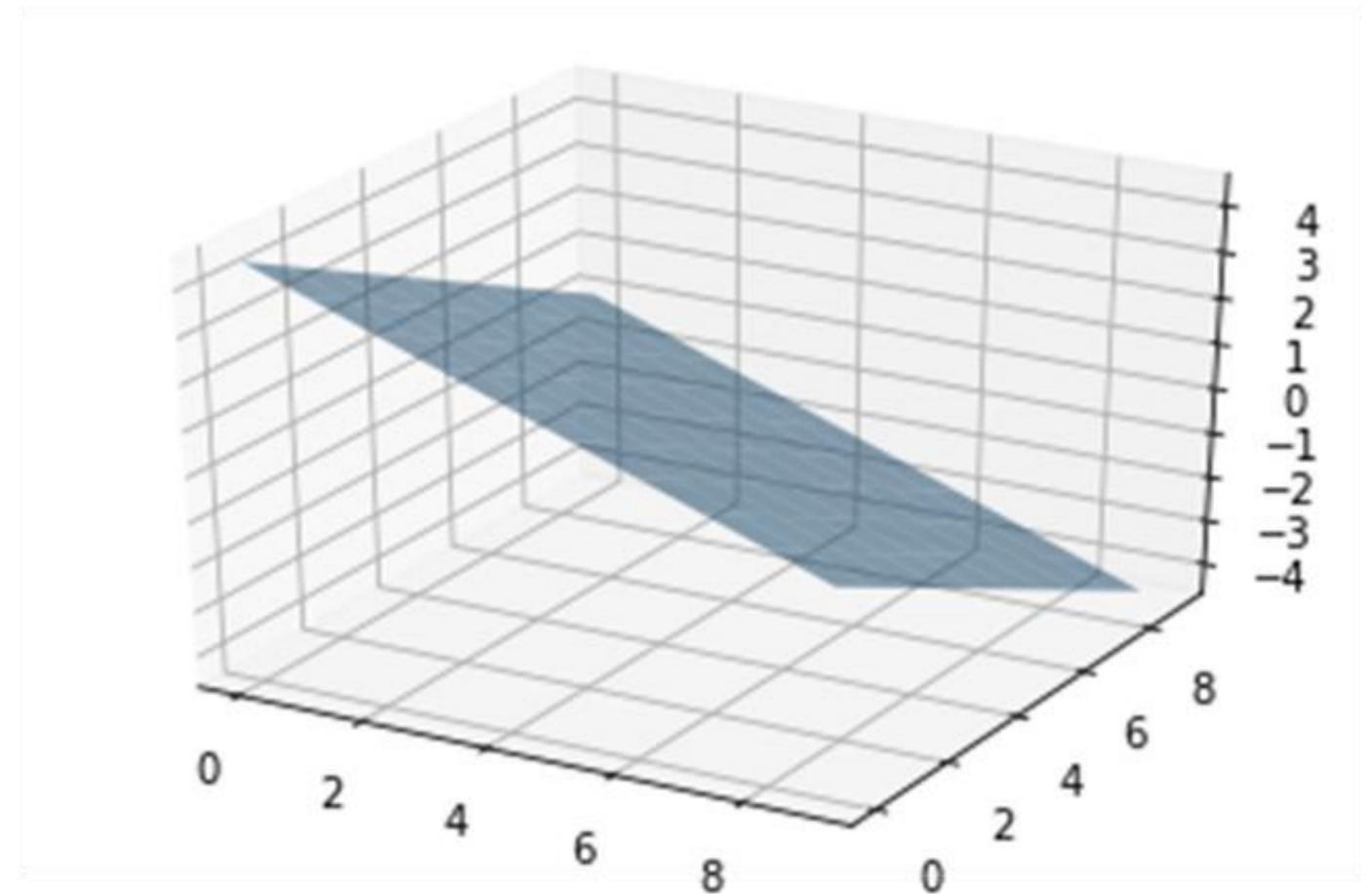
- Finite difference:

$$H_k \cdot v \approx \frac{1}{\epsilon} [g(x_k + \epsilon \cdot v) - g_k],$$

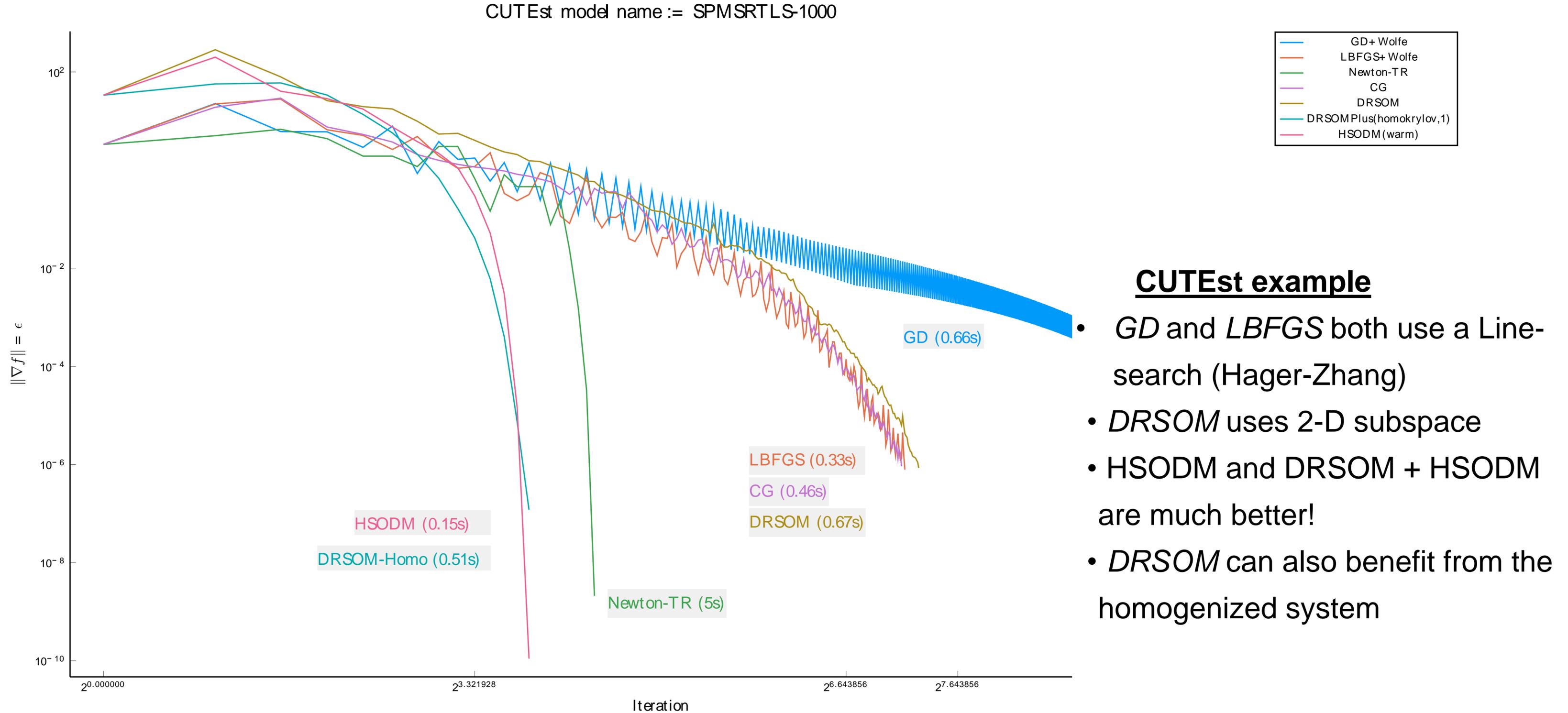
- Analytic approach to fit modern automatic differentiation,

$$H_k g_k = \nabla \left(\frac{1}{2} g_k^T g_k \right), H_k d_k = \nabla (d_k^T g_k),$$

- **Zero-Order: Three(-or more)-Point Interpolation**

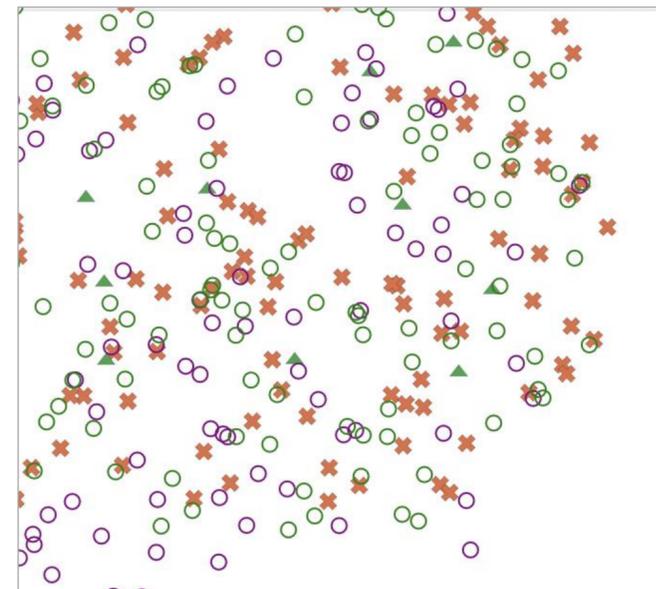
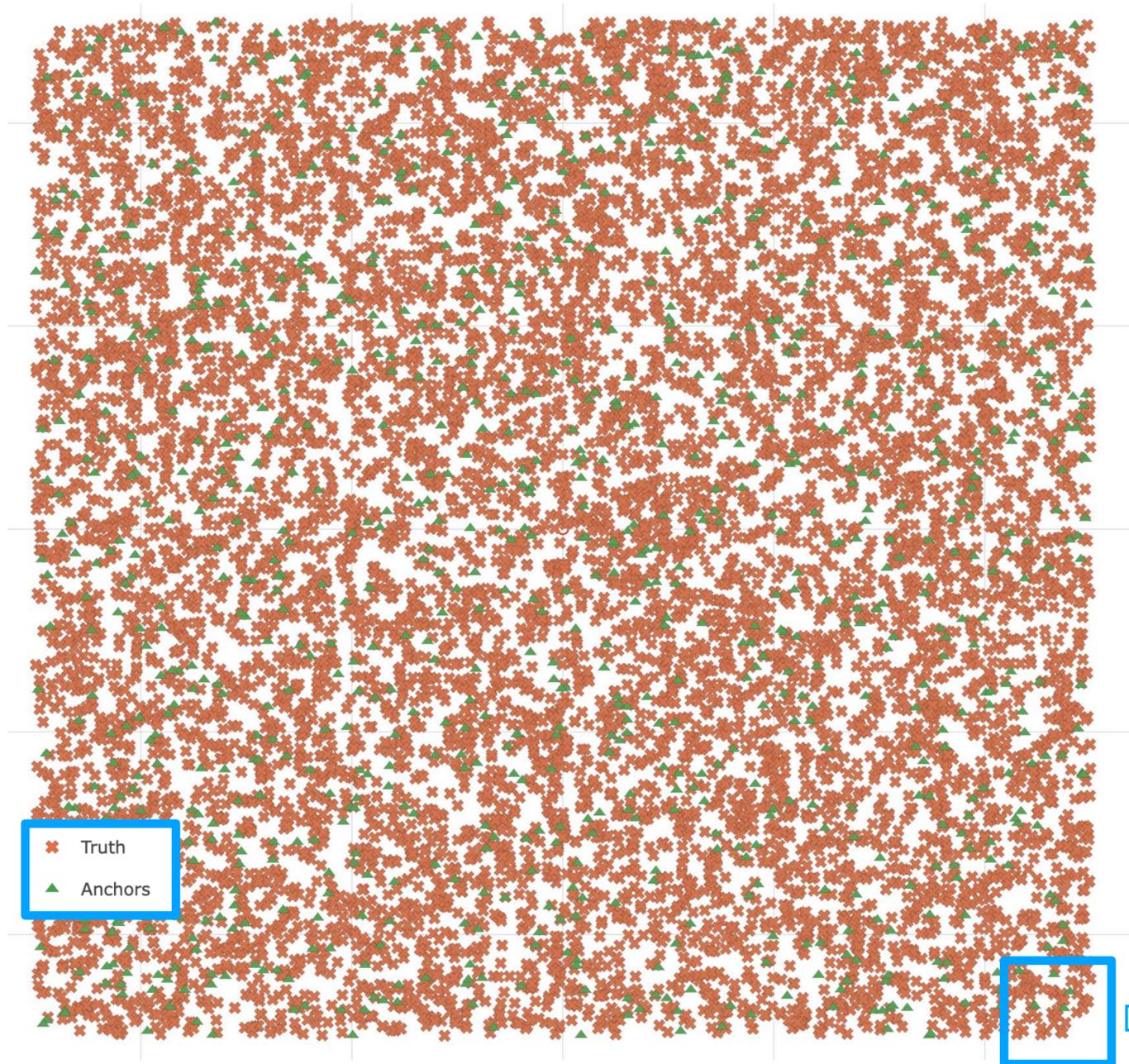


Preliminary Results: HSODM, DRSOM and DRSOM+HSODM

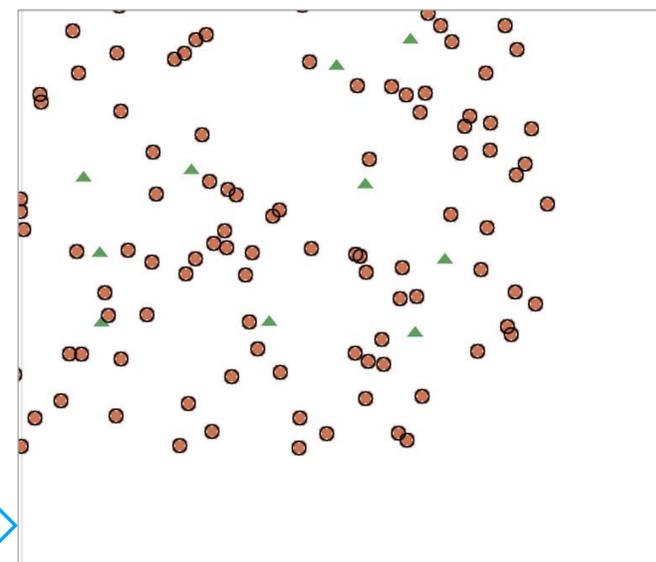


Sensor Network Location again, Large-Scale Instances

- Graphical results with 10,000 nodes and 1000 anchors (no noise) **within 3,000 seconds**

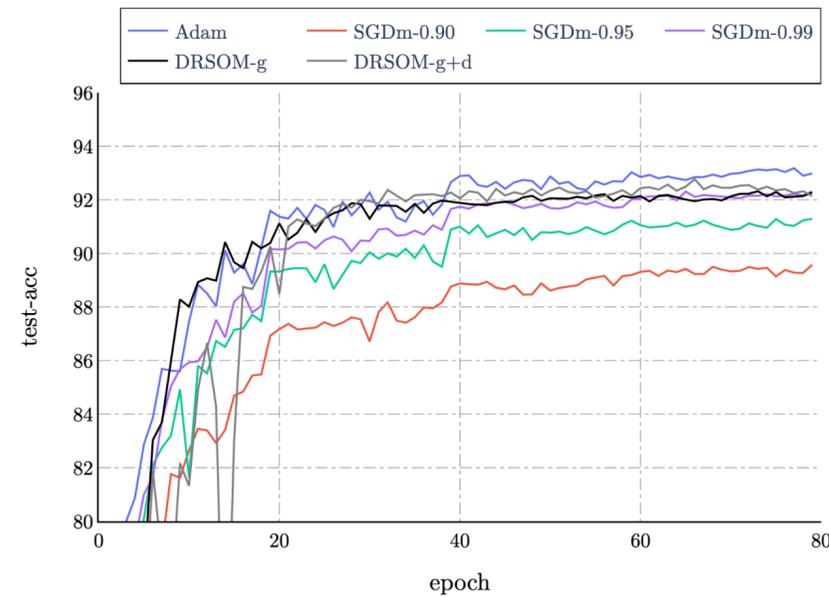
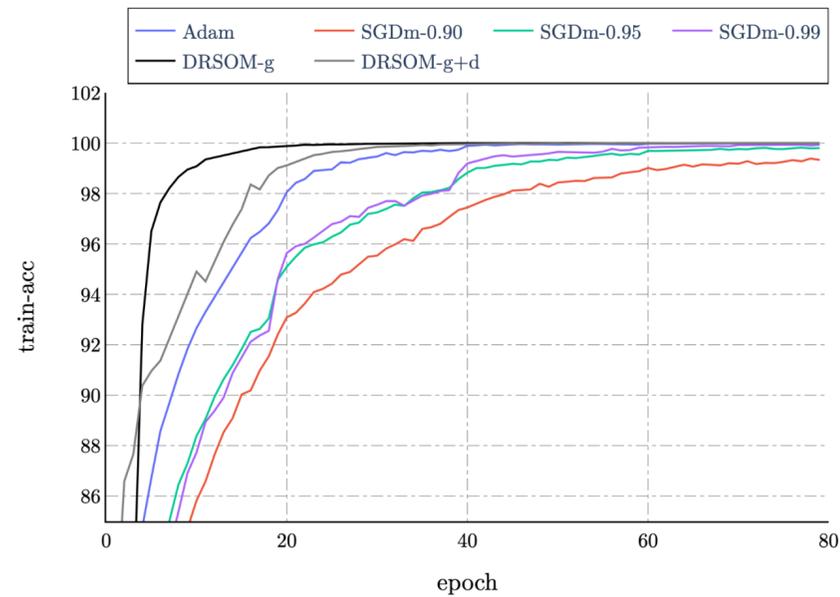


- GD with Line-search and Hager-Zhang CG both timeout**

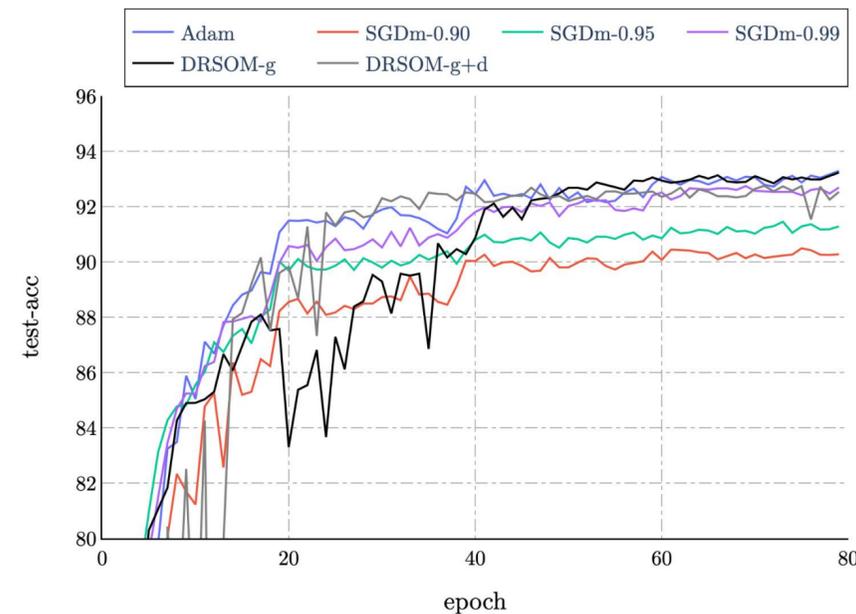
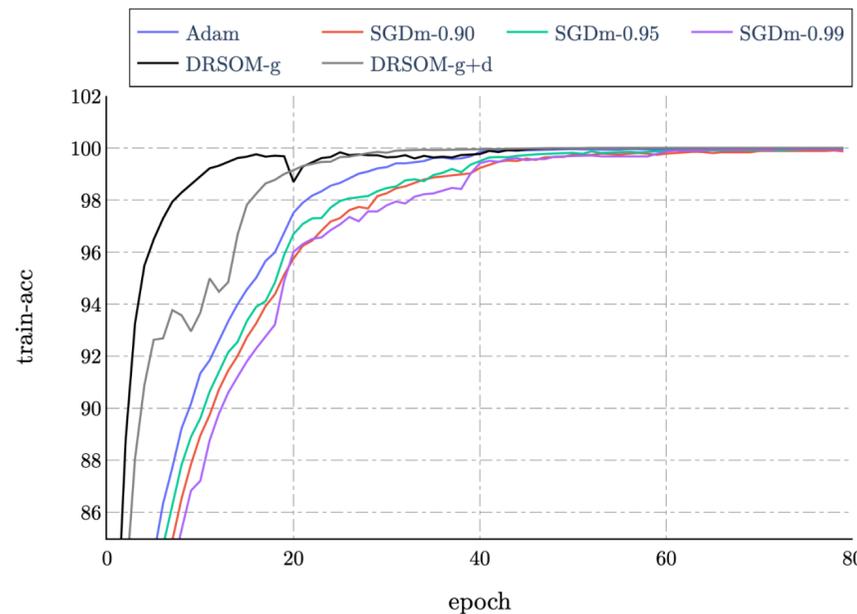


- DRSOM can converge to $|g_k| \leq 1e^{-5}$ in 2,200s**

App. V: Neural Networks and Deep Learning



Training and test results for ResNet18 with DRSOM and Adam



Training and test results for ResNet34 with DRSOM and Adam

Pros

- DRSOM has rapid convergence (30 epochs)
- DRSOM needs little tuning

Cons

- DRSOM may over-fit the models
- Running time can benefit from Interpolation
- Single direction DRSOM is also good

Good potential to be a standard optimizer for deep learning!

Today's Sharing

1. Online Linear Programming Algorithms and Applications

2. Accelerated Second-Order Methods for Nonlinear Optimization and Applications

3. Zero-Order Optimization Algorithms and Applications

4. Pre-Trained Mixed Integer Linear Programming Solvers and Applications

SOLNP+: History

- First proposed by Y in 1989.
- Originally implemented (SOLNP) in **Matlab**, 1989.
- **R** implementation (**Rsolnp**) by Alexios Ghalanos and Stefan Theussl, 2011.
- **New and C implementation (SOLNP+) with improvements, 2022; and addition of Randomized BCG and DRSSOM for unconstrained optimization by Tan et al., 2023**
- **Github link: https://github.com/COPT-Public/SOLNP_plus**
- Use **forward difference** to calculate the approximated gradient.

$$[\nabla_{\delta} f(x)]_i = \frac{f(x + \delta e_i) - f(x)}{\delta}, \quad e_i = [0, \dots, 1, \dots, 0].$$

Shortcoming of Finite-Difference

- For finite-difference methods and other traditional zero-order methods, the number of function query **increases linearly** with the problem dimension.
- Traditional methods may not perform well for solving **high dimensional** problems.
- The dimension in machine learning problems can be **extremely high**.

RMP: Multi-Point Random Perturbation

- Multipoint ZO Gradient Estimates

$$\hat{\nabla} f(x) := \frac{\phi(n)}{\delta b} \sum_{i=1}^b [(f(x + \delta u_i) - f(x)) u_i]$$

$$\mathbb{E}[\hat{\nabla} f(x)] = \nabla f_\delta(x)$$

$$f_\delta(x) = \mathbb{E}_u[f(x + \delta u)]$$

where u_i is i.i.d. **random direction**.

- Advantage: **Fewer** function queries to evaluate the gradient.

SOLNP+: Adopt Two Strategies

- With gradient estimates, SOLNP+ implements ZO version of
 - ZO-RMP (Ghadimiet al., 2013; Duchi et al., 2014), or
 - ZO-BCD, more recent research see (Cai et al., 2021)
- DRSOM (Zhang et al., 2022) with interpolation

S. Ghadimi and G. Lan, "Stochastic first-and zeroth-order methods for nonconvex stochastic programming," *SIAM J. Optimiz.*, vol. 23, no. 4, pp. 2341–2368, 2013. doi: 10.1137/120880811

J.C.Duchi, M. I. Jordan, M. J. Wainwright, and A. Wibisono, "Optimal rates for zero-order convex optimization: The power of two function evaluations," *IEEE Trans. Inf Theory*, vol. 61, no. 5, pp. 2788–2806, 2015. doi: 10.1109/TIT.2015.2409256.

Cai, HanQin, et al. "A zeroth-order block coordinate descent algorithm for huge-scale black-box optimization." *International Conference on Machine Learning*. PMLR, 2021.

Zhang, Chuwen, et al. "DRSOM: A Dimension Reduced Second-Order Method and Preliminary Analyses." *arXiv preprint arXiv:2208.00208* (2022).

SOLNP+: Apply RMP or RBCD plus DRSOM

- Randomized Multiple Point Estimator:
 - $u_i \sim N(0, R^d)$
 - u_i Rademacher Random Variable (each element $\in \{-1, +1\}$)
- Randomized Block Coordinate Descent Estimator:
 - Randomly select some block
 - Use e_i (vector with zero components except that dimension i is 1), $i \in \text{Block}$ to estimate gradient
- Update rule: $x_{k+1} = x_k - \eta \hat{\nabla} f(x_k)$

R. Sun and Y. Y, "Worst-case complexity of cyclic coordinate descent: $O(n^2)$ gap with randomized version." *Mathematical Programming*, Volume 185, 487-520, 2021.

Experiments in Large Problems: Rosenbrock I

- Rosenbrock function is a well-known nonconvex functions in the form of

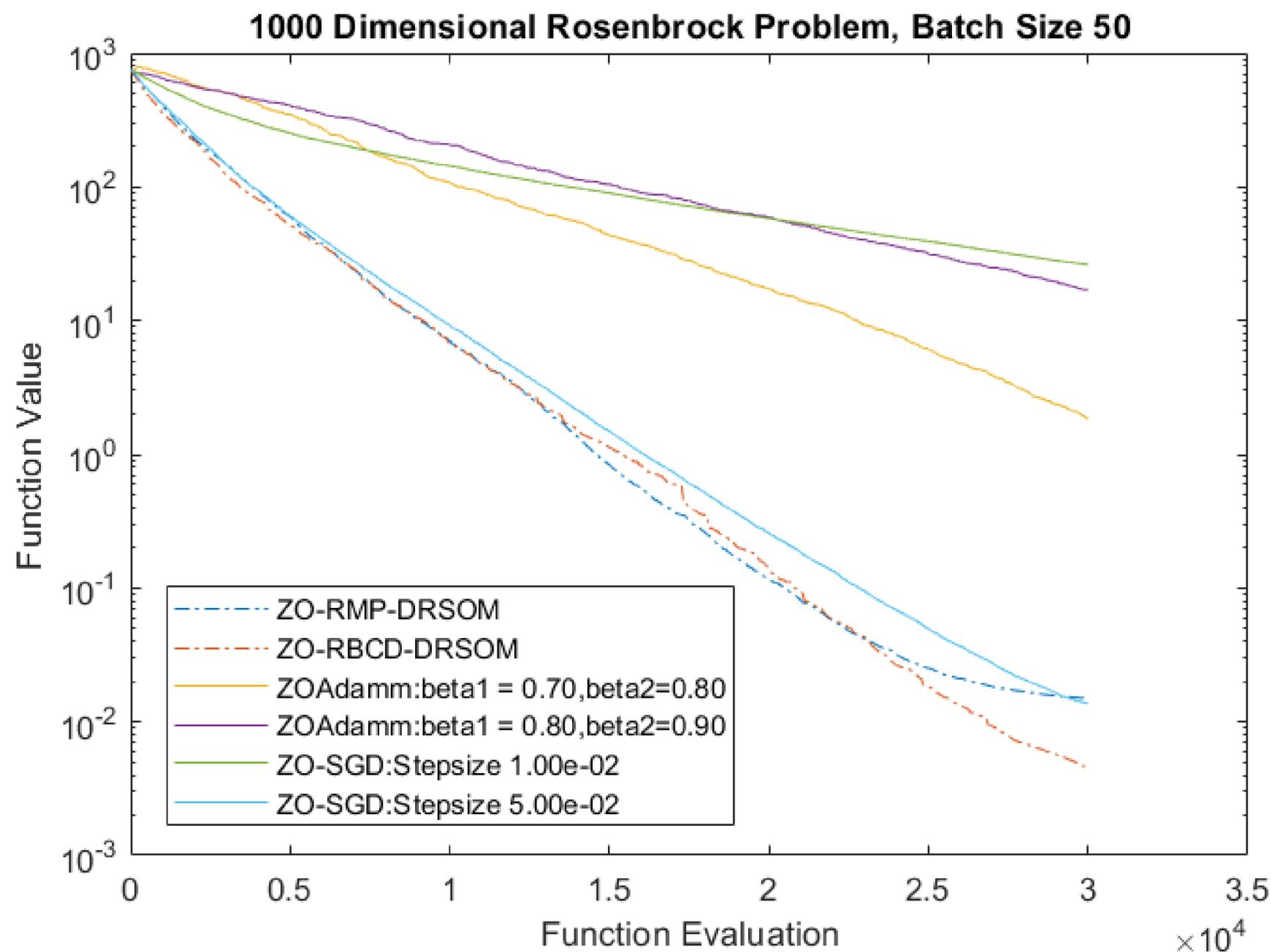
$$f(\mathbf{x}) = \sum_{i=1}^{n-1} 100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2$$

- ZO-Adamm, ZO-SGD and ZO-DRSOM are tested in a 1200 dimensional Rosenbrock problem.
 - Batch size 50
 - Each experiment is repeated for 10 times.

Chen, Xiangyi, et al. "Zo-adamm: Zeroth-order adaptive momentum method for black-box optimization." *Advances in neural information processing systems* 32 (2019).

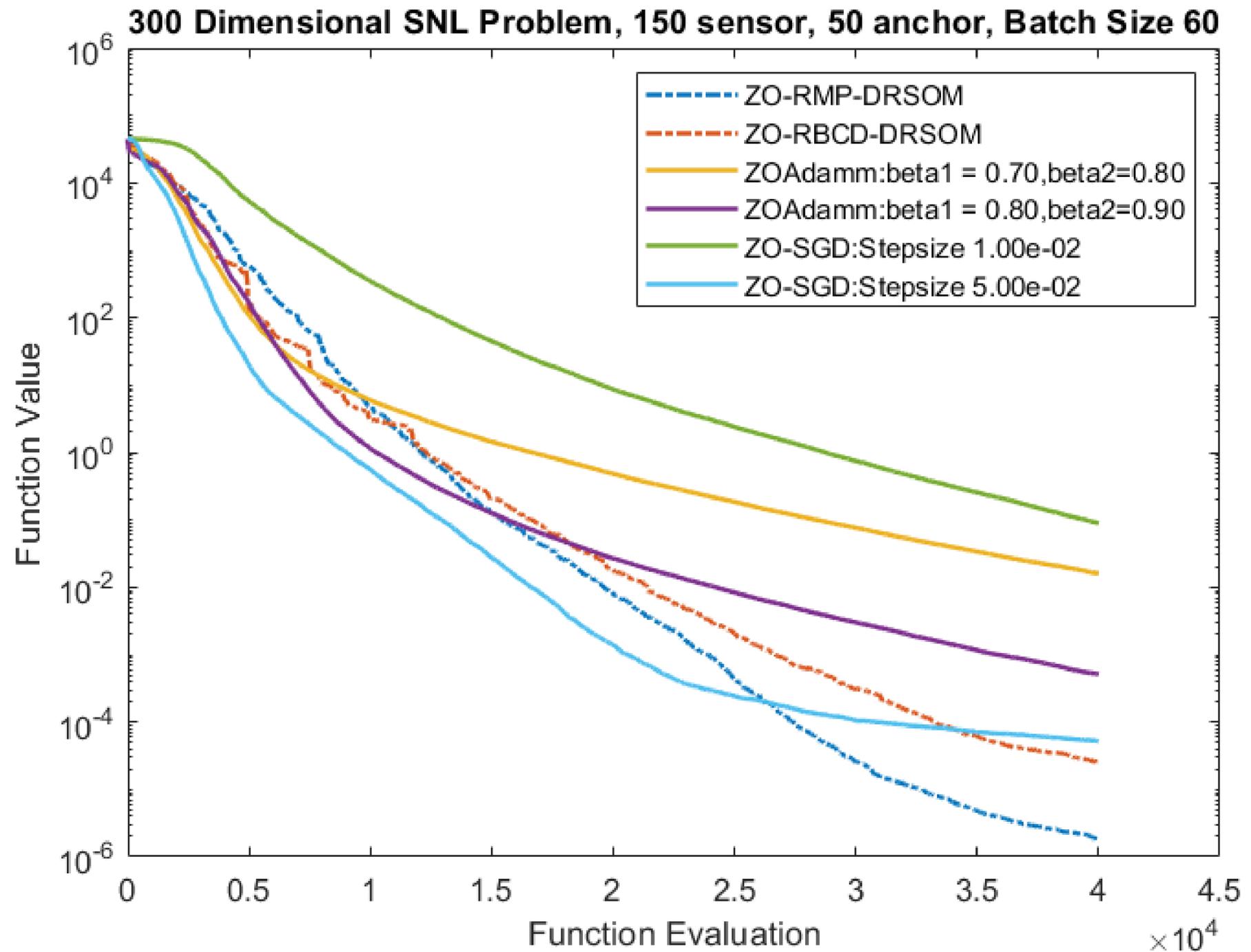
Experiments in Large Problems: Rosenbrock II

- ZO-RMP-DRSOM, ZO-RBCD-DRSOM and ZO-SGD decrease most smoothly. However, inappropriate parameters lead to worse performance of ZO-SGD and ZO-ADAMM.



Experiments in Large-Scale Problems: SNL I

- ZO-Adam, ZO-SGD, ZO-RBCD-DRSOM and ZO-RMP-DRSOM are tested in a 150-sensor SNL problem.
 - Batch size 60
 - Each experiment is repeated for 10 times.



Today's Sharing

1. Online Linear Programming Algorithms and Applications

2. Accelerated Second-Order Methods for Nonlinear Optimization and Applications

3. Zero-Order Optimization Algorithms and Applications

4. Pre-Trained Mixed Integer Linear Programming Solvers and Applications

App. VI: Unit Commitment Problem

- Electricity is generated from units (**various** generators)
- Transmitted **safely** and **stably** through power grids
- Consumed at **minimum (reasonable)** price



Optimization has its role to play

minimize **Cost of electricity**
subject to **Safety and Stability**

Adaptivity to various units

Unit commitment problem dispatches the units **safely** and **stably** at **minimum** cost

App. VII: Beijing Public Transport Intelligent Urban Bus Operations Management with Mixed Fleet Types and Charging Schedule

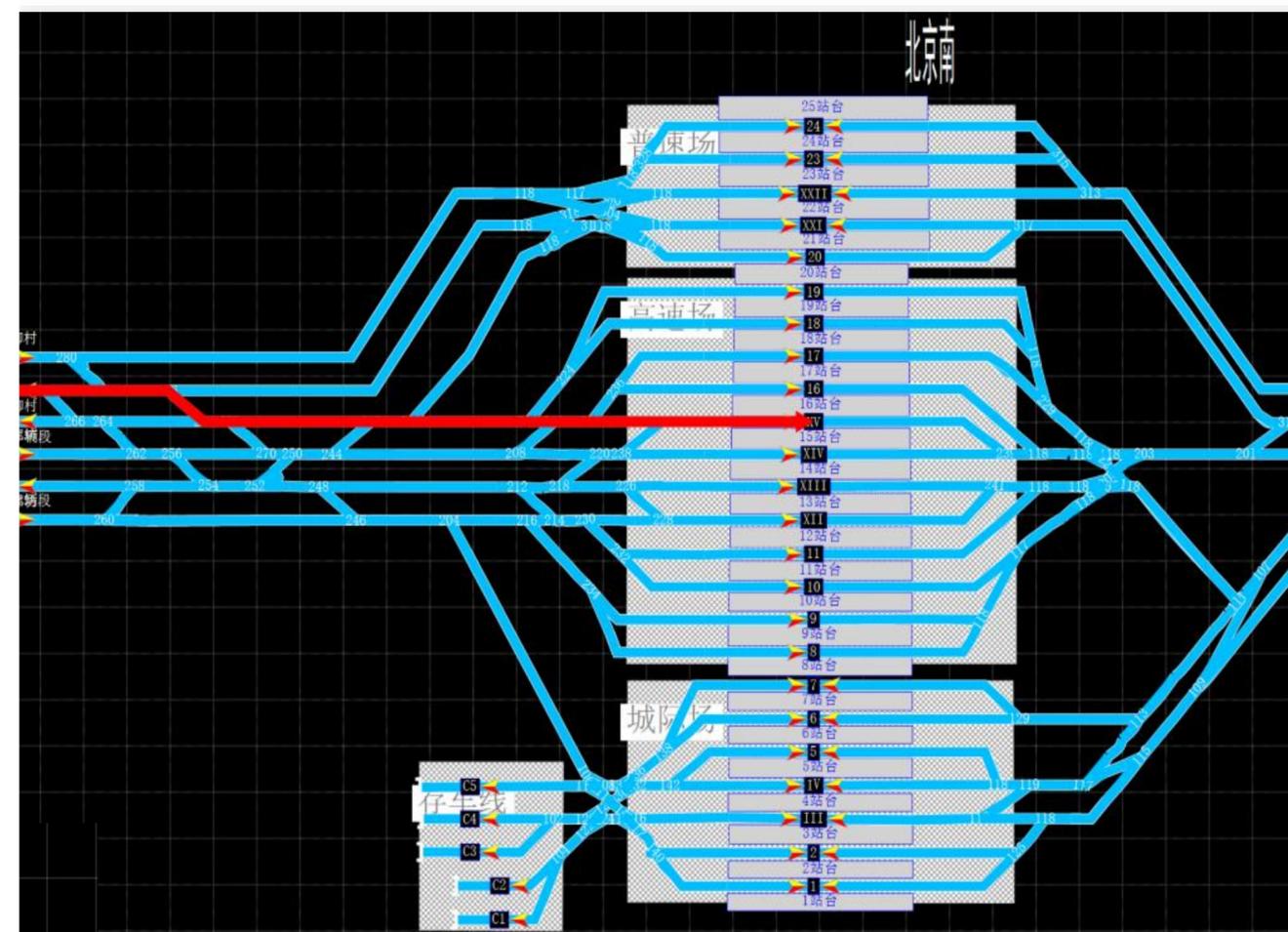
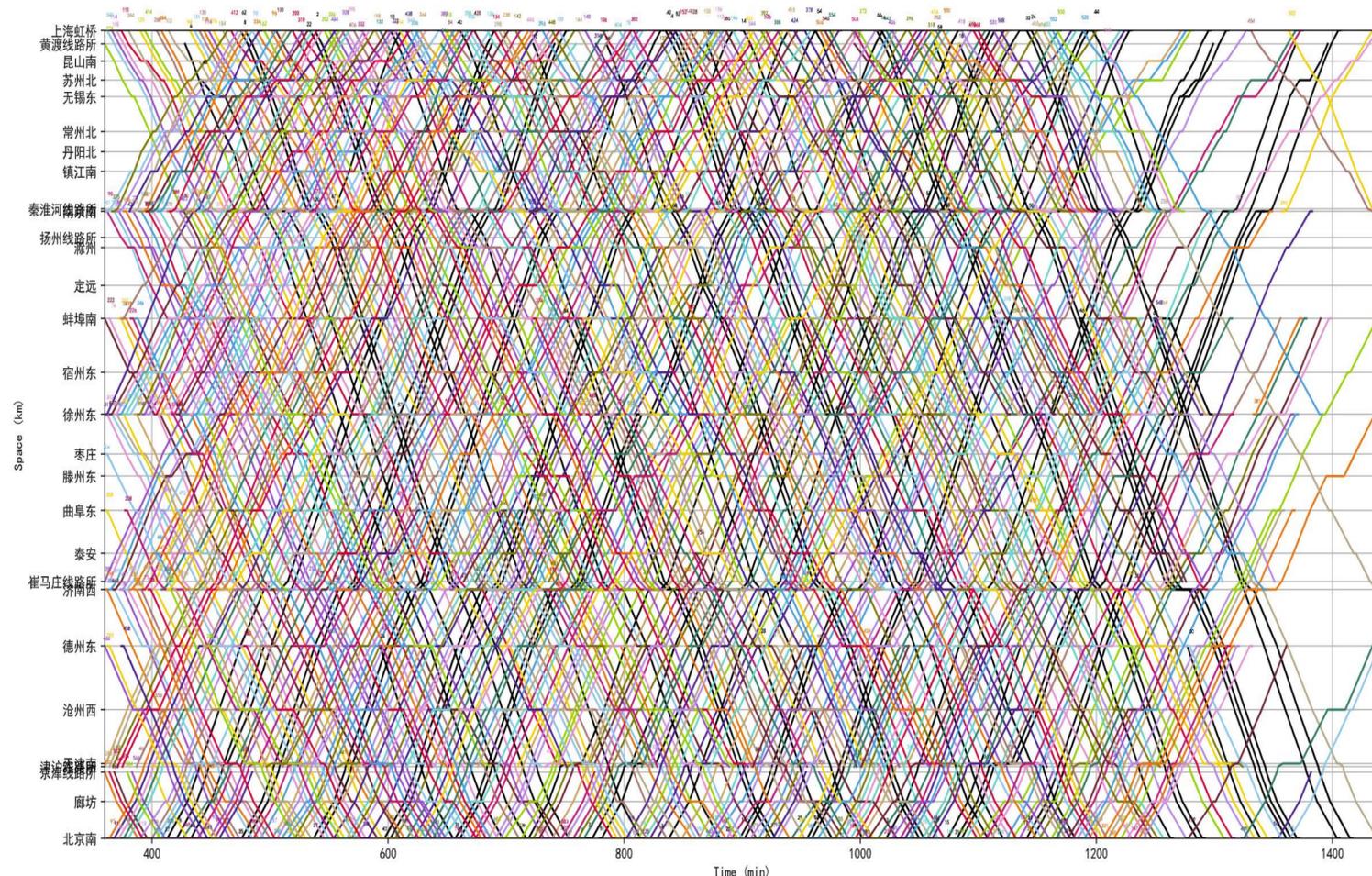


Kickoff 2022.8



App. VIII: Beijing-Shanghai High-speed Railway Scheduling Optimization

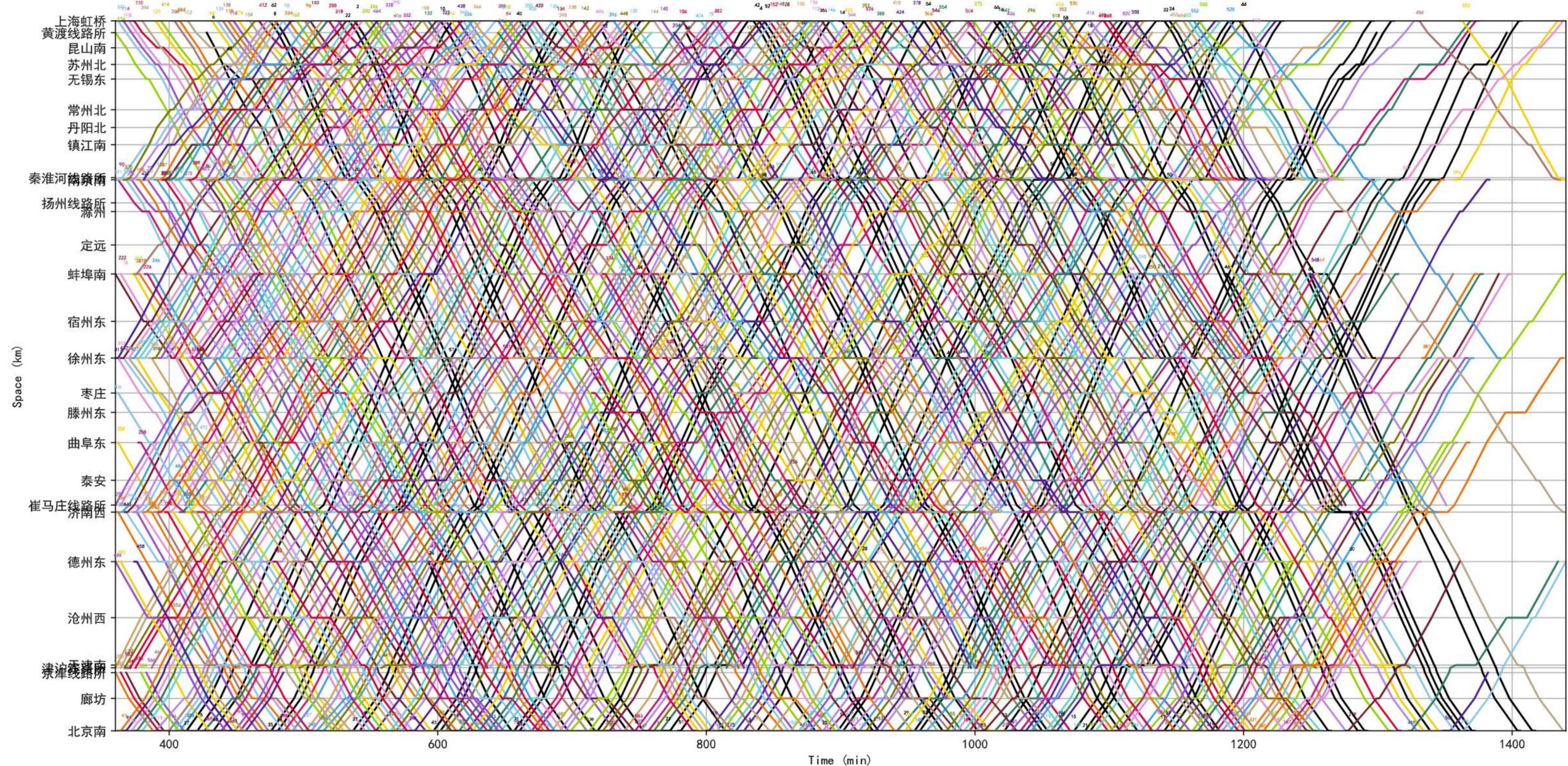
COPT, Cardinal Operations 2022



Numerical Results: TTP for Beijing-Shanghai



- We solve the TTP for Beijing-Shanghai high-speed railway using Cardinal Optimizer (COPT).
- COPT is the first fully independently developed mathematical programming solver in China with strong solving ability of MIP problem. It also has excellent performance in solving this problem.
- The result is presented in the following figure. We only need about **1000 seconds** to schedule 584 train in two directions.



Risk-Pooling Cuts in MILP

- Given an MILP, the interior point solution of the LP relaxation tells us

$$\begin{pmatrix} \hat{y}_1(\xi) \\ \hat{y}_2(\xi) \\ \vdots \\ \hat{y}_n(\xi) \end{pmatrix} = \begin{pmatrix} 0.99 \\ 0.12 \\ \vdots \\ 0.38 \end{pmatrix}$$

- Each \hat{y} is the likelihood a variable takes 1 or 0 in the optimal solution
- Each variable introduces some **risk/variance** of **such rounding**
so that dealing them separately results in **extremely risk outcomes**

Q: What should we do seeing a set of risky guesses? A: **Put them in a pool!**

Risk-Pooling Cuts via Moment Ambiguity of DRO

- Pooling the binary variables by adding “confidence” cardinality cuts

$$\sum_{i \in \mathcal{U} = \{j: \hat{y}_j(\xi) \geq 0.9\}} y_i^*(\xi) \geq \alpha \cdot |\mathcal{U}| \quad \sum_{i \in \{\mathcal{L} = \{j: \hat{y}_j(\xi) \leq 0.1\}\}} y_i^*(\xi) \leq \beta \cdot |\mathcal{L}|$$

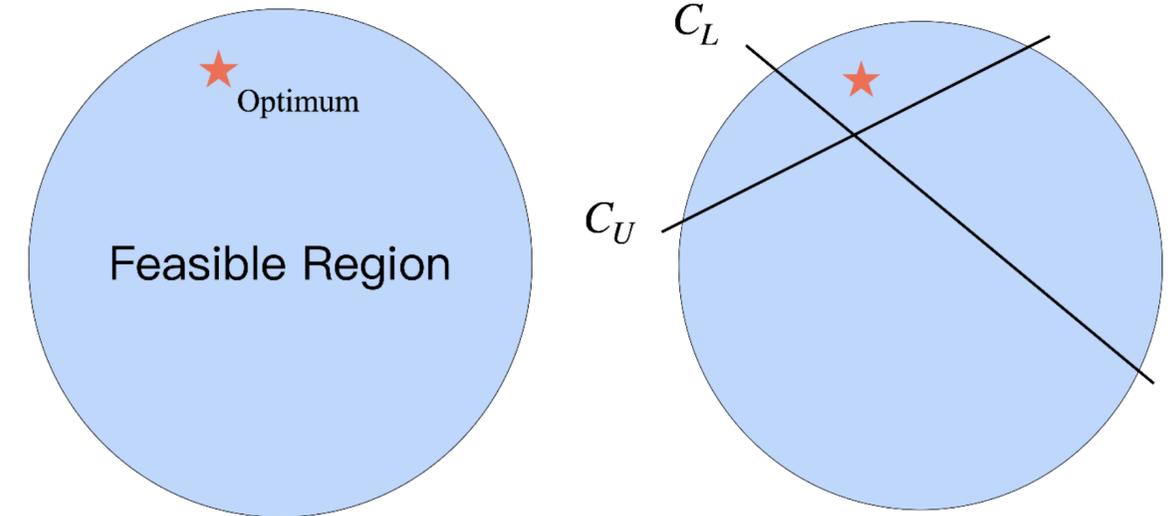
- Intuitively we know that the above two inequalities are **expectedly** to hold for $\alpha \rightarrow 0.9$ and $\beta \rightarrow 0.1$
- These two inequalities are exactly *cutting planes* for MILP
- The last issue is how to choose α, β **to increase the confidence level:**
Interpret y_j^* as some random variables with expectation \hat{y}_j ,
then justify by **concentration inequalities**

Statistical Confidence Cut Generation (Gao et al. SHUFE, 2023)

Theorem 2. Given independent random variables $\{y_1^*(\xi), \dots, y_n^*(\xi)\}$ such that $\mathbb{E}[y_i^*(\xi)|\xi] = \hat{y}_i(\xi)$, letting $\mathcal{U} := \{i: \hat{y}_i(\xi) \geq \tau\}$ and $\mathcal{L} := \{i: \hat{y}_i(\xi) < 1 - \tau\}$ for $0.5 \leq \tau \leq 1$. Then w.p. $1 - \delta$, each of the inequalities below holds.

$$C_{\mathcal{U}}: \sum_{i \in \mathcal{U}} y_i^*(\xi) \geq \sum_{i \in \mathcal{U}} \hat{y}_i(\xi) - \sqrt{\frac{|\mathcal{U}|}{2} \log(1/\delta)}$$

$$C_{\mathcal{L}}: \sum_{i \in \mathcal{L}} y_i^*(\xi) \leq \sum_{i \in \mathcal{L}} \hat{y}_i(\xi) + \sqrt{\frac{|\mathcal{L}|}{2} \log(1/\delta)}$$



- Overall, the two cuts (and their complement) split the whole feasible region into four regions
- Solving the **most likelihood** region of two cuts often gives a satisfying solution with **confidence**
- Branching over all four regions **independently** will not miss the optimal solution

Data-Driven Approaches to Mixed Integer Optimization in Practice

Learn from the **past** and **predict the future** such as the unit commitment problem in Electrical Power Generation

- Many real-life MIO applications are solved on a regular basis

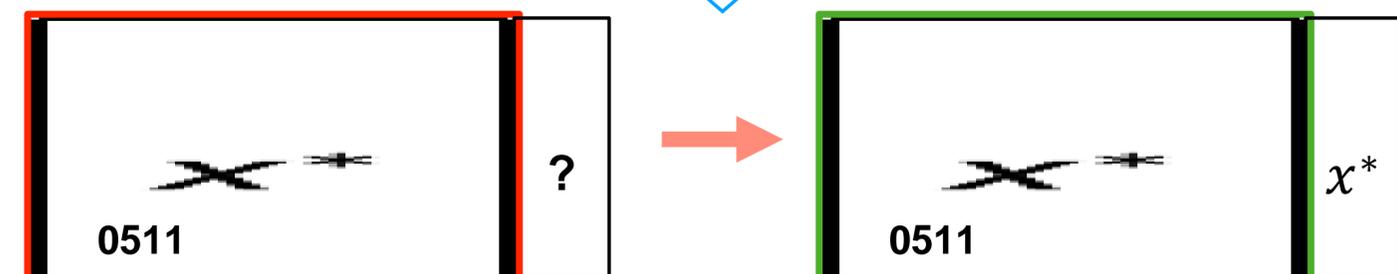
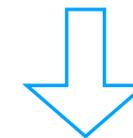
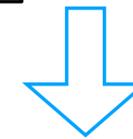
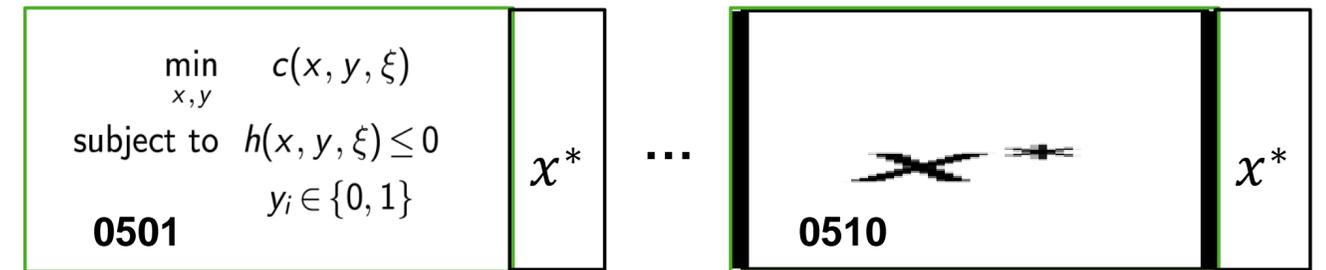
Unit commitment; portfolio; scheduling...

- Large amount of data and solutions collected from the **past**

Future instances are similar to the past

- A natural idea: use machine learning to learn from history

A Pre-trained Data-driven Approaches

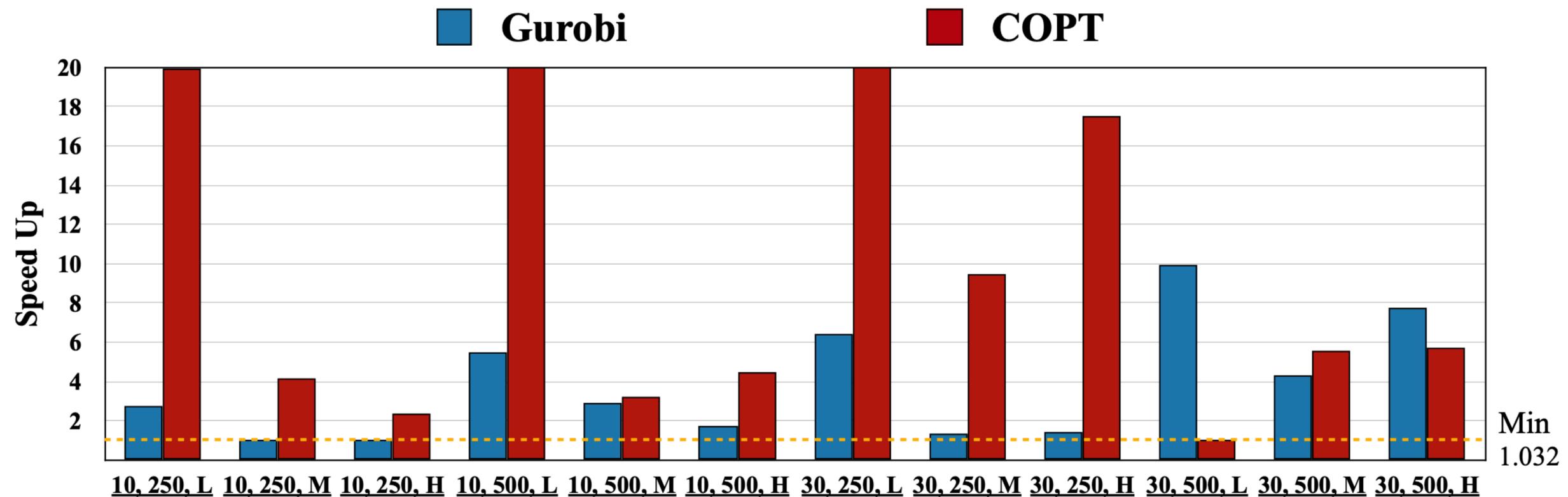


Numerical Test Results I

- The method is tested on multi-knapsack, set-covering and unit-commitment problems

Train from 500 instances and test on 20 instances

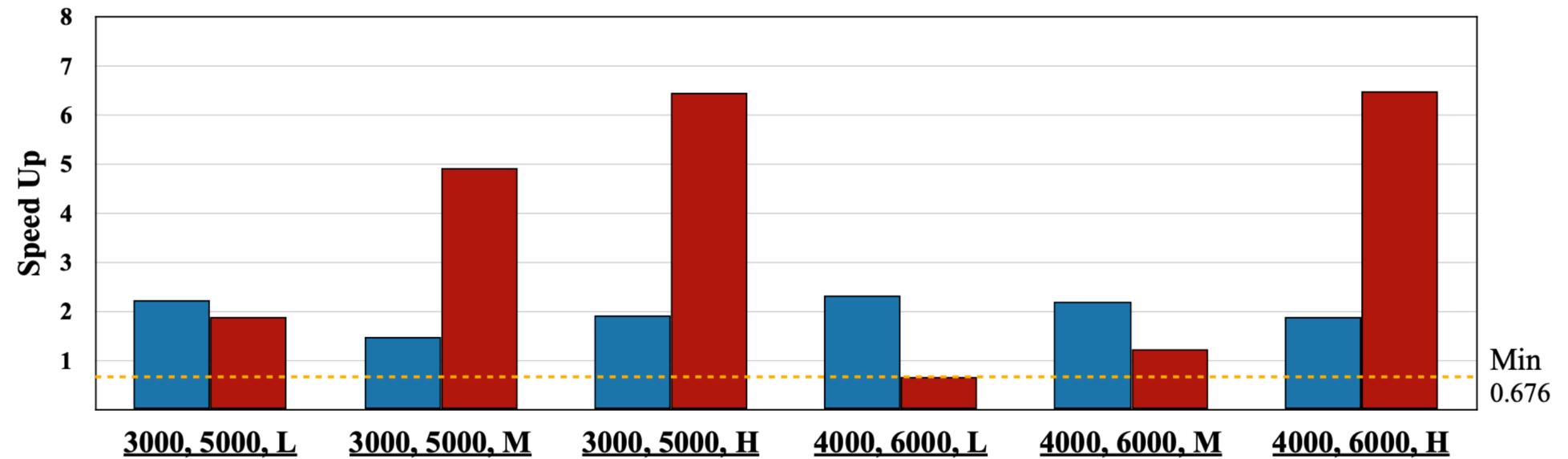
- Measure the speedup of finding a good solution on in the region formed by two cuts



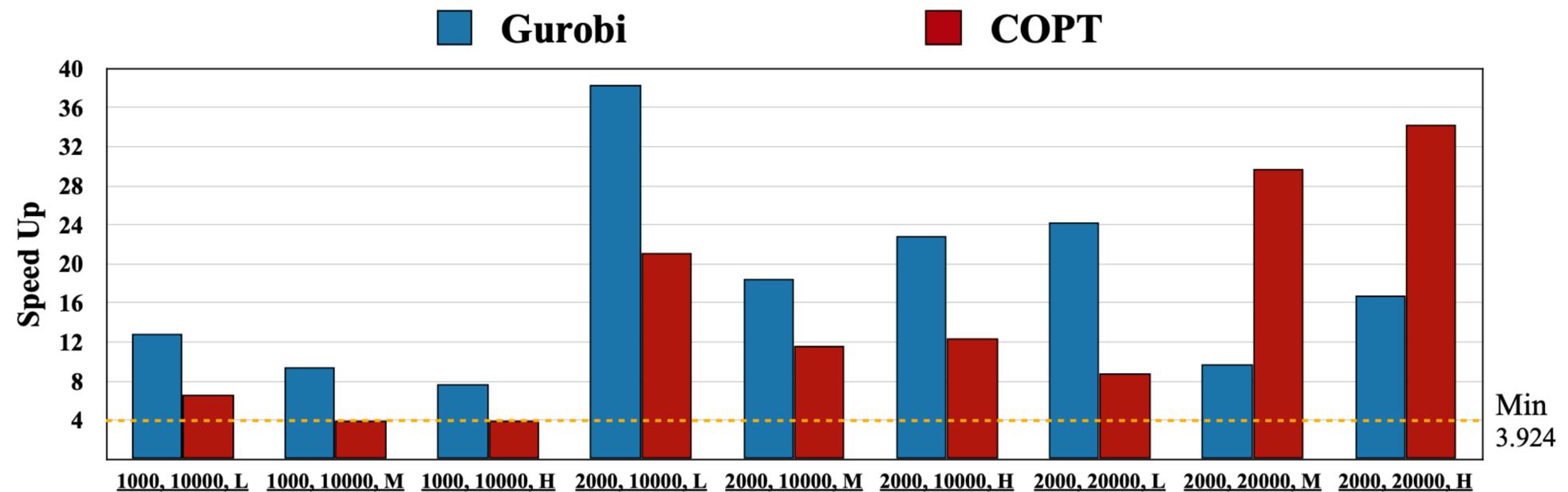
Average speedup on knapsack instances

Numerical Test Results II

- Acceleration by two lines of code
- Remarkable speedup on primal solution finding for both the state of art MIP solvers Gurobi and COPT
- No loss of optimality

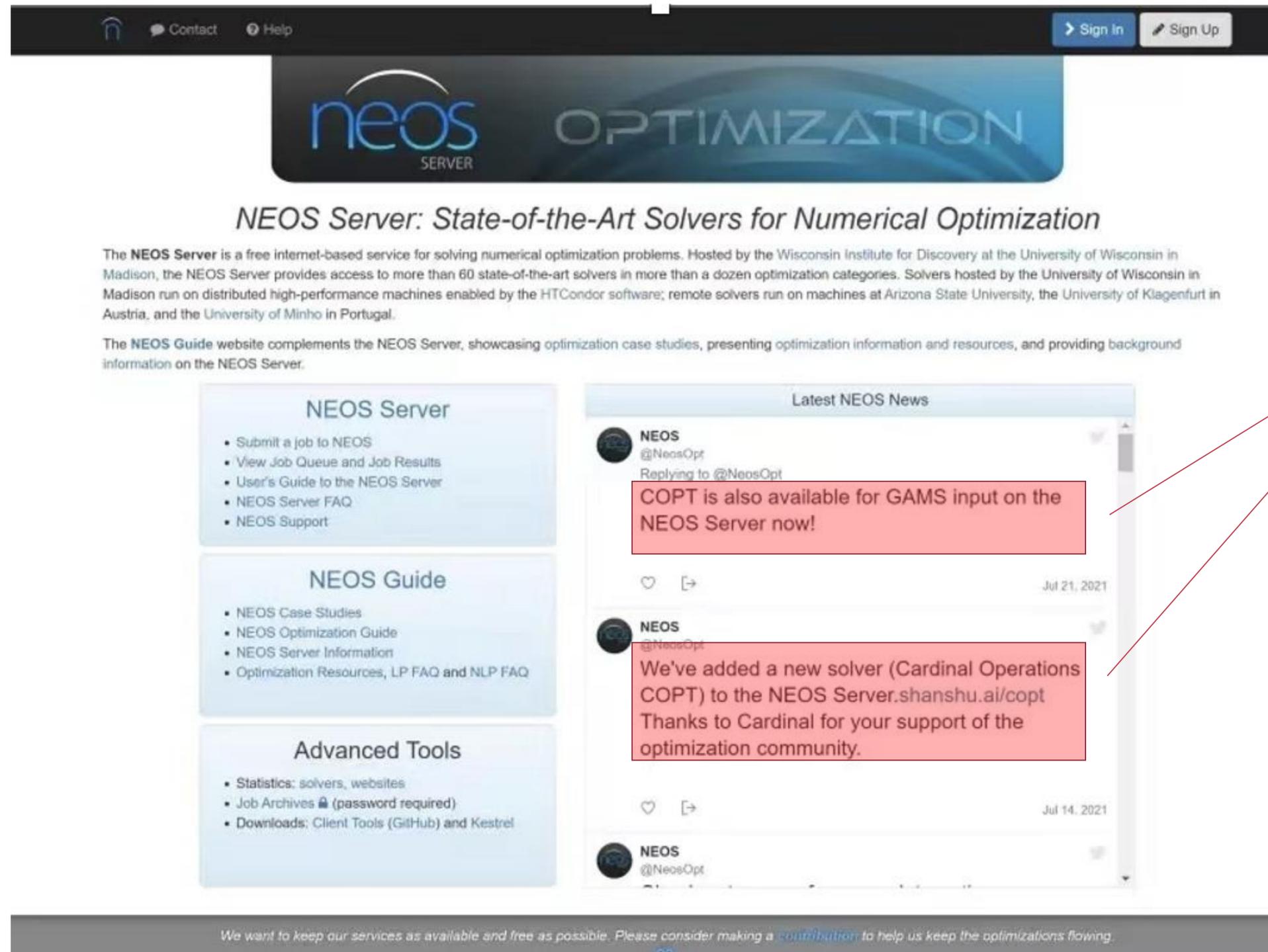


Unit Commitment



Set-Covering

求解问题类型	问题难点与创新内容	国外求解器	ASU榜上国内求解器	比较结果	目前位置
线性规划	矩阵预处理与矩阵分解高效实现，大规模并行等	GUROBI/CPLEX/XPRESS/MATLAB/GOOGLE	杉数/华为/阿里	第一，比美国最快的GUROBI快18%	国际领先
整数规划	数百种算法的调用强度调优和协同工作调优，百万行级软件工程协作	GUROBI/CPLEX/XPRESS/S CIP	杉数	第二，慢于GUROBI，与CPLEX XPRESS基本相当	与世界主流求解器水平基本相当，填补国内空白
半正定规划	针对特殊结构问题的算法切换	GUROBI/MOSEK/SEDUMI	杉数/阿里	第一	国际领先
二阶锥规划	针对稀疏结构的探索和高效算法实现	GUROBI/MOSEK/SEDUMI	杉数	第二	国际一流，填补国内空白
凸二次规划	识别问题是否为凸，并将凸问题并转化为二阶锥规划求解	GUROBI	杉数	第一，比GUROBI快103%	国际领先，填补国内空白
混合整数凸二次规划	抛物面拆分算法	GUROBI	杉数	第二，比GUROBI慢9%	国际领先，填补国内空白
混合整数二阶锥规划	充分利用二阶锥规划松弛	GUROBI/MOSEK	杉数	第二，仅比GUROBI慢	国际一流，填补国内空白



NEOS Server: State-of-the-Art Solvers for Numerical Optimization

The NEOS Server is a free internet-based service for solving numerical optimization problems. Hosted by the Wisconsin Institute for Discovery at the University of Wisconsin in Madison, the NEOS Server provides access to more than 60 state-of-the-art solvers in more than a dozen optimization categories. Solvers hosted by the University of Wisconsin in Madison run on distributed high-performance machines enabled by the HTCCondor software; remote solvers run on machines at Arizona State University, the University of Klagenfurt in Austria, and the University of Minho in Portugal.

The NEOS Guide website complements the NEOS Server, showcasing optimization case studies, presenting optimization information and resources, and providing background information on the NEOS Server.

- NEOS Server**
 - Submit a job to NEOS
 - View Job Queue and Job Results
 - User's Guide to the NEOS Server
 - NEOS Server FAQ
 - NEOS Support
- NEOS Guide**
 - NEOS Case Studies
 - NEOS Optimization Guide
 - NEOS Server Information
 - Optimization Resources, LP FAQ and NLP FAQ
- Advanced Tools**
 - Statistics: solvers, websites
 - Job Archives (password required)
 - Downloads: Client Tools (GitHub) and Kestrel

Latest NEOS News

- NEOS @NeosOpt Replying to @NeosOpt
COPT is also available for GAMS input on the NEOS Server now!
Jul 21, 2021
- NEOS @NeosOpt
We've added a new solver (Cardinal Operations COPT) to the NEOS Server. shanshu.ai/copt
Thanks to Cardinal for your support of the optimization community.
Jul 14, 2021

We want to keep our services as available and free as possible. Please consider making a [contribution](#) to help us keep the optimizations flowing.

- 2021年7月，COPT求解器获得全球最大的优化求解平台NEOS主动邀请，**成为历史上第一个，也是至今唯一的入驻此平台的的中国求解器。**
- NEOS由美国阿贡国家实验室与西北大学在1996年开发。
- 每年执行数百万次来自全球学术和实业机构的计算申请。是全世界最有影响力的数学优化求解云平台。

RESELLER AGREEMENT AMPL Selling (COPT – AMPL)

THIS RESELLER AGREEMENT ("Agreement") is entered into as of (December 1, 2021) ("Effective Date"), by and between Shanghai Cardinal Operations Network Technology Co., Ltd., a Chinese Company with its principal offices at (Room 109, Building 5, No.8, Jinian Road, Yangpu District, Shanghai, China) ("Cardinal Operations"), and AMPL Optimization Inc., a California corporation, with its principal offices at 211 Hope Street #339, Mountain View, CA 94041 USA ("Reseller").

WHEREAS, Cardinal Operations owns and markets the COPT mathematical optimization software with accompanying documentation, demonstration programs and examples ("COPT"); and

WHEREAS, Reseller owns and markets the AMPL modeling language and system with accompanying documentation, demonstration programs and examples ("AMPL"); and

WHEREAS, Cardinal Operations has built, in collaboration with Reseller, a version of (COPT) specifically designed to be used with the AMPL system ("COPT - AMPL"); and

WHEREAS, Reseller wishes to market and distribute COPT - AMPL in conjunction with its marketing and distribution of AMPL;

NOW THEREFORE, in consideration of the premises and mutual obligations herein, the parties agree as follows:

Gertjan de Lange
AIMMS, www.aimms.com

查看更多



GAMS-COPT-Final-Signed-with-price-lists.pdf
Page 1 of 13

Bill Wells October 19, 2021 at 03:39
Re: AMPL COPT Cooperation
To: Huangfu Qi, [Hide](#)
Cc: 4er@ampl.com, Dongdong Ge, 饶倩雯, Christian Valente,
Martin Laskowski, Filipe Brandão

Dear Qi,

Thank you for the note. It comes at a very good time for us as I was just preparing to send you a note on the same topic.

Since we last spoke there have been several discussions and a great deal of product testing here at AMPL, all focused on COPT. Needless to say, we have a high level of interest in your product and we believe that there might already be a market for it within AMPL's prospective customers.

Regarding the testing of COPT, we put it through a fairly extensive set of performance tests. We are very pleased with the results of these tests. They speak well to your own development efforts and how far you have already come with the product even before version 3.0. The high level summary of these results is:

- COPT's performance on LP models is outstanding. It is on par, or perhaps even exceeding, the performance of the very best commercial solvers on the market.
- COPT's performance on MIP models is good. It lags that of the top commercial MIP solvers but is substantially better than the best free solvers and actually closer to the top commercial solvers than the best free solvers.

I would be prepared to share some level of detail regarding our testing process and results if that would be of interest to you. In order to move in that direction we would need to put an NDA in place as this type of information can be very sensitive (as I am sure you understand).

We would be very happy to redo our testing using version 3.0 of COPT to see how great the performance gain is on our own test suites.

Given the results that we have seen so far, we believe that COPT could be a successful product within the AMPL customer base right now. This is something that we would like to talk about in greater detail with you.

Looking in the other direction, (Cardinal selling AMPL), I have gone through our current agreements with our reseller in China. I am happy to let you know that there does not seem to be any impediment to your organization acting as a distributor of AMPL within China, either to academic or commercial customers or both.

There is much to talk about. Perhaps we could set up a discussion later this week involving myself, Martin Laskowski (Director of AMPL marketing), and your team at Cardinal. If that is of interest to you, could you suggest a few times that might work for you? I am looking forward to moving our relationship forward.

King Regards,

Bill Wells
CEO
AMPL Optimization, Inc..
Ph: +1 (773) 336-2675

2021年，优化建模语言和系统
集成最著名的三家国际机构，

- 德国的GAMS
- 美国的AMPL
- 荷兰的AIMMS

分别写信来，主动邀请COPT与
其系统进行集成，进行全球服
务。

**COPT也是至今唯一一个集成
在这几个系统中的中国求解器。
已经销售到美国，英国，韩国，
东欧多个国家。**

Overall Takeaways

It is possible to make online decision/learning for quantitative models with performance guarantees close to that of the offline decision-making with perfect/complete information

Second-Order Derivative information matters and better to integrate FOM and SOM for nonlinear optimization!

Zero-Order Methods are competitive, especially for “Black-Box” optimization

Pre-training greatly improves Mixed Integer LP solvers that benefit real economy

- THANK YOU