A Convex Parimutuel Formulation for Contingent Claim Markets*

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ABSTRACT

In this paper we study the problem of centrally organizing a market where the participants submit bids for contingent claims over the outcome of a future event and the market organizer must determine which bids to accept. The bidder will select a set of future states and a price at which he is willing to buy the contingent claims. By accepting a bid, the market organizer agrees to pay the bidder a fixed amount if one of the bidder's selected states is realized. We will specifically study markets which are run as call auctions where the organizer holds the auction open until a certain time and then determines the bids to accept and reject. This type of market has broad usage in financial markets, betting markets and general prediction markets.

Lange and Economides [8] have developed a parimutuel mechanism for solving such a market with many positive characteristics. However, one drawback of their formulation is that their mathematical model is not convex and no efficient algorithm is known to solve it. In this paper, we introduce a new mathematical formulation called the Convex Parimutuel Call Auction Mechanism (CPCAM). This formulation produces many of the same advantageous properties of the Lange and Economides model but can more easily be solved due to its convexity. In particular, our model yields the first fully polynomial–time approximation scheme (FPTAS) for the problem. Moreover, we show that our model actually produces identical state prices as the Lange and Economides model. As a corollary, we show that by first obtaining the state prices from our model, the Lange and Economides model becomes a linear program and hence can be solved in polynomial time.

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1. INTRODUCTION

1.1 Background

Contingent claim markets enjoy tremendously widespread usage in the financial and betting worlds. In these markets, traders exchange claims to future payouts which are event–based. Some simple examples of contingent claim securities include a call option on a share of Google's stock, a derivative based on United Airlines' potential to default on its outstanding debt, or a bet on which team will win the 2006 Superbowl.

Due to the high degree of flexibility that traders have in their ability to specify event–based payouts, liquidity can become an issue in some markets. If the market is organized so that the types of payouts are restricted (such as requiring that all claims must be call options on stock x with a strike price of $\$y$) and there is a sufficient number of traders, then the market can achieve reasonable liquidity. In these types of markets, it is more effective for traders to interact directly. The transaction costs should be relatively low since all traders are trading the same type of claim.

However, in instances where the number of traders interested in trading claims over some event is smaller and there is not a small set of claim formats that everyone is interested in trading, it may make sense to centrally organize this market. By central organization, we mean that all traders will interact with one market organizer who will conduct trades. The market organizer will issue and guarantee all claims for the market. Without central organization, reasonable liquidity may be difficult to achieve since the cost of making transactions may be high as individual traders may need to conduct several trades to create the specific claim payouts that they desire.

In a centrally organized market, we can think of the traders as making orders to the market organizer. To actually run this type of market, we need to design a mechanism which would inform the market organizer of which orders to accept and which orders to reject. When considering potential mechanisms, there are several features that we would like to capture. First, it would be valuable to allow market traders to place limit orders. Most actual financial trading includes limit orders where traders express a limit on the price they are willing to pay or a limit on the number of shares that they desire. This makes trading more efficient by reducing the number of interactions that the market organizer has with the traders. Secondly, we would want to make sure that the market organizer has a "balanced book" in the sense that she is never exposed to a financial loss for any particular outcome. Furthermore, we need to design an objective for the market organizer that she is trying to optimize. A reasonable objective of the market organizer might be to accept the largest number of orders or to accept the greatest dollar value of orders.

From the individual trader or market participant's point of view, she would like to know that her order is being duly considered by the market organizer. One way to ensure that is for the market organizer to announce state prices at the time of accepting and rejecting orders. An order's calculated state price is simply the sum of the prices of the individual states that are specified in the order. Then, the market organizer agrees to fully accept any order with a limit price greater than the calculated state price of the order while rejecting any order with a lower limit price. We will refer to this requirement as the *price consistency requirement*.

There are many available approaches to solve this problem. We will focus our attention on parimutuel mechanisms due to the fact that these mechanisms possess the key characteristic that they are self-funding. The market organizer will never have to pay out more for realized claims than the amount that he has collected. A parimutuel mechanism is defined as a mechanism where all the promised payouts to traders are funded exclusively by the accepted orders. The most prevalent use of parimutuel mechanisms is in horse racing betting (see, e.g., [3]). In a parimutuel mechanism, the market organizer has no risk of suffering a loss regardless of the outcome of the event in question. We will make a distinction between traditional parimutuel and limit order parimutuel mechanisms.

In a traditional parimutuel approach, the market organizer would charge the market participants a fixed amount of money to make an order containing a claim over one particular state. All orders would be accepted. When the market organizer closes the market and one of the states is realized, the total money collected will be divided out to the holders of claims on that state in proportion of the number of orders that they hold (the market organizer could take out his commission before this distribution). This mechanism exposes the market organizer to no risk and has the advantage of accepting all orders. However, one major drawback is that the actual payout to a participant with an order for the realized state will be uncertain. When the participant's order is accepted, the market organizer could tell her what the payout would be if there were no more orders in the market. However, subsequent orders will change the payouts for realized states. This result does not fit well with the desire of market participants to hold contingent claim securities with known state–dependent payouts.

1.2 Previous Work

In our opinion, the more interesting class of solution is the limit order parimutuel. As we will detail later, the limit order parimutuel mechanism will pay each holder of a claim containing the realized state a fixed payment and it allows participants to submit price and quantity limits for their orders. Lange and Economides [8] have provided a parimutuel model for contingent claims market that is run by a call auction. The market organizer will receive orders for a period of time until the market is closed. At this point, their mechanism will determine an implied price for an order on each state and inform the organizer which orders to accept. The distinction between this mechanism and the traditional parimutuel is that the market organizer guarantees a fixed payout if an order is accepted and one of its specified states is realized. Each market participant will specify a limit price corresponding to the maximum amount she is willing to pay for a contingent claim order. The market organizer will then determine whether to accept their orders and what price to charge. This mechanism will satisfy the price consistency constraints that we referred to earlier in the Introduction.

Lange and Economides' model produces a self–funded auction and creates more liquidity by allowing multi–lateral order matching. A multi–lateral order matching means that the mechanism will attempt to use portions of multiple orders to balance the market organizer's book. This is a more efficient way to offset risk than requiring that orders be matched with only one other order to form a balanced position. The ability to implement multi–lateral order matching allows participants to submit orders with any possible desired claim structure rather than requiring a smaller set of formats to increase liquidity. This model has been implemented by Goldman Sachs and Deutsche Bank to run markets on options for economic data [2].

Despite its many positive characteristics, the Lange and Economides model suffers from two problems. First, the main drawback of their model is that some of the required constraints are not convex. Thus, their model requires special techniques to find the global optimum, and there is no guarantee that those techniques will yield a solution in polynomial time. Secondly, in order to produce unique state prices, the market organizer is required to seed the market with starting orders. While these starting orders can be made small, it is unclear what impact they have on the state prices.

On another front, Yang and Ng [10] have recently developed an alternative limit order parimutuel model named the Qualified–Bound–Pricing Method. They have created a linear parimutuel model with a different objective function and a two–stage solution procedure. Their mechanism has many of the same positive characteristics as the Lange and Economides model such as self–funding, price consistency maintained and providing a guaranteed payout to accepted orders which include the realized state. However, one drawback with this approach is that the solution procedure is iterative and may require one to solve a linear program many times to determine the solution. Another, and perhaps more serious issue is that this model can produce an optimal solution which contains negative state prices.

Table 1: Notations Used in this Paper

Variable	Name	Description
$a_{i,j}$	State Bid	Participant j's bid on state i
q_j	Limit Quantity	Participant j's maximum number of bids requested
π_j	Limit Price	Participant j's maximum price for bid
p_i	Price	Organizer's price level for state i
x_j	Order Fill	Number of participant j's bids accepted
c_j	Bid Price	The price of participant j's bid

1.3 Our Contribution

In this paper, we will describe a new mechanism for centrally organizing a contingent claim market to maximize liquidity. Our model is called the Convex Parimutuel Call Auction Mechanism (CPCAM). We utilize the limit order structure to ensure market participants will know with certainty their payout if their order is accepted and one of their specified states is realized. In particular, we will present the following results:

- We give a convex formulation of the parimutuel call auction mechanism. Using the path–following algorithm recently developed in [11], this results in the first fully polynomial–time approximation scheme (FPTAS) for the problem.
- Our mechanism is parimutuel in the sense that the payouts made by the market organizer will be completely funded by accepted orders.
- We show that our mechanism will satisfy the desired price consistency constraints.
- We prove that the mechanism will produce unique state prices.
- While we require non–zero starting orders to generate the unique state prices, we prove that the state prices converge to a unique state price vector as we drive the magnitude of the starting orders to zero. Moreover, such a price vector can be found using the aforementioned algorithm.
- Our mechanism will find the same state prices as the model of Lange and Economides [8]. Furthermore, the nonlinear, nonconvex model of Lange and Economides can be solved as a linear program after obtaining the state prices from our mechanism.

Thus, the CPCAM provides an easy–to–solve mechanism that can be used to run contingent claim markets. The mechanism is attractive to market organizers due to its solvability and the parimutuel property. Market participants will value the certainty of payouts, the low transaction costs (they merely need to submit a limit order for their desired claim), and the fact that price consistency restrictions are satisfied.

In Section 2, we will give an overview of the Lange and Economides model. In Section 3, we will present our convex formulation and detail some properties that make it attractive to market organizers and participants. We will also show that the solution set of our model is closely related to that of the Lange and Economides model. In Section 4, we present a short application of the CPCAM. Finally, a discussion of some ideas for further analysis will be presented in Section 5.

2. MODELS

Consider a market with one organizer and n participants. There are S states of the world in the future on which the market participants are submitting bids for contingent claims. For each bid that is accepted by the organizer and contains the realized future state, the organizer will pay the participant some fixed amount of money w, which, without loss of generality, equals 1 in this paper. The participants will submit bids to the organizer which specify the states which they want contingent claims over, the price at which they are willing to pay for the bid, and the number of identical bids that they will buy. The organizer will then decide whether to accept or reject each bid. If the bid is accepted, the organizer also decides the number of bids to accept and the price per bid to be collected from the participant.

The market will not be run as a continuous auction. Instead, the market organizer will collect all bids, close the market and then announce the accepted bids, quantities and prices.

Throughout the analysis, we will use the notations in Table 1. The participants will supply the values of $a_{i,j}$, q_j and π_j for all i, j, which are denoted by the matrix A and vectors q and π. Thus, these data are considered given for the models which we will discuss. The market organizer will need to determine the decision variables p_i and x_j for all i, j.

Parimutuel Market Microstructure (PMM) Model

The PMM model was developed by Lange and Economides [8]. They prove that a unique state price vector exists for a contingent claim market when the organizer prices according to their principles. They have designed a nonlinear model which maximizes the total money raised by the market while maintaining constraints for price consistency, market self–funding and fill order feasibility. Their model can be formulated as follows:

$$
\begin{array}{lll}
\text{maximize} & M = c^T x + e^T \theta & \\
\text{subject to} & \sum_i p_i = 1 & \\
& c_j = \sum_i a_{i,j} p_i & \text{for } 1 \le j \le n \\
& M = \sum_j x_j a_{i,j} + \frac{\theta_i}{p_i} & \text{for } 1 \le i \le S \\
& c_j - \pi_j + y_j \ge 0 & \text{for } 1 \le j \le n \\
& x_j (c_j - \pi_j + y_j) = 0 & \text{for } 1 \le j \le n \\
& y_j (q_j - x_j) = 0 & \text{for } 1 \le j \le n \\
& 0 \le x \le q & \\
& p > 0 & \\
& y \ge 0 &
\end{array}
\qquad (1)
$$

In this formulation, θ represents a starting order needed to guarantee uniqueness of the state prices in the solution. The starting orders are not decision variables — the market organizer will provide an order for all of the possible states.

The simplest representation of θ is as an S–dimensional vector composed of all ones. In effect, the market organizer is seeding the market with this order. In some outcomes, the market organizer could actually lose some of this seed money (thus the auction is not completely risk–free). However, the values of the components of the θ vector can be made arbitrarily small. Any changes to the value of the starting orders will result in a new value for the state prices. It is not clear what happens to the state prices as we reduce the magnitude of the starting orders in this model.

Note that the participants will pay $p^T a_j$, where a_j is the j–th column of A, for their order if it is accepted instead of π_j (the limit price). As we will shortly see in the price consistency constraints, for any accepted order, we will have $p^T a_j \leq \pi_j$. Thus, the market organizer could decide to charge a higher price for any accepted order (ensuring that the new price is still less than or equal to the limit price) and keep the additional proceeds as profit.

The third constraint is used to ensure that the money collected is greater than or equal to the money paid out.

The price consistency constraints are equivalent to the following conditions:

$x_j = 0$	$\Longrightarrow p^T a_j = c_j \geq \pi_j$
$0 < x_j < q_j$	$\Longrightarrow p^T a_j = c_j = \pi_j$
$x_j = q_j$	$\Longrightarrow p^T a_j = c_j \leq \pi_j$

The fourth, fifth and sixth constraints are equilibrium or complementarity constraints used to ensure the consistency of the prices. In these three constraints, we use y as a dummy variable to ensure the consistency of the prices. If $x_j = 0$, we have $y_j = 0$ by the sixth constraint, then by the fourth constraint, we have $c_j \geq \pi_j$. If $0 < x_j < q_j$, then we have $y_j = 0$ by the sixth constraint. By the fifth constraint, we have $c_j = \pi_j$. If $x_j = q_j$, then we have $c_j - \pi_j + y_j = 0$ by the fifth constraint. Now, since $y \geq 0$, we have $c_j \leq \pi_j$.

While this formulation captures the constraints of the PMM model adequately, unfortunately, this is not a convex program. In particular, the complementarity constraints are not generally convex. We do not believe that there is a manner to incorporate the price consistency conditions as constraints without making the formulation non–convex. Thus, it is not clear whether there exists an efficient algorithm for this formulation.

3. CONVEX PARIMUTUEL CALL AUCTION MECHANISM (CPCAM)

We would like to find an alternative formulation of this problem which have similar constraints as the PMM model but is also a convex program. The primary constraints are to ensure that the market is self–funding and that the quantities granted to each participant are consistent based on the relationship of their limit price and the calculated state price of the order. Furthermore, it is valuable that the model has a unique optimum. Below is our alternative parimutuel formulation, again, with $w = 1$:

$$\begin{array}{lll} \text{maximize} & \pi^T x - M + \sum_i \theta_i \log(s_i) & \\ \text{subject to} & \sum_j a_{i,j} x_j + s_i = M & \text{for } 1 \leq i \leq S \\ & 0 \leq x \leq q & \\ & s \geq 0 & \end{array} \tag{2}$$

The objective function in this formulation has the following interpretation. First, recall that θ is the vector of starting orders. Now, the term $\pi^T x - M$ is the profit to the market organizer. On the other hand, the term

$$\sum_i \theta_i \log(s_i) = \sum_i \theta_i \log\left(M - \sum_j a_{i,j} x_j\right)$$

can be viewed as a disutility function (or weighted logarithmic penalty function) for the market organizer that ensures she will find an allocation of accepted orders that is parimutuel. Thus, intuitively, the model (2) is trying to maximize the profit of the market organizer while remaining parimutuel.

In this section, we will show that the CPCAM model possesses the following characteristics:

- the CPCAM is a convex program
- the CPCAM creates a self–funding market
- the CPCAM satisfies the price consistency constraints
- the CPCAM produces a unique price vector
- the price vector solution to the CPCAM remains optimal if we charge the participants the calculated state price of their orders instead of their limit price
- for any given starting order θ, there is a unique limit to the state price vector as we reduce θ to zero with each element reduced by the same proportion

To begin, it is easy to see that (2) is a convex program, as the logarithmic function is concave and the constraints are linear. In particular, it can be solved (up to any prescribed accuracy) in polynomial time using standard techniques. Now, let p_i be the Lagrange multipliers for the first market self–funding constraints and γ_j be the Lagrange multipliers for the constraints $x \leq q$. Then, we have the following Lagrangian function:

$$\begin{aligned} L(x, M, s) &= \pi^T x - M + \sum_i \theta_i \log(s_i) \\ &\quad - \sum_i p_i \left(\sum_j a_{i,j} x_j + s_i - M\right) \\ &\quad + \sum_j \gamma_j (x_j - q_j) \end{aligned}$$

From this Lagrangian, we can derive the following KKT conditions for optimality:

$$\begin{array}{rl} \pi_j - \sum_i p_i a_{i,j} + \gamma_j \leq 0 & \text{for } 1 \leq j \leq n \\ x_j \left(\pi_j - \sum_i p_i a_{i,j} + \gamma_j\right) = 0 & \text{for } 1 \leq j \leq n \\ \sum_i p_i = 1 & \\ \frac{\theta_i}{s_i} - p_i \geq 0 & \text{for } 1 \leq i \leq S \\ s_i \left(\frac{\theta_i}{s_i} - p_i\right) = 0 & \text{for } 1 \leq i \leq S \\ \gamma_j (x_j - q_j) = 0 & \text{for } 1 \leq j \leq n \\ \gamma \leq 0 & \end{array} \tag{3}$$

Since $s_i > 0$ for any optimal solution, this implies that $p_i = \frac{\theta_i}{s_i}$, or $s_i = \frac{\theta_i}{p_i}$ for all i. Thus, the first constraint in the

CPCAM model is equivalent to $M = \sum_j a_{i,j}x_j + \frac{\theta_i}{p_i}$, which is precisely the self–funding constraint from the PMM model.

We note that the KKT conditions require that the p_i's be summed to one and, furthermore, we can derive the following relationships from the KKT conditions:

$x_j = 0$	$\Longrightarrow p^T a_j \geq \pi_j$
$0 < x_j < q_j$	$\Longrightarrow p^T a_j = \pi_j$
$x_j = q_j$	$\Longrightarrow p^T a_j \leq \pi_j$

Again, these are simply the consistency constraints of the PMM model.

Note that the CPCAM model involves the market organizer collecting π_j (the limit price) for each *accepted* order of the j–th participant. On the other hand, the PMM model collects $p^T a_j$ (the parimutuel price) per accepted order. We would like to determine whether the CPCAM model would have a different optimum if it was modified so it charged the participants the parimutuel price rather than the limit price. The answer is summarized in the following proposition.

PROPOSITION 1. *The optimal solution (x^*, p^*) from the CPCAM model would also be optimal if we replaced π_j with $(p^*)^T a_j$ in the objective function. Furthermore, the solution will remain optimal if we replace π_j with any c_j where $\pi_j \leq c_j \leq (p^*)^T a_j$.*

PROOF. If we replace π_j with $(p^*)^T a_j$ in the KKT conditions (3) and set $x = x^*$, $p = p^*$, $s = s^*$ and $\gamma = 0$, then our optimal solution (x^*, p^*, s^*) from the CPCAM model, with $\gamma = 0$, will still satisfy all the KKT conditions. Thus, the optimal solution found using the CPCAM model would be equivalent to the optimal solution found if the objective function included $(p^*)^T a_j$ instead of π_j. In fact, the solution (x^*, p^*) will be optimal if $\pi^T x$ is replaced by $c^T x$ where c is any vector that satisfies $\pi_j \leq c_j \leq (p^*)^T a_j$ for all j. This is due to the fact that we can simply adjust the value of γ while leaving x and p unchanged to satisfy the KKT conditions. $\square$

Next, we would like to determine whether the prices in the CPCAM model will be unique as they are in the PMM model.

THEOREM 1. *The state price vector p^* for any optimal solution to the CPCAM model is unique.*

PROOF. We know from the KKT conditions that $p_i s_i = \theta_i$ for all i. Now, if the s_i's are unique then the p_i's must also be unique. Thus, we will show that the s_i's must be unique.

We will first state a general lemma applying to the following problem:

$$\begin{array}{ll} \text{minimize} & c^T x - \sum_{j=1}^n w_j \log(x_j) \\ \text{subject to} & \bar{A}x = b \\ & x \geq 0 \end{array} \tag{4}$$

where $\bar{A}$ is an $m \times n$ matrix with full row rank and the feasible region is non–empty and has an interior. Our proof is similar to that found in [11].

LEMMA 1. *Let $x^* = (x_1^*, \ldots, x_n^*)$ be an optimal solution to (4), and let $\mathcal{J} = \{1 \leq j \leq n : w_j > 0\}$. Then, x_j^* is unique for all $j \in \mathcal{J}$, i.e. if $x' = (x_1', \ldots, x_n')$ is another optimal solution to (4), then we have $x_j^* = x_j'$ for all $j \in \mathcal{J}$.*

PROOF. Without loss of generality, we may assume that $w_1, \ldots, w_k > 0$ and the remaining w_j are zero. Then, the necessary and sufficient optimality conditions are:

$$\begin{aligned} z_j x_j &= w_j && \text{for } 1 \leq j \leq k \\ z_j x_j &= 0 && \text{for } k+1 \leq j \leq n \\ z + \bar{A}^T y &= c \\ \bar{A}x &= b \\ x, z &\geq 0 \end{aligned}$$

The proof is by contradiction. Let (x^*, y^*, z^*) be an optimal solution and (x', y', z') be another optimal solution where $x_1^* \neq x_1'$. Since we have:

$$x_1^* z_1^* = x_1' z_1' = w_1 > 0$$

we must have $z_1^* \neq z_1'$, and

$$(x_1^* - x_1')(z_1^* - z_1') < 0$$

On the other hand, since $(x^* - x')$ is in the nullspace of $\bar{A}$ and $(z^* - z')$ is in the row–space of $\bar{A}$, we have:

$$(x^* - x')^T (z^* - z') = \sum_{j=1}^n (x_j^* - x_j')(z_j^* - z_j') = 0$$

Thus, there must be at least one $\bar{j} \in \{2, \ldots, n\}$ such that:

$$(x_{\bar{j}}^* - x_{\bar{j}}')(z_{\bar{j}}^* - z_{\bar{j}}') > 0$$

Without loss of generality, suppose that $x_{\bar{j}}^* > x_{\bar{j}}' \geq 0$. If $\bar{j} \leq k$, then we must have $z_{\bar{j}}^* \geq z_{\bar{j}}' > 0$ and $w_{\bar{j}} = x_{\bar{j}}^* z_{\bar{j}}^* > x_{\bar{j}}' z_{\bar{j}}' = w_{\bar{j}}$, which is a contradiction. If $\bar{j} > k$, then we must have $z_{\bar{j}}^* \geq z_{\bar{j}}' \geq 0$ which, from $x_{\bar{j}}^* z_{\bar{j}}^* = 0$, implies that $0 = z_{\bar{j}}^* = z_{\bar{j}}'$. This contradicts the fact that $(x_{\bar{j}}^* - x_{\bar{j}}')(z_{\bar{j}}^* - z_{\bar{j}}') > 0$.

Therefore, we must have $(x_1^* - x_1')(z_1^* - z_1') = 0$ which, together with $x_1^* z_1^* = x_1' z_1' = w_1 > 0$, imply that:

$$x_1^* = x_1' \quad \text{and} \quad z_1^* = z_1'$$

In a similar fashion, we can show that $x_j^* = x_j'$ and $z_j^* = z_j'$ for $j = 2, \ldots, k$. This completes the proof. $\square$

Now, we can rewrite the CPCAM model in the form of the above problem. First, let us add a slack variable for the constraint $x \leq q$. We can rewrite this constraint as $x + u = q$ with $u \geq 0$. Let

$$\bar{x} = \begin{pmatrix} s \\ x \\ M \\ u \end{pmatrix}, \quad \bar{\theta} = \begin{pmatrix} \theta \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ -\pi \\ 1 \\ 0 \end{pmatrix} \tag{5}$$

$$b = \begin{pmatrix} 0 \\ q \end{pmatrix}, \quad \bar{A} = \begin{bmatrix} I & A & -e & 0 \\ 0 & I & 0 & I \end{bmatrix}$$

where e is the vector of all ones. Then, we can rewrite the CPCAM model as:

$$\begin{array}{ll} \text{minimize} & c^T \bar{x} - \sum_j \bar{\theta}_j \log(\bar{x}_j) \\ \text{subject to} & \bar{A}\bar{x} = b \\ & \bar{x} \geq 0 \end{array}$$

From the lemma above, it follows that all $s_i^* = \bar{x}_i^*$, where $i = 1, \ldots, S$, will be unique since $\bar{\theta}_i = \theta_i > 0$ for $1 \leq i \leq S$. Thus, the state prices $p_i^* = \theta_i / s_i^*$ for any optimal solution to the CPCAM model will be unique as well. This completes the proof of Theorem 1.

3.1 Elimination of Starting Orders Sizes

One somewhat unnatural part of this formulation is the use of the starting orders. In essence, the market organizer needs to seed the market with these starting orders to guarantee the uniqueness of the optimal state price vector. However, the market organizer could actually lose this seed money in some outcomes. In practice, we can set the starting orders to be very small so that this is not an issue. On the other hand, it is natural to ask whether the starting orders can be removed altogether from the model. In this section, we will show that this is indeed possible.

For each starting order θ, there will be a unique state price vector. Now, imagine that we set the starting order to be equal to $\mu\theta$ with $0 < \mu < 1$. Now, as we reduces μ to zero, we would like to know what happens to the resulting state price vector. We will show below that the state price vector will converge to a unique limit point as $\mu \searrow 0$. In fact, we shall establish this result under a more general setting.

To begin, let $1 \leq k < n$ and $\theta_1, \ldots, \theta_k > 0$ be such that $\sum_i \theta_i = 1$. Let $\mu > 0$ be given, and consider the following parametric problem:

$$(P_\mu): \quad \begin{array}{ll} \text{minimize} & c^T x - \mu \sum_{i=1}^k \theta_i \log(x_i) \\ \text{subject to} & \bar{A}x = b \\ & x \geq 0 \end{array}$$

It is easy to see that the *parametric* CPCAM model:

$$\begin{array}{lll} \text{maximize} & \pi^T x - M + \mu \sum_i \theta_i \log(s_i) & \\ \text{subject to} & \sum_j a_{i,j} x_j + s_i = M & \text{for } 1 \leq i \leq S \\ & 0 \leq x \leq q & \\ & s \geq 0 & \end{array}$$

can be cast into the form above. Indeed, using the notations of (5) we can rewrite the parametric CPCAM model as follows:

$$\begin{array}{ll} \text{minimize} & c^T \bar{x} - \mu \sum_{i=1}^S \bar{\theta}_i \log(\bar{x}_i) \\ \text{subject to} & \bar{A}\bar{x} = b \\ & \bar{x} \geq 0 \end{array}$$

Henceforth, we shall restrict our attention to (P_μ) where, for notational simplicity, we shall denote $\bar{A}$ by A. Now, the dual associated with (P_μ) is given by:

$$(D_\mu): \quad \begin{array}{ll} \text{maximize} & b^T y + \mu \sum_{i=1}^k \theta_i \log(z_i) \\ \text{subject to} & A^T y + z = c \\ & z \geq 0 \end{array}$$

Let $x(\mu)$ be the minimizer for (P_μ) and $(y(\mu), z(\mu))$ be the maximizer for (D_μ). We are interested in the behavior of $(x(\mu), y(\mu), z(\mu))$ as $\mu \searrow 0$. The optimality conditions for (P_μ) are given by:

$$\begin{array}{rl} c_i - (A^T y)_i - \dfrac{\mu\theta_i}{x_i} = 0 & \text{for } 1 \leq i \leq k \\ c_i - (A^T y)_i \geq 0 & \text{for } k < i \leq n \\ x_i \left(c_i - (A^T y)_i \right) = 0 & \text{for } k < i \leq n \end{array}$$

Similarly, the optimality conditions for (D_μ) are given by:

$$\begin{array}{rl} b = Ax & \\ \dfrac{\mu\theta_i}{z_i} - x_i = 0 & \text{for } 1 \leq i \leq k \\ x_i \geq 0 & \text{for } k < i \leq n \\ x_i z_i = 0 & \text{for } k < i \leq n \end{array}$$

Note that $z = c - A^T y$. It follows that $x(\mu)^T z(\mu) = \mu$ (recall that $\sum_i \theta_i = 1$). The following lemma is straightforward.

LEMMA 2. *Let $x(\mu) = (x^1(\mu), x^2(\mu))$, where:*

$$\begin{array}{rcl} x^1(\mu) & = & (x_1(\mu), \ldots, x_k(\mu)) \\ x^2(\mu) & = & (x_{k+1}(\mu), \ldots, x_n(\mu)) \end{array}$$

Let $z(\mu) = (z^1(\mu), z^2(\mu))$ be defined analogously. Let $\mu^0 > 0$ be given. Then, for any $0 < \mu \leq \mu^0$, we have $(x^1(\mu), z^1(\mu))$ bounded.

PROOF. Since $x(\mu^0) - x(\mu)$ belongs to the nullspace of A and $z(\mu^0) - z(\mu)$ belongs to the row–space of A, we have $(x(\mu^0) - x(\mu))^T (z(\mu^0) - z(\mu)) = 0$. It follows that:

$$\begin{array}{rl} & \displaystyle\sum_{j=1}^k x(\mu^0)_j z(\mu)_j + x(\mu)_j z(\mu^0)_j \\ \leq & \displaystyle\sum_{j=1}^n x(\mu^0)_j z(\mu)_j + x(\mu)_j z(\mu^0)_j \\ \leq & 2\mu^0 \end{array}$$

Note that for $j = 1, \ldots, k$, we have $x(\mu^0)_j z(\mu^0)_j = \mu^0 > 0$. It follows that:

$$\sum_{j=1}^k \theta_j \left(\frac{z(\mu)_j}{z(\mu^0)_j} + \frac{x(\mu)_j}{x(\mu^0)_j} \right) \leq 2$$

which completes the proof. □

Now, let $\{\mu_n\}$ be such that $\mu_n \searrow 0$. Lemma 2 implies that each of the sequences $\{x^1(\mu_n)\}$ and $\{z^1(\mu_n)\}$ has an accumulation point. We will show that $x^1(\mu_n)$ (resp. $z^1(\mu_n)$) actually converges to a unique limit point $x^1(0)$ (resp. $z^1(0)$). To do this, we first recall a well–known fact from the theory of linear programming and introduce some notations. Consider the following primal–dual pair of linear programs:

$$(\text{LP–P}): \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

$$(\text{LP–D}): \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + z = c \\ & z \geq 0 \end{array}$$

By a result of Goldman and Tucker [5], there exists a pair of strictly complementary solutions (x^*, z^*) to (LP–P) and (LP–D). Let (P^*, D^*) be the strict complementarity partition of the index set $\{1, \ldots, n\}$, i.e. $P^* = \{j : x_j^* > 0\}$ and $D^* = \{j : z_j^* > 0\}$, and set $\overline{P^*} \equiv P^* \cap \{1, \ldots, k\}$ and $\overline{Z^*} \equiv D^* \cap \{1, \ldots, k\}$. Note that such a partition is unique, i.e. every pair of strictly complementary solutions to (LP–P)

Table 2: Bidding Information

Order	State 1	State 2	State 3	State 4	State 5	Price Limit	Quantity Limit
1	0	0	0	1	1	0.4032	100
2	1	0	0	1	1	0.95	100
3	0	0	1	0	0	0.5486	100
4	0	0	0	1	1	0.40	100
5	0	1	0	1	1	0.95	100
6	0	1	0	0	0	0.50	100
7	0	1	1	0	0	0.40	100
8	0	1	0	0	0	0.5938	100

and (LP–D) defines the same partition. Now, for $x, z \in \mathbb{R}^n$, we define a potential function Λ as follows:

$$\Lambda\left(x_{\overline{P^*}}, z_{\overline{D^*}}\right) = \sum_{j \in \overline{P^*}} \theta_j \log(x_j) + \sum_{j \in \overline{D^*}} \theta_j \log(z_j)$$

where $x_{\overline{P^*}}$ and $z_{\overline{D^*}}$ are the vectors $(x_j)_{j \in \overline{P^*}}$ and $(z_j)_{j \in \overline{D^*}}$, respectively. Since Λ is strictly concave, it has a unique maximizer $\left(x^*_{\overline{P^*}}, z^*_{\overline{D^*}}\right)$ over all pairs of strictly complementary solutions to (LP–P) and (LP–D). We now show that:

THEOREM 2. *As $\mu \searrow 0$, the solutions $x^1(\mu)$ and $z^1(\mu)$ converge to the unique limit points $x^1(0) = x^*_{\overline{P^*}}$ and $z^1(0) = z^*_{\overline{D^*}}$, respectively.*

PROOF. Let $(x^*_{P^*}, z^*_{D^*})$ be a pair of strictly complementary solutions to (LP–P) and (LP–D) that maximizes the potential function Λ. By strict complementarity, we have:

$$\sum_{j=1}^{n} \left(x(\mu)_j z^*_j + x^*_j z(\mu)_j\right) = \mu$$

from which it follows that:

$$\sum_{j=1}^{k} \theta_j \left(\frac{z^*_j}{z(\mu)_j} + \frac{x^*_j}{x(\mu)_j}\right) + \sum_{j>k} \frac{1}{\mu}\left(x(\mu)_j z^*_j + x^*_j z(\mu)_j\right) = 1 \tag{6}$$

In particular, since each of the summands is non–negative, we have:

$$\begin{aligned} x(\mu)_j &\geq \theta_j x^*_j > 0 \qquad &\text{for } j \in \overline{P^*} \\ z(\mu)_j &\geq \theta_j z^*_j > 0 \qquad &\text{for } j \in \overline{D^*} \end{aligned}$$

from which it follows that $x(\mu)_j \to 0$ for $j \in \overline{D^*}$ and $z(\mu)_j \to 0$ for $j \in \overline{P^*}$ as $\mu \searrow 0$. Moreover, by (6), we have:

$$\sum_{j=1}^{k} \theta_j \left(\frac{z^*_j}{z(\mu)_j} + \frac{x^*_j}{x(\mu)_j}\right) = \sum_{j \in \overline{P^*}} \frac{\theta_j x^*_j}{x(\mu)_j} + \sum_{j \in \overline{D^*}} \frac{\theta_j z^*_j}{z(\mu)_j} \leq 1$$

By the weighted arithmetic–mean–geometric–mean inequality, it follows that:

$$\left[\prod_{j \in \overline{P^*}} \left(\frac{x^*_j}{x(\mu)_j}\right)^{\theta_j}\right] \left[\prod_{j \in \overline{D^*}} \left(\frac{z^*_j}{z(\mu)_j}\right)^{\theta_j}\right] \leq 1$$

or equivalently,

$$\begin{aligned} &\left[\prod_{j \in \overline{P^*}} \left(x^*_j\right)^{\theta_j}\right] \left[\prod_{j \in \overline{D^*}} \left(z^*_j\right)^{\theta_j}\right] \\ \leq \quad &\left[\prod_{j \in \overline{P^*}} \left(x(\mu)_j\right)^{\theta_j}\right] \left[\prod_{j \in \overline{D^*}} \left(z(\mu)_j\right)^{\theta_j}\right] \end{aligned} \tag{7}$$

Now, note that the LHS is simply the maximum value of Λ over all pairs of strictly complementary solutions to (LP–P) and (LP–D), and $(x(0)_{\overline{P^*}}, z(0)_{\overline{D^*}})$ is one such pair of solutions. It follows that (7) holds with equality, with $x(0)_{\overline{P^*}} = x^*_{\overline{P^*}}$ and $z(0)_{\overline{D^*}} = z^*_{\overline{D^*}}$ by the uniqueness of the maximizer. This completes the proof. □

In particular, Theorem 2 allows us to eliminate the starting orders altogether by taking $\mu \searrow 0$, and our model will still yield a unique price vector p such that the market is self–funding and the prices are consistent. Moreover, such an p can be computed efficiently using the path–following algorithm developed recently in [11]. Specifically, from (5), we see that $\bar{x} \in \mathbb{R}^N$, and $\bar{A}$ is an $M \times N$ matrix, where $M = 2n$ and $N = 2n + S + 1$. By the result of [11], we can compute, for any $\epsilon > 0$, a solution $(\bar{x}, \bar{y}, \bar{z})$ that satisfies:

$$\begin{aligned} \|\bar{Z}\bar{x} - \bar{\theta}\| &\leq \epsilon \\ \bar{A}\bar{x} &= \bar{b} \\ -\bar{A}^T\bar{y} + \bar{z} &= 0 \\ \bar{x}, \bar{z} &\geq 0 \end{aligned}$$

in $O(\sqrt{N}\log(\epsilon^{-1} N \max(\theta))$ iterations, and each iteration solves a system of linear equations in $O(NM^2 + M^3)$ arithmetic operations. Here, we have $\bar{Z} = \text{Diag}(\bar{z})$ and $\max(\theta) = \max_i \theta_i$. To the best of our knowledge, this is the first fully polynomial–time approximation scheme (FPTAS) for the contingent claim markets problem.

3.2 Example of Limiting Behavior

As stated previously, we have introduced the starting orders θ into our model to guarantee the uniqueness of the state prices. However, such uniqueness is no longer guaranteed when $\theta = 0$, as there are instances with multiple optimal solutions. For these instances, an interesting question is how the selection of θ impacts the limiting solution when we examine the model with $\mu\theta$ as the starting orders and gradually drive μ to zero.

Below we construct an example where we examine two different θ vectors (θ_1 and θ_2) that lead to different state price vectors as $\mu \searrow 0$. The allocations, $x_{\mu\theta_1}$ and $x_{\mu\theta_2}$, in these two cases will converge to the same point as $\mu \searrow 0$. However, we will see that the limiting prices will be different in the two cases.

In our example, there are 5 states and 8 bids on those states. Table 2 shows the bidding information.

Now, if we set $\theta = 0$ and solve the model, we will find the following prices:

p_1	p_2	p_3	p_4	p_5
0	0.5	0.05	0.45	0

However, notice that these prices are not unique. In particular, note that the bids always include both states 4 and 5 or neither state 4 nor 5. Thus, the model cannot distinguish between these states. In fact, any price vector p that satisfies $p_4 + p_5 = 0.45$ and with the other prices as stated above will be optimal.

Now, we examine the convergence of the state price vector for two different θ vectors as $\mu \searrow 0$. We will normalize each θ vector such that the sum of its components will be equal to one. Here are our two θ vectors:

–	State 1	State 2	State 3	State 4	State 5
θ_1	0.2	0.2	0.2	0.2	0.2
θ_2	0.167	0.167	0.167	0.333	0.167

Now, let us examine the paths and limiting state prices for each of these θ vectors as $\mu \searrow 0$. Below is a table of the state prices for various values of μ when θ_1 is used:

μ	p_1	p_2	p_3	p_4	p_5
1000	0.184	0.226	0.184	0.203	0.203
100	0.081	0.434	0.081	0.202	0.202
10	0.025	0.500	0.025	0.225	0.225
1	0.025	0.500	0.025	0.225	0.225
0.1	0.025	0.500	0.025	0.225	0.225

We see that p_4 and p_5 both converge to 0.225. This is an optimal solution to the problem when $\theta = 0$. However, the introduction of the non–zero starting orders results in a unique price vector.

Next, we have a similar table for θ_2:

μ	p_1	p_2	p_3	p_4	p_5
1000	0.148	0.202	0.163	0.324	0.163
100	0.071	0.456	0.071	0.269	0.135
10	0.025	0.500	0.025	0.300	0.150
1	0.025	0.500	0.025	0.300	0.150
0.1	0.025	0.500	0.025	0.300	0.150

Again, this is an optimal solution to the problem when $\theta = 0$. However, since we have different weights on the components of θ_2, we converge to a different price vector than θ_1. In terms of the order allocation, it turns out that both $x_{\mu\theta_1}$ and $x_{\mu\theta_2}$ converge to the same order allocation x as $\mu \searrow 0$. The limiting allocation x is given in the following table:

Order	x
1	0
2	100
3	100
4	0
5	0
6	0
7	0
8	100

In summary, this example shows the limiting behavior of the price vector with different θ vectors. In particular, one can view the entries of the θ vector as *relative* weights assigned to each state, and the limiting state prices will depend only on the *proportion* of these weights and not on their magnitudes. This gives the market organizer additional flexibility in her pricing, as she is free to choose the θ vector. In addition, the limiting allocations $x_{\mu\theta}$ will also converge to an optimal allocation. This is an important fact for the market organizer since the allocations are important to her as they determine her outcomes. She does not necessarily care about what the state prices are.

3.3 Solution Similarity

Our CPCAM model provides many of the stated benefits of the PMM model but is also a convex program and, thus, easier to solve. In this section, we show that in fact the feasible solutions of the PMM model correspond precisely to the optimal solutions of the CPCAM model. To begin, let us consider the PMM model (1) and the CPCAM model (2). It is clear that the constraints in the PMM model are precisely the KKT conditions (3) of our CPCAM model. Since there is a unique state price vector p^* that satisfies (3) by Theorem 1, we see that the PMM model will necessarily generate the same state price vector as the CPCAM model. Now, let $s^* > 0$ be the (unique) vector that satisfies $p_i^* s_i^* = \theta_i$ for $i = 1, 2, \ldots, S$. Then, we see that (1) is equivalent to the following problem:

$$\begin{array}{lll} \text{maximize} & (p^*)^T Ax & \\ \text{subject to} & Ax - Me = s^* & \\ & 0 \le x \le q & \\ & A^T p^* - \pi + y \ge 0 & \\ & x_j((p^*)^T a_j - \pi_j + y_j) = 0 & \text{for } 1 \le j \le n \\ & y_j(q_j - x_j) = 0 & \text{for } 1 \le j \le n \\ & y \ge 0 & \end{array} \tag{8}$$

Since any feasible solution x of (8) must have $x_j = 0$ if $\pi_j < (p^*)^T a_j$, we see that problem (8) can be further relaxed to the following problem:

$$\begin{array}{lll} \text{maximize} & (p^*)^T Ax & \\ \text{subject to} & Ax - Me = s^* & \\ & x_j = 0 & \text{if } \pi_j < (p^*)^T a_j \\ & 0 \le x \le q & \end{array} \tag{9}$$

On ther other hand, every feasible soluion of (9) is also feasible for (8) by assigning $y_j = 0$ if $0 \le x_j < q_j$ and $y_j = \pi_j - (p^*)^T a_j$ if $x_j = q_j$. Thus, we summarize as follows:

THEOREM 3. *The set of feasible solutions of the PMM model (1) coincides with the set of optimal solutions of the CPCAM model (2) and they produce the identical state price vector. Furthermore, the PMM model can be solved as a linear program (cf. (9)) after obtaining the state price vector p^*.*

In summary, we have shown that the CPCAM model will produce unique prices if the market organizer provides positive starting orders for each state. In addition, we have seen from the KKT conditions that these prices will be consistent and non–negative. The CPCAM model will produce a market that is self–funding (except in the case of the starting orders). We have also shown that the same prices will be found if we had modified the objective function to contain $(p^*)^T a_j$ instead of π_j. Therefore, the CPCAM model shares several key characteristics of the PMM model. However, the CPCAM model also happens to be a convex program and hence can be solved (up to any prescribed accuracy) in polynomial time.

4. SAMPLE APPLICATION

To demonstrate the CPCAM, we ran a limit order call auction for students in a graduate optimization course at Stanford. The auction was organized around the 2004 NFL playoffs. Students were bidding on claims for the Superbowl winner. At the time that we ran the auction, there were 8 teams remaining in the playoffs: Atlanta, Indianapolis, Minnesota, New England, New York Jets, Philadelphia, Pittsburgh and St. Louis. If a student had a bid accepted and had specified the winning team, they would be awarded one extra credit point. The student's net extra credit would be the total extra credit points won minus the limit prices of his accepted orders. We placed restrictions on the number of orders that each student could submit (100). During the course of the auction, we received orders from 48 students with a total of 4,375 orders. After solving the CPCAM model, we accepted 1,980 orders. The following prices were calculated for each state:

Team	Price
Atlanta	0.001
Indianapolis	0.045
Minnesota	0.002
New England	0.400
New York Jets	0.001
Philadelphia	0.049
Pittsburgh	0.500
St. Louis	0.002

One can think of the state prices as an estimate of the probability of each state being realized. It is interesting in this case that the CPCAM calculated rather high state prices for New England (the 2003 Superbowl champion who also ended up winning the 2004 Superbowl) and Pittsburgh (the team with the best regular season record in the NFL). Clearly, our students tended to include these "favorites" in their orders and the resulting prices reflect that demand.

5. DISCUSSION

The CPCAM is a valuable mechanism for centrally organizing contingent claim markets. It possesses many characteristics that are beneficial to both market organizers and market participants. It is interesting to note that, while the mechanism will produce unique state prices p, the order fills x are not unique in general. Thus, there are some degrees of freedom concerning how to allocate order fills when market participants submit orders whose calculated state price is equal to their limit price. We could choose to build a set of order fill preference rules into the CPCAM to ensure a unique solution for order fills. In the second stage of their model, Yang and Ng [10] use an order fill preference rule where the earliest orders are filled first. It would be worthwhile to consider other order fill preference rules (such as giving preference to the largest orders) and study how they could be effectively incorporated into the CPCAM.

In real market situations, market participants may want to submit different types of orders to create hedged positions. However, to ensure proper hedging, the participant will want to make sure that different orders are accepted in some appropriate proportion. We can easily allow this type of conditional order specification in the CPCAM formulation. For example, if a market participant had submitted two separate order types but wanted the number of accepted orders of the first type to be no larger than twice the number of accepted orders of the second type, we could simply introduce the linear constraint $x_1 \leq 2x_2$ into the model. The convexity of the model would be retained and we could easily solve it using the aforementioned algorithm. Such flexibility of the CPCAM could be of great use to market participants.

It is also interesting to note the similarity between the CPCAM model and the Qualified–Bound–Pricing model of Yang and Ng [10]. Our CPCAM formulation is similar to their first stage optimization problem (where they qualify orders) other than the fact that the CPCAM has an additional barrier function in the objective function. However, there are several advantages of the CPCAM formulation over the Yang and Ng formulation. First, the CPCAM can be solved in one step. Secondly, it appears that the Yang and Ng formulation is only self–funding if the participants are charged their limit prices instead of the calculated state prices for their bids. The CPCAM is self–funding (with the exception of the starting orders) for any price charged to participants that is greater than or equal to the calculated state prices. Note that the limit prices for accepted bids will always be greater than or equal to the calculated state prices.

Another critique that we had of the Yang and Ng model was that it can lead to negative state prices which is not desirable. Negative state prices could be confusing to market participants — implying that they would actually be paid if they had submitted an order for a particular state while still holding a claim for a payment if that state were realized. However, by removing the restriction that the state prices must be non–negative, the market organizer may be able to accept more orders and thus create a more liquid market. The market may suffer because the market organizer may choose not to announce state prices to avoid confusion but the market may actually function more effectively. It would be interesting to better understand the types of conditions that lead to negative state prices.

One can interpret the state prices calculated by the CPCAM as representing the probability that a certain state is realized. An interesting avenue for research would be to determine whether the state prices calculated by the CPCAM formulation have any predictive power. There have been some experimental and empirical work done to better understand the value of market mechanisms for aggregating information and predicting uncertain events (see Berg et al. [1] and Plott [9]). Additionally, there has been a good amount of research done to study the predictive value of market prices in standard parimutuel markets where bidders are only bidding on one state and the value of their potential payout is not fixed (see Feeney and King [4], Koessler and Ziegelmeyer [6] and Koessler et al. [7]). However, it would be interesting to determine whether the type of call auction and mechanism that we have studied would lead participants to bid truthfully. Finally, it would be valuable to know if the mechanism aggregates the bidding information in a manner that is useful for making predictions based on the calculated state prices.

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