A Note on Exchange Market Equilibria with Leontief's Utility: Freedom of Pricing Leads to Rationality

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Abstract: We extend the analysis of [26] to handling more general utility functions: linear substitution functions, which include the Leontief utility. We show that the problem reduces to the general analytic center model discussed in [26]. Thus, the same complexity bound applies to approximating the Fisher equilibrium problem with linear substitution utilities. More importantly, we apply the model to show that the solution to a (pairing) class of Arrow-Debreu problems with Leontief's utility, a more difficult exchange market problem, can be decomposed to solutions of two systems of linear equalities and inequalities, and the price vector is the Perron-Frobenius eigen-vector of a scaled Leontief utility matrix. Consequently, if all input data are rational, then there always exists a rational Arrow-Debreu equilibrium, that is, the entries of the equilibrium vector are rational numbers. Furthermore, the size (bit-length) of the equilibrium solution is bounded by the size of the input data. The result is interesting since rationality does not hold for Leontief's utility in the general model, and it implies, for the first time, that this class of Leontief's exchange market problems can be solved as a linear complementarity problem.

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1 Introduction

Consider the Fisher exchange market problem where players are divided into two sets: producer and consumer; see Brainard and Scarf [2, 22]. Consumers have money to buy goods and maximize their individual utility functions; producers sell their goods for money. The price equilibrium is an assignment of prices to goods so that when every consumer buys a maximal bundle of goods then the market clears, meaning that all the money is spent and all the goods are sold.

Eisenberg and Gale [10, 13] gave a convex optimization setting to formulate Fisher's problem with linear utilities. They constructed an aggregated concave objective function that is maximized at the equilibrium. Thus, finding an equilibrium became solving a convex optimization problem, and it could be obtained by using the Ellipsoid method or interior-point algorithms in polynomial time. Here, polynomial time means that one can compute an ϵ-approximate equilibrium in a number of arithmetic operations bounded by polynomial in n and $\log \frac{1}{\epsilon}$. The best arithmetic operation bound for solving the Fisher problem with linear utilities is $O(n^4 \log \frac{1}{\epsilon})$; see [26]. Moreover, if the input data are rational, then an exact solution can be obtained by solving a system of linear equations and inequalities when $\epsilon < 2^{-L}$, where L is the bit length of the input data. Thus, the arithmetic operation bound becomes $O(n^4 L)$, which is in line with the best complexity bound for linear programming of the same dimension and size.

In this note, we extend the analysis of [26] to handling more general utility functions: linear substitution functions, which include the Leontief utility. We show that the problem reduces to the same general analytic center model discussed in [26]. Thus, the same complexity bound applies to approximating the Fisher problem with linear substitution utilities. More importantly, we apply a theorem on the model in [26] to show that the solution to an Arrow-Debreu problem with Leontief's utilities, a more difficult exchange market problem, is the Perron-Frobenius eigen-vector to a scaled Leontief utility matrix, and the equilibrium vector is a solution to a system of linear equations and inequalities of the original data. Therefore, if all input data are rational, then there always exists a rational Arrow-Debreu equilibrium, that is, the entries of the equilibrium vector are rational numbers; and the size (bit-length) of the solution vector is bounded by the size of the data.

2 The Fisher equilibrium problem

Without loss of generality, assume that there is 1 unit good from each producer $j \in P$ with $|P| = n$. Let consumer $i \in C$ (with $|C| = m$) has an initial endowment w_i to spend and buy goods to maximize his or her individual linear substitution utility:

$$u_i(x_i) = \min_k \{u_i^k(x_{i1}, ..., x_{in})\}, \tag{1}$$

where $u_i^k(x_i)$ is a linear function in x_{ij}—the amount of good bought from producer j by consumer i. More precisely,

$$u_i^k(x_i) = (u_i^k)^T x_i = \sum_{j \in P} u_{ij}^k x_{ij}.$$

In particular, the Leontief utility function is the one with

$$u_i^k(x_i) = \frac{x_{ik}}{a_{ik}}, \;\; k = j \in P$$

for a given $a_{ik} > 0$, that is, vector u_i^k is an all zero vector except for the kth entry that equals $1/a_{ik}$.

Through out this note, we make the following assumptions:

Assumption 1. *Every consumer's initial money endowment $w_i > 0$, at least one $u_{ij}^k > 0$ for every k and $i \in C$ and at least one $u_{ij}^k > 0$ for every k and $j \in P$.*

This is to say that every consumer in the market has money to spend and he or she likes at least one good; and every good is valued by at least one consumer. We will see that, with these assumptions, each consumer can have a positive utility value at equilibria. If a consumer has zero budget or his or her utility has zero value for every good, then buying nothing is an optimal solution for him or her so that he or she can be removed from the market; if a good has zero value to every consumer, then it is a "free" good with zero price in a price equilibrium and can be arbitrarily distributed among the consumers so that it can be removed from the market too.

For given prices p_j on good j, consumer i's maximization problem is

$$\begin{array}{ll} \text{maximize} & u_i(x_{i1}, ..., x_{in}) \\ \text{subject to} & \sum_{j \in P} p_j x_{ij} \le w_i, \\ & x_{ij} \ge 0, \quad \forall j. \end{array} \tag{2}$$

Let x_i^* denote a maximal solution vector of (2). Then, vector p is called a Fisher price equilibrium if there is x_i^* for each consumer such that

$$\sum_{i \in C} x_i^* = e$$

where e is the vector of all ones representing available goods on the exchange market.

Problem (2) can be rewritten as an linear program, after introducing a scalar variable u_i, as

$$\begin{array}{ll} \text{maximize} & u_i \\ \text{subject to} & \sum_{j \in P} p_j x_{ij} \le w_i, \\ & u_i - \sum_{j \in P} u_{ij}^k x_{ij} \le 0, \ \forall k, \\ & u_i, \ x_{ij} \ge 0, \quad \forall j. \end{array} \tag{3}$$

Besides (u_i, x_i) being feasible, the optimality conditions of (3) are

$$\begin{array}{rcl} \lambda_i p_j - \sum_k \pi_i^k u_{ij}^k & \ge & 0, \ \forall j \in P \\ \sum_k \pi_i^k & = & 1 \\ \lambda_i w_i & = & u_i. \end{array} \tag{4}$$

for some $\lambda_i, \ \pi_i^k \ge 0$.

It has been shown by Eisenberg and Gale [10, 9, 13] (independently later by Codenotti et al. [3]) that a Fisher price equilibrium is an optimal Largrange multiplier vector of an aggregated convex optimization problem:

$$\begin{array}{ll} \text{maximize} & \sum_{i \in C} w_i \log u_i \\ \text{subject to} & \sum_{i \in C} x_{ij} = 1, \quad \forall j \in P, \\ & u_i - \sum_{j \in P} u_{ij}^k x_{ij} \le 0, \quad \forall k, i \in C, \\ & u_i, \ x_{ij} \ge 0, \quad \forall i, j. \end{array} \tag{5}$$

Conversely, an optimal Largrange multiplier vector is also a Fisher price equilibrium, which can be seen from the optimality conditions of (5):

$$
\begin{array}{rcl}
p_j - \sum_k \pi_i^k u_{ij}^k & \geq & 0, \ \forall i, j \\
\pi_i^k (\sum_{j \in P} u_{ij}^k x_{ij} - u_i) & = & 0, \ \forall i, k \\
x_{ij} (p_j - \sum_k \pi_i^k u_{ij}^k) & = & 0, \ \forall i, j \\
u_i \sum_k \pi_i^k & = & w_i, \ \forall i.
\end{array}
\tag{6}
$$

for some p_j, the Largarange multiplier of equality constraint of $j \in P$, and some $\pi_i^k \geq 0$, the Largarange multiplier of inequality constraint of $i \in C$ and k. Summing the second constraint over k we have

$$
w_i = \sum_k \pi_i^k u_i = \sum_k \pi_i^k \sum_{j \in P} u_{ij}^k x_{ij} = \sum_{j \in P} \left(x_{ij} \sum_k \pi_i^k u_{ij}^k \right), \ \forall i;
$$

then summing the third constraint over j we have

$$
\sum_{j \in P} p_j x_{ij} = \sum_{j \in P} \left(x_{ij} \sum_k \pi_i^k u_{ij}^k \right) = w_i.
$$

This implies that x_i from the aggregate problem is feasible for (3). Moreover, note that π_i^k in (6) equals π_i^k / λ_i in (4). Thus, finding a Fisher price equilibrium is equivalent to finding an optimal Largrange multiplier of (5).

In particular, if each $u_i^k(x_i)$ has the Leontief utility form, i.e.,

$$
u_i^k(x_i) = \frac{x_{ik}}{a_{ik}}, \ \forall k = j \in P
$$

for a given $a_{ik} > 0$. Then, upon using u_i to replace variable x_{ij}, the aggregated convex optimization problem can be simplified to

$$
\begin{array}{ll}
\text{maximize} & \sum_i w_i \log u_i \\
\text{subject to} & A^T u \leq e, \\
& u \geq 0.
\end{array}
\tag{7}
$$

where the Leontief matrix

$$
A = \begin{pmatrix}
a_{11} & a_{12} & \dots & a_{1n} \\
a_{21} & a_{22} & \dots & a_{2n} \\
\dots & \dots & \dots & \dots \\
a_{m1} & a_{m2} & \dots & a_{mn}
\end{pmatrix}
\quad \text{and variable vector} \quad
u = \begin{pmatrix}
u_1 \\
u_2 \\
\dots \\
u_m
\end{pmatrix}
\tag{8}
$$

Therefore, the jth entry of $A^T u$ is the total amount of good j purchased by all m consumers, and a Fisher equilibrium is an optimal Largrange multiplier vector of the simplified problem. If $(A^T u^*)_j < 1$ at the optimal solution u^*, then the equilibrium price of the jth good will be 0.

3 The weighted analytic center problem

In [26] the Eisenberg-Gale aggregated problem was related to the (linear) analytic center problem studied in interior-point algorithms

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{n} w_j \log(x_j) \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \tag{9}$$

where the given A is an $m \times n$-dimensional matrix with full row rank, b is an m-dimensional vector, and w_j is the nonnegative weight on the jth variable. Any x who satisfies the constraints is called a primal feasible solution, while any optimal solution to the problem is called a weighted analytic center.

If the weighted analytic center problem has an optimal solution, the optimality conditions are

$$\begin{array}{rcl} Sx & = & w, \\ Ax & = & b, \ x \geq 0, \\ -A^T y + s & = & 0, \ s \geq 0, \end{array} \tag{10}$$

where y and s are the Largrange or KKT multipliers or dual variable and slacks of the dual linear program:

$$\min \ b^T y \quad \text{subject to} \quad s = A^T y \geq 0,$$

and S is the diagonal matrix with slack vector s on its diagonals.

Let the feasible set of (9) be bounded and has a (relative) interior, i.e., has a strictly feasible point $x > 0$ with $Ax = b$ (clearly holds for problem (5) and (7)). Then, there is a strictly feasible dual solution $s > 0$ with $s = A^T y$ for some y. Furthermore, [26], based on the literature of interior-point algorithms (e.g., Megiddo and Kojima et al. [19, 18] and Güler [14]), has shown that

Theorem 1. *Let A, b be fixed and consider a solution $(x(w), y(w), s(w))$ of (10) as a mapping of $w \geq 0$ with $\sum_j w_j = 1$. Then,*

- *The mapping of $S^n_{++} = \{x > 0 \in R^n : \ e^T x = 1\}$ to $F_{++} = \{(x > 0, y, s > 0) : \ Ax = b, \ s = A^T y\}$ is one-to-one, continuously and differentiable.*
- *The mapping of $S^n_{+} = \{x \geq 0 \in R^n : \ e^T x = 1\}$ to $F_{+} = \{(x \geq 0, y, s \geq 0) : \ Ax = b, \ s = A^T y\}$ is upper semi-continuous.*
- *The pair $(x_j(w), s_j(w))$ is unique for any $j \in W = \{j : \ w_j > 0\}$, and*

$$x'_j(w) s''_j(w) = x''_j(w) s'_j(w) = 0, \ \forall j \notin W$$

and for any two solutions $(x'(w), y'(w), s'(w))$ and $(x''(w), y''(w), s''(w))$ of (10).

From this theorem, we see that, in the Fisher equilibrium problem (5) or (7), $u_i(w)$, the utility value of each consumer, is unique; but the price vector $p(w)$ can be non-unique.

In addition, a modified primal-dual path-following algorithm was developed in [26], for computing an ϵ-solution for any $\epsilon > 0$:

$$\begin{aligned} \|Sx - w\| &\leq \epsilon, \\ Ax &= b, \ x \geq 0, \\ -A^T y + s &= 0, \ s \geq 0. \end{aligned} \tag{11}$$

Theorem 2. *The primal-dual interior-point algorithm solves the weight analytic center problem (9) in $O(\sqrt{n} \log(n \max(w)/\epsilon))$ iterations and each iteration solves a system of linear equations in $O(nm^2 + m^3)$ arithmetic operations. If Karmarkar's rank-one update technique is used, the average arithmetic operations per iteration can be reduced to $O(n^{1.5} m)$.*

A rounding algorithm is also developed for certain types of problems possessing a rational solution, and the total iteration bound would be $O(\sqrt{n} L)$ and the average arithmetic operation bound would be $O(n^{1.5}m)$ per iteration, where L is the bit-length of the input data A, b, w. These results indicate, for the first time, that the complexity of the Fisher equilibrium problem with linear substitution utility functions is completely in line with linear programming of the same dimension and size.

4 The Arrow-Debreu equilibrium problem

The Arrow-Debreu exchange market equilibrium problem which was first formulated by Leon Walras in 1874 [24]. In this problem everyone in a population of m players has an initial endowment of a divisible good and a utility function for consuming all goods—their own and others. Every player sells the entire initial endowment and then uses the revenue to buy a bundle of goods such that his or her utility function is maximized. Walras asked whether prices could be set for everyone's good such that this is possible. An answer was given by Arrow and Debreu in 1954 [1] who showed that such equilibrium would exist if the utility functions were concave.

We consider a special class of Arrow-Debreu's problems, where each of the $m = n$ players have exactly one unit of a divisible good for trade (e.g., see [15, 26]), and let player i, $i = 1, ..., m$, bring good $j = i$ and have the linear substitution utility function of (1). We call this class of problems the pairing class. The main difference between Fisher's and Arrow-Debreu' models is that, in the latter, each player is both producer and consumer and the initial endowment w_i of player i is *not* given and will be the price assigned to his or her good i. Nevertheless, we can still write a (parametric) convex optimization model

$$\begin{array}{ll} \text{maximize} & \sum_i w_i \log u_i \\ \text{subject to} & \sum_i x_{ij} = 1, \quad \forall j, \\ & u_i \leq \sum_j u_{ij}^k x_{ij}, \quad \forall i, k, \\ & u_i,\ x_{ij} \geq 0, \quad \forall i, j, \end{array}$$

where we wish to select weights w_i's such that an optimal Largrange multiplier vector p

equals w. It is easily seen that any optimal Largrange multiplier vector p satisfies

$$p \geq 0 \quad \text{and} \quad e^T p = e^T w.$$

For fixed u_{ij}^k, consider p be a map from w. Then, the mapping is from S_+^n to S_+^n, and the mapping is upper semi-continuous from Theorem 1. Thus, there is a $w \in S_+^n$ such that an Largrange multiplier vector $p(w) = w$ from the Kakutani fixed-point theorem (see, e.g., [22, 23, 25]). This may be seen as an alternative, restricted to the case of the linear substitution utility functions, to Arrow-Debreu's general proof of the existence of equilibria.

We now focus on the Arrow-Debreu equilibrium with the (complete) Leontief utility function:

$$u_i^k(x_i) = \frac{x_{ik}}{a_{ik}}, \ \forall k = j = 1, ..., m$$

for a given $a_{ik} > 0$. The parametric convex optimization model becomes

$$\begin{array}{ll} \text{maximize} & \sum_i w_i \log u_i \\ \text{subject to} & A^T u \leq e, \\ & u \geq 0. \end{array}$$

where the Leontief matrix A of (8) is a $m \times m$ positive matrix or each entry of A is positive. Let $p \in R^m$ be an optimal Largrange multiplier vector of the constraints. Then, we have

$$\begin{array}{rcl} u_i \sum_j a_{ij} p_j & = & w_i, \ \forall i \\ p_j (1 - \sum_i a_{ij} u_i) & = & 0, \ \forall j \\ \sum_i a_{ij} u_i & \leq & 1, \ \forall j \\ u_i, \ p_j & \geq & 0, \ \forall i, j. \end{array}$$

Thus, the Arrow-Debreu equilibrium is a $p \in R^m$, together with $u \in R^m$, satisfy

$$\begin{array}{rcl} u_i \sum_j a_{ij} p_j & = & p_i, \ \forall i \\ p_j (1 - \sum_i a_{ij} u_i) & = & 0, \ \forall j \\ \sum_i a_{ij} u_i & \leq & 1, \ \forall j \\ u_i, \ p_i & \geq & 0, \ \forall i. \end{array}$$

In the matrix form, they become

$$\begin{array}{rcl} UAp & = & p, \\ P(e - A^T u) & = & 0, \\ A^T u & \leq & e, \\ u,\ p & \geq & 0, \end{array} \tag{12}$$

where U and P are diagonal matrices whose diagonal entries are u and p, respectively. The Arrow-Debreu theorem implies that nonzero p and u exist for this system of equalities and inequalities, even in general case where $A \geq 0$, that is, some $a_{ik} = 0$ in the Leontief matrix.

5 Characterization of an Arrow-Debreu equilibrium

If $u_i > 0$ at a solution $(u, p \neq 0)$ of system (12), we must have $p_i > 0$, that is, player i's good must be priced positively in order to have a positive utility value. On the other hand, $p_i > 0$ implies that $\sum_k^m a_{ki} u_k = 1$, that is, good i must be all consumed and gone. Conversely, if $p_i > 0$, we must have $u_i > 0$, that is, player i's utility value must be positive. Thus, there is a partition of all players (or goods) such that

$$B = \{i : \ p_i > 0\} \quad \text{and} \quad N = \{i : \ p_i = 0\}$$

where the union of B and N is $\{1, 2, ..., m\}$. Then, (u, p) satisfies

$$\begin{array}{rcl} (U_B A_{BB}) p_B & = & p_B, \\ A_{BB}^T u_B & = & e, \\ A_{BN}^T u_B & \leq & e, \\ u_B,\ p_B & > & 0. \end{array}$$

Here A_{BB} is the principal submatrix of A corresponding to the index set B, A_{BN} is the submatrix of A whose rows in B and columns in N. Similarly, u_B and p_B are subvectors of u and p with entries in B, respectively.

Since the scaled Leontief matrix $U_B A_{BB}$ is a (column) stochastic matrix (i.e., $e^T U_B A_{BB} = e^T$), p_B must be the (right) Perron-Frobenius eigen-vector of $U_B A_{BB}$. Moreover, A_{BB} is (generally) irreducible because $U_B A_{BB}$ is (generally) irreducible and $u_B > 0$, and $U_B A_{BB}$ is (generally) irreducible because $p_B > 0$. (Stochastic matrix A is (generally) irreducible if it has a strictly positive distribution p such that $Ap = p$, and p is a strictly convex combination of all irreducible subclass distributions.) To summarize, we have

Theorem 3. *Let $B \subset \{1, 2, ..., n\}$, $N = \{1, 2, ..., n\} \setminus B$, A_{BB} be (generally) irreducible, and u_B satisfy the linear system*

$$\begin{aligned} A_{BB}^T u_B &= e, \\ A_{BN}^T u_B &\le e, \\ u_B &> 0. \end{aligned}$$

Then the (right) Perron-Frobenius eigen-vector p_B of $U_B A_{BB}$ together with $p_N = 0$ will be an Arrow-Debreu equilibrium. And the converse is also true. Moreover, there is always a rational Arrow-Debreu equilibrium for every such B, that is, the entries of price vector are rational numbers, if the entries of A are rational. Furthermore, the size (bit-length) of the equilibrium is bounded by the size of A.

Proof. We only need to prove $p_B > 0$. But this is the result of the Perron-Frobenius theorem on $U_B A_{BB}$ since it is (generally) irreducible. Conversely, if $(p_B > 0, p_N = 0)$ is an Arrow-Debreu price, then $u_B > 0$ and $A_{BB}^T u_B = e$ from the complementarity, and A_{BB} is (generally) irreducible from $p_B > 0$. To prove the rationality, we see that there is a rational vector u_B to the linear system, so that matrix $U_B A_{BB}$ will be rational, so that there will be a rational solution p_B to the linear system

$$(U_B A_{BB} - I)\, p_B = 0,\ e^T p_B = 1,\ p_B > 0.$$

The size result is due to that the sizes of these two linear systems are bounded by the size of A. ∎

Our theorem implies that the players in block B can trade among them self and keep others goods "free." In particular, if one player likes his or her own good more than any other good, that is, $a_{ii} \ge a_{ij}$ for all j. Then, $u_i = 1/a_{ii}$, $p_i = 1$, and $u_j = p_j = 0$ for all $j \ne i$, that is, $B = \{i\}$, makes an Arrow-Debreu equilibrium. The theorem thus establishes, for the first time, a combinatorial algorithm to compute an Arrow-Debreu equilibrium with Leontief's utility by finding a right block $B \neq \emptyset$, which is actually a non-trivial complementarity solution to a *linear complementarity problem* (LCP)

$$A^T u + v = e, \ u^T v = 0, \ 0 \neq u, v \geq 0. \tag{13}$$

If $A > 0$, then any complementarity solution $u \neq 0$ and $B = \{j : \ u_j > 0\}$ of (13) induce an equilibrium that is the (right) Perron-Frobenius eigen-vector of $U_B A_{BB}$, and it can be computed in polynomial time by solving a linear equation. If A is not strictly positive, then any complementarity solution $u \neq 0$ and $B = \{j : \ u_j > 0\}$, as long as A_{BB} is (generally) irreducible, induces an equilibrium. The equivalence between the pairing Arrow-Debreu model and the LCP also implies

Corollary 1. *Let square matrix $A \geq 0$ and have no all-zero row nor all zero-column. Then, LCP (13) has a complementarity solution $u \neq 0$ such that A_{BB} is (generally) irreducible where $B = \{j : \ u_j > 0\}$.*

The pairing class of Arrow-Debreu's problems is a rather restrictive class of problems. Consider a general supply matrix $0 \leq G \in R^{m \times n}$ where row i of G represents the multiple goods brought to the market by player i, $i = 1, ..., m$. Without loss of generality, we assume $e^T G = e^T \in R^n$, that is, each of the n goods has exactly one unit in the market. The pairing model represents the case that $G = I$, the identity matrix, or $G = P$ where P is any permutation matrix of $m \times m$.

What to do if one player brings two different goods? One solution is to copy the same player's utility function twice and treat the player as two players with an identical Leontief utility function, where each of them brings only one type of good. Then, the problem reduces to the pairing model. Thus, we have

Corollary 2. *If all goods are different from each other in the general Arrow-Debreu problem with Leontief's utility, that is, each column of $G \in R^{m \times n}$ has exactly one nonzero entry, then there is always a rational equilibrium, that is, the entries of price vector are rational numbers, if the entries of Leontief's coefficient matrix are rational. Furthermore, the size (bit-length) of the equilibrium is bounded by the size Leontief's coefficient matrix.*

6 An illustrative example

The rationality result is interesting since the existence of a rational equilibrium is not true for Leontief's utility in Fisher's setting with rational data, see the following example, with three consumers and three goods, adapted from Codenotti et al. [3] and Eaves [11].

$$A^T = \begin{pmatrix} 1 & 1/2 & 1/4 \\ 1/2 & 1 & 1/5 \\ 1/2 & 1 & 1/5. \end{pmatrix}.$$

Note that goods 2 and 3 have identical coefficients from each of the three consumers.

Let the initial endowments of three consumers be $w_1 = w_2 = w_3 = 1$ in Fisher's setting. Then, the maximal utility values of the three consumers are

$$u_1^* = \frac{2}{3\sqrt{3}},\ u_2^* = \frac{1}{3} + \frac{1}{3\sqrt{3}},\ u_3^* = \frac{10}{3} - \frac{10}{3\sqrt{3}},$$

the Fisher equalibrium price for good 1 is

$$p_1^* = 3(\sqrt{3} - 1),$$

and the (multiple) Fisher equilibrium prices for goods 2 and 3 are

$$\{(p_2^*, p_3^*) :\ p_2^* + p_2^* = 3(2 - \sqrt{3}),\ p_2^* + p_3^* \geq 0\}.$$

However, let player i owns good i, $i = 1, 2, 3$, in the Arrow-Debreu model. Then, there are multiple rational equilibria:

1. $B = \{1\}$, with $u_1^* = 1$ and $p_1^* = 1$, and $u_2^* = u_3^* = p_2^* = p_3^* = 0$.
2. $B = \{2\}$, with $u_2^* = 1$ and $p_2^* = 1$, and $u_1^* = u_3^* = p_1^* = p_3^* = 0$.
3. $B = \{1, 2\}$, with $u_1* = u_2^* = \frac{2}{3}$ and $p_1^* = p_2* = \frac{1}{2}$, and $u_3^* = p_3^* = 0$.
4. $B = \{2, 3\}$, with an equilibrium $u_2^* = \frac{1}{2}$, $u_3^* = \frac{5}{2}$, $p_2^* = \frac{1}{2}$, $p_3^* = \frac{1}{2}$, and $u_1^* = p_1^* = 0$.
5. $B = \{1, 2, 3\}$, with an equilibrium $u_1^* = \frac{7}{15}$, $u_2^* = \frac{17}{30}$, $u_3^* = 1$, $p_1^* = \frac{7}{23}$, $p_2^* = \frac{221}{460}$, and $p_3^* = \frac{99}{460}$.

Now what to do if two players bring the same type of good? In our present pairing class, they are being treated as two different goods. However, one can set the same utility coefficients to them so that they receive an identical appreciation from all the players (as illustrated in the example). Again, the problem reduces to the pairing class, which leads to rationality. The problem is that these two "same" goods may receive two different prices; for example, one is priced higher and the other is at a discount level. I guess this could happen in the real world since two "same" goods may not be really the same and the market has "freedom" to price. Another open question: is there a polynomial-time algorithm to solve the pairing class of Arrow-Debreu's problems?

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