A Unified Theorem on SDP Rank Reduction and its Applications

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Outline

- Problem Statement
- SDP Rank Reduction Theorem and Algorithm
- Sketch of Proof
- Applications
- More Questions

Semidefinite Programming Problem

Consider the Semidefinite Programming problem:

$$\begin{array}{lll} (SDP) & \text{minimize} & C \bullet X \\ & \text{subject to} & A_i \bullet X = b_i \qquad i = 1, \ldots, m, \\ & & X \succeq \mathbf{0} \end{array}$$

where $C, A_1, \ldots, A_m$ are given $n \times n$ symmetric matrices and $b_1, \ldots, b_m$ are given scalars, and

$$A \bullet X = \sum_{i,j} a_{ij} x_{ij} = \text{trace} A^T X.$$

An SDP Example

$$(SDP) \quad \text{minimize} \quad 2x_1 + x_2 + x_3$$

$$\text{subject to} \quad x_1 + x_2 + x_3 = 1,$$

$$\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq \mathbf{0}.$$

$$(LP) \quad \text{minimize} \quad 2x_1 + x_2 + x_3$$

$$\text{subject to} \quad x_1 + x_2 + x_3 = 1,$$

$$(x_1, x_2, x_3) \geq \mathbf{0}.$$

The Dual of SDP

The dual problem to (SDP) can be written as:

$$(SDD) \quad \text{sup} \quad \mathbf{b}^T\mathbf{y}$$

$$\text{subject to} \quad \sum_i^m y_i A_i + S = C, \; S \succeq \mathbf{0},$$

where $\mathbf{y} \in \mathcal{R}^m$.

Let X^* and S^* be a solution pair with zero duality gap. Then

$$\text{rank}(X^*) + \text{rank}(S^*) \leq n.$$

If there is S^* such that $\text{rank}(S^*) \geq n - d$, then the max rank of X^* is bounded by d.

Computational Complexity and Rank of SDP Solution

- The SDP interior-point algorithm finds an ϵ-approximate solution where solution time is linear in $\log(1/\epsilon)$ and polynomial in m and n.
- Barvinok 95 showed that if the problem is solvable, then there exists a solution X whose rank r satisfies $r(r+1) \leq 2m$. (A constructive proof can be based on Carathéodory's theorem.)
- And the rank bound is essentially tight.
- A such low-rank solution can be found in polynomial time; Pataki (1999), and Alfakih/Wolkowicz (1999).

SDP Feasibility Problem

For simplicity, consider finding X satisfies

$$A_i \bullet X = b_i \qquad i = 1, \ldots, m, \quad X \succeq \mathbf{0}$$

where $A_1, \ldots, A_m$ are positive semidefinite matrices and scalars $(b_1, \ldots, b_m) \geq \mathbf{0}$.

$$x_1 + x_2 + x_3 = 1,$$
$$\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq \mathbf{0}.$$

Problem Statement

- We are interested in finding a fixed low–rank (say d) solution to the above system.
- However, there are some issues:
  - Such a solution may not exist!
  - Even if it does, one may not be able to find it efficiently.
- So we consider an approximation of the problem.

Approximating the Problem

We consider the problem of finding an $\hat{X} \succeq 0$ of rank at most d that satisfies the SDP constraints approximately:

$$\beta(m,n,d) \cdot b_i \leq A_i \bullet \hat{X} \leq \alpha(m,n,d) \cdot b_i \quad \forall\, i = 1, \ldots, m.$$

Here, $\alpha(\cdot) \geq 1$ and $\beta(\cdot) \in (0,1]$ are called the distortion factors.

Clearly, the closer are both to 1, the better the solution quality.

Our Main Result

Theorem 1. *(So, Y and Zhang 07) Let* $r = \max\{\textit{rank}(A_i)\}$*. Then, for any* $d \geq 1$*, there exists an* $\hat{X} \succeq \mathbf{0}$ *with* $\textit{rank}(\hat{X}) \leq d$ *such that*

$$\alpha(m,n,d) = \begin{cases} 1 + \dfrac{12\ln(4mr)}{d} & \text{for } 1 \leq d \leq 12\ln(4mr) \\ 1 + \sqrt{\dfrac{12\ln(4mr)}{d}} & \text{for } d > 12\ln(4mr) \end{cases}$$

and

$$\beta(m,n,d) = \begin{cases} \dfrac{1}{e(2m)^{2/d}} & \text{for } 1 \leq d \leq 4\ln(2m) \\ \max\left\{ \dfrac{1}{e(2m)^{2/d}},\ 1 - \sqrt{\dfrac{4\ln(2m)}{d}} \right\} & \text{for } d > 4\ln(2m) \end{cases}$$

Moreover, there exists an efficient randomized algorithm for finding such an $\hat{X}$.

Some Remarks

- There is always a low-rank, or sparse, approximate SDP solution with respect to a bounded relative residual distortion. As the allowable rank increases, the distortion bounds become smaller and smaller.
- The lower distortion factor is independent of n and the rank of A_is.
- The factors can be improved if we only consider one–sided inequalities.
- This result contains as special cases several well-known results in the literature.

Early Result: Metric Embedding

- Given an n–point set $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in $\mathbf{R}^l$, we would like to embed it into a low–dimensional Euclidean space as faithfully as possible.
- Specifically, a map $f : V \rightarrow \mathbf{R}^d$ is an α–embedding (where $\alpha \geq 0$) if

$$(1-\alpha)\|\mathbf{v}_i - \mathbf{v}_j\|_2 \leq \|f(\mathbf{v}_i) - f(\mathbf{v}_j)\|_2 \leq (1+\alpha) \cdot \|\mathbf{v}_i - \mathbf{v}_j\|_2$$

The goal is to find an f such that α is as small as possible. This is a case of the SDP with $A_{ij} = (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T$.

- It is known that: for any $\epsilon > 0$, an ϵ–embedding into $\mathbf{R}^{O(\epsilon^{-2}\log n)}$ exists (Johnson–Lindenstrauss 84).

Early Result: Approximating QPs

- Let $A_1, \ldots, A_m$ be positive semidefinite. Consider the following QP:

$$v^* = \text{maximize} \;\; \mathbf{x}^T A \mathbf{x} \quad \text{s.t.} \;\; \mathbf{x}^T A_i \mathbf{x} \le 1 \quad i = 1, \ldots, m$$

and its natural SDP relaxation:

$$v^*_{sdp} = \text{maximize} \;\; A \bullet X \quad \text{s.t.} \;\; A_i \bullet X \le 1 \quad i = 1, \ldots, m; \;\; X \succeq 0$$

- Let X^* be an optimal solution to the SDP.

- Nemirovskii et al. 99 showed that one can randomly extract a rank–1 matrix $\hat{X}$ from X^* such that it is feasible for the SDP and that $\mathbb{E}[A \bullet \hat{X}] \ge \Omega(\log^{-1} m) v^*$.

Early Result: Approximating QPs (Cont'd)

- Luo et al. 06 considered the following real (complex) QP:

$$\text{minimize } \mathbf{x}^T A\mathbf{x} \quad \text{s.t. } \mathbf{x}^T A_i\mathbf{x} \geq 1 \quad i = 1, \ldots, m$$

  and its natural SDP relaxation:

$$\text{minimize } A \bullet X \quad \text{s.t. } A_i \bullet X \geq 1 \quad i = 1, \ldots, m; \quad X \succeq 0$$

- They showed how to extract a solution $\hat{x}$ from an optimal solution matrix to the SDP so that it is feasible for the SDP and that it is within a factor $O(m^{-2})$ ($O(m^{-1})$) of the optimal.
- Again, we can obtain the same results from our Theorem on both real ($d = 1$) and complex ($d = 2$) spaces.

How Sharp are the Bounds?

For metric embedding, it is known that:

- for any $d \geq 1$, there exists an n–point set $V \subset \mathbf{R}^{d+1}$ such that any embedding of V into $\mathbf{R}^d$ requires $D = \Omega(n^{1/\lfloor (d+1)/2 \rfloor})$ (Matousek 90);

- there exists an n–point set $V \subset \mathbf{R}^l$ for some l such that for any $\epsilon \in (n^{-1/2}, 1/2)$, say, an $(1+\epsilon)$–embedding of V into $\mathbf{R}^d$ will require $d = \Omega((\epsilon^2 \log(1/\epsilon))^{-1} \log n)$ (Alon 03).

Thus, from the metric embedding perspective, the ratio of our upper and lower bounds is almost tight for $d \geq 3$.

How Sharp are the Bounds? (Cont'd)

For the QP:

$$v^* = \text{maximize} \;\; \mathbf{x}^T A \mathbf{x} \quad \text{s.t.} \;\; \mathbf{x}^T A_i \mathbf{x} \le 1 \quad i = 1, \ldots, m$$

and its natural SDP relaxation:

$$v^*_{sdp} = \text{maximize} \;\; A \bullet X \quad \text{s.t.} \;\; A_i \bullet X \le 1 \quad i = 1, \ldots, m; \quad X \succeq 0$$

Nemirovskii et al. 99 showed that the ratio between v^* and v^*_{sdp} can be as large as $\Omega(\log m)$.

For the minimization version, Luo et al. 06 showed that the ratio can be as small as $\Omega(m^{-2})$.

Thus, from the QP perspective, the ratio of our upper and lower bounds is almost tight for $d = 1$.

Sketch of Proof of the Theorem

We only need to prove: Let $A_1, \ldots, A_m \in \mathcal{M}^n$ be symmetric PSD matrices. Then, for any $d \geq 1$, there exists an $\hat{X} \succeq \mathbf{0}$ with $\text{rank}(\hat{X}) \leq d$ such that:

$$\beta(m,n,d)\cdot\text{Tr}(A_i) \leq A_i \bullet \hat{X} \leq \alpha(m,n,d)\cdot\text{Tr}(A_i) \quad \text{for } i = 1, \ldots, m \tag{1}$$

where $\alpha(m,n,d)$ and $\beta(m,n,d)$ are given in the main Theorem, respectively.

Note that I is a feasible solution to (1) with zero distortion.

The general theorem can be reduced to this form. (How?)

Sketch of Proof of the Theorem (Cont'd)

The proof is constructive: we use a simple randomized construction procedure to generate $\hat{X}$:

- Generate i.i.d. Gaussian random variables ξ_i^j with mean 0 and variance $1/d$, and define $\xi^j = (\xi_1^j, \ldots, \xi_n^j)$, where $i = 1, \ldots, n;\ j = 1, \ldots, d$.
- Return $\hat{X} = \sum_{j=1}^{d} \xi^j \left(\xi^j\right)^T$.

Cearly, the rank of $\hat{X}$ is d.

The rest of proof is based on careful analyses of various probability bounds.

Sketch of Proof of the Theorem (Cont'd)

The analysis makes use of the following Markov inequality:

Lemma 1. *Let* $\xi_1, \ldots, \xi_n$ *be i.i.d. standard Gaussian RVs. Let* $\alpha \in (1, \infty)$ *and* $\beta \in (0, 1)$ *be constants, and Chi-square* $U_n = \sum_{i=1}^{n} \xi_i^2$*. Then, the following hold:*

$$\Pr\left(U_n \geq \alpha n\right) \leq \exp\left[\frac{n}{2}\left(1 - \alpha + \log \alpha\right)\right]$$

$$\Pr\left(U_n \leq \beta n\right) \leq \exp\left[\frac{n}{2}\left(1 - \beta + \log \beta\right)\right]$$

Sketch of Proof of the Theorem (Cont'd)

Lemma 2. *Let* $H \in \mathcal{M}^n$ *be a symmetric PSD matrix with* $r \equiv \textit{rank}(H) \geq 1$. *Then, for any* $\beta \in (0, 1)$, *we have:*

$$\Pr\left(H \bullet \hat{X} \leq \beta \, \textit{Tr}(H)\right) \leq \exp\left(\frac{d}{2}(1 - \beta + \ln \beta)\right) \quad (2)$$

Lemma 3. *Let* $H \in \mathcal{M}^n$ *be a symmetric PSD matrix with* $r \equiv \textit{rank}(H) \geq 1$. *Then, for any* $\alpha > 1$, *we have:*

$$\Pr\left(H \bullet \hat{X} \geq \alpha \, \textit{Tr}(H)\right) \leq r \cdot \exp\left(\frac{d}{2}(1 - \alpha + \ln \alpha)\right) \quad (3)$$

Low Rank SDP Applications

The low–rank SDP problem arises in many applications, e.g.:

- graph realization/sensor network localization (e.g., Biswas and Y 03, So and Y 04)
- metric embedding/dimension reduction (e.g., Johnson and Lindenstrauss 84, Matousek 90)
- approximating non-convex (complex, quaternion) quadratic optimization (e.g., Nemirovskii, Roos and Terlaky 99, Luo, Sidiropoulos, Tseng and Zhang 06, Faybusovich 07)
- graph rigidity/distance matix (e.g., Alfakih, Khandani and Wolkowicz 99, etc.)

Graph Realization

Given a graph $G = (V, E)$ and sets of non–negative weights, say $\{d_{ij} : (i, j) \in E\}$, the goal is to compute a realization of G in the Euclidean space $\mathbf{R}^d$ for a given low dimension d, i.e.

- to place the vertices of G in $\mathbf{R}^d$ such that
- the Euclidean distance between every pair of adjacent vertices (i, j) equals (or bounded) by the prescribed weight $d_{ij} \in E$.

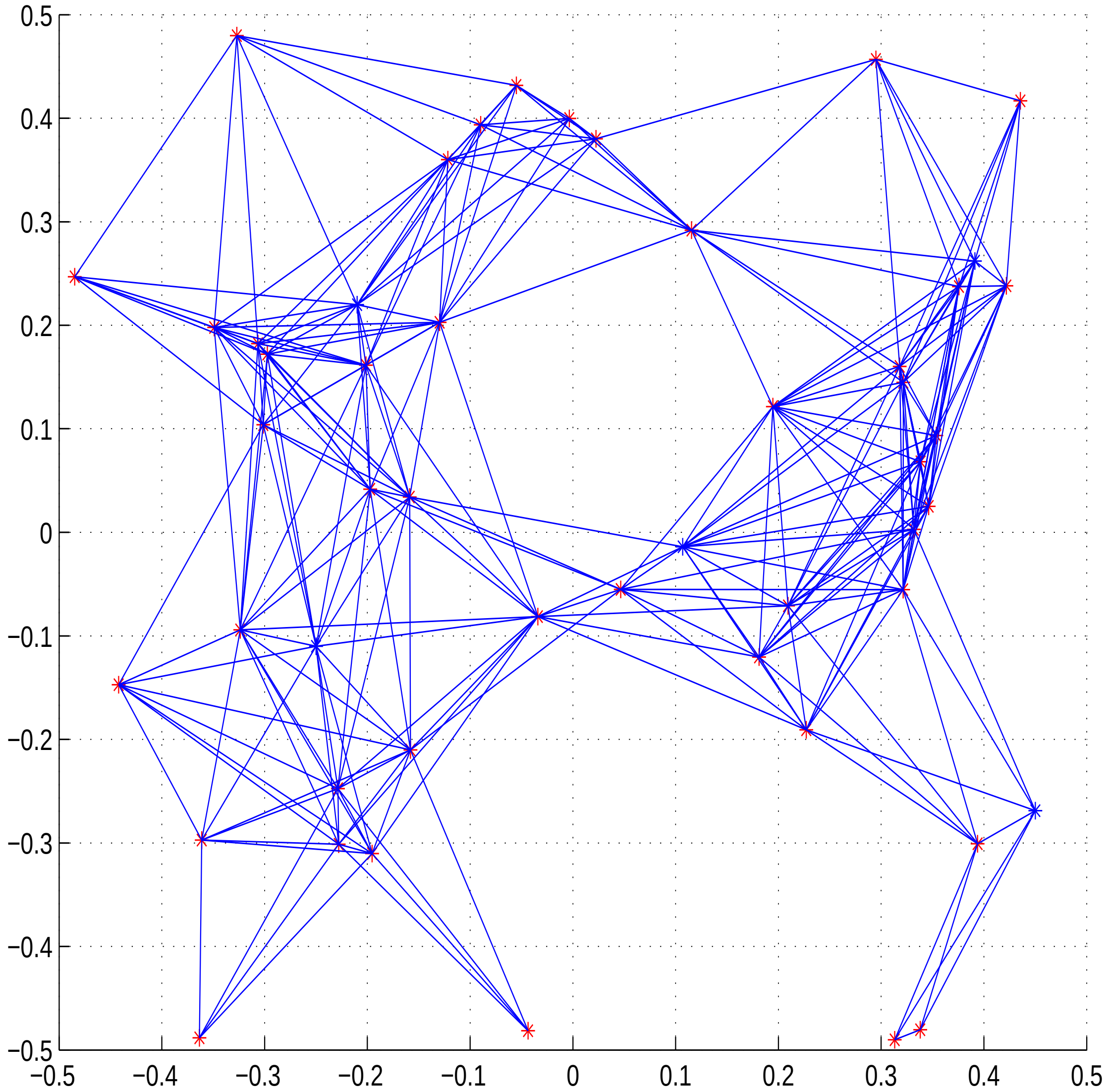


Figure 1: 50-node 2-D Sensor Localization

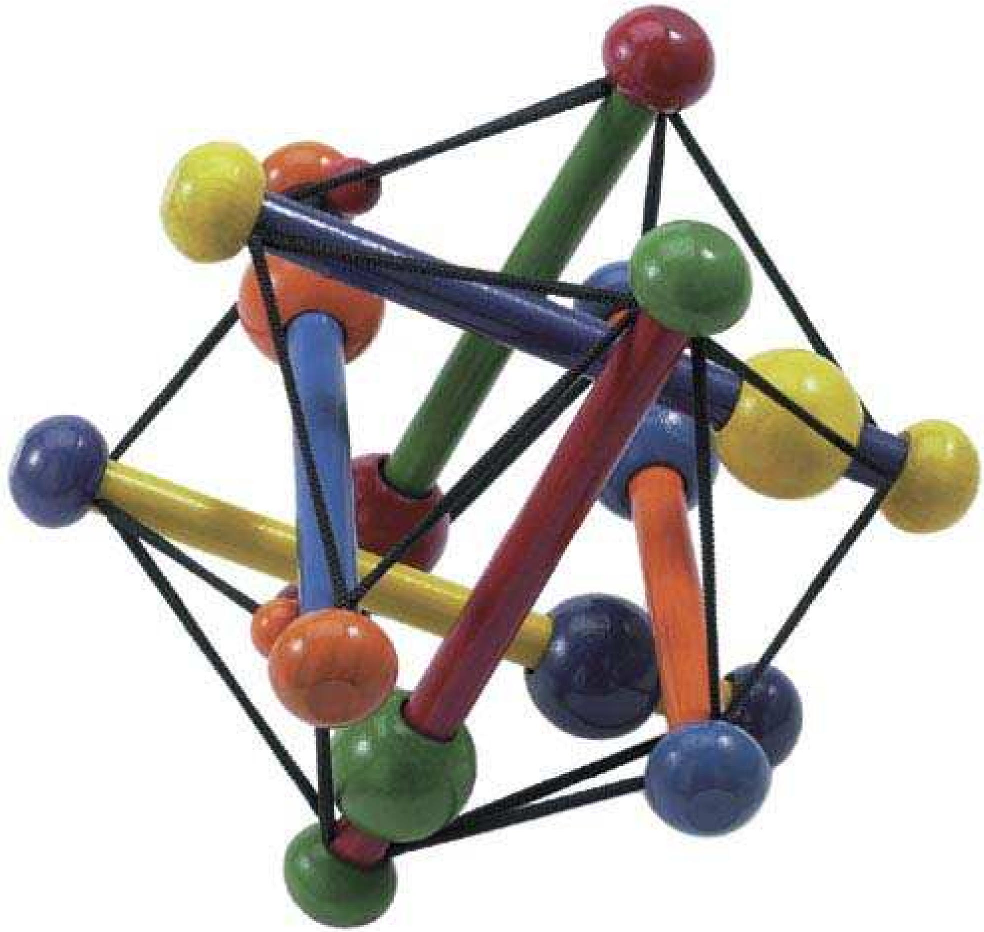


Figure 2: A 3-D Tensegrity Graph Realization; provided by Anstreicher

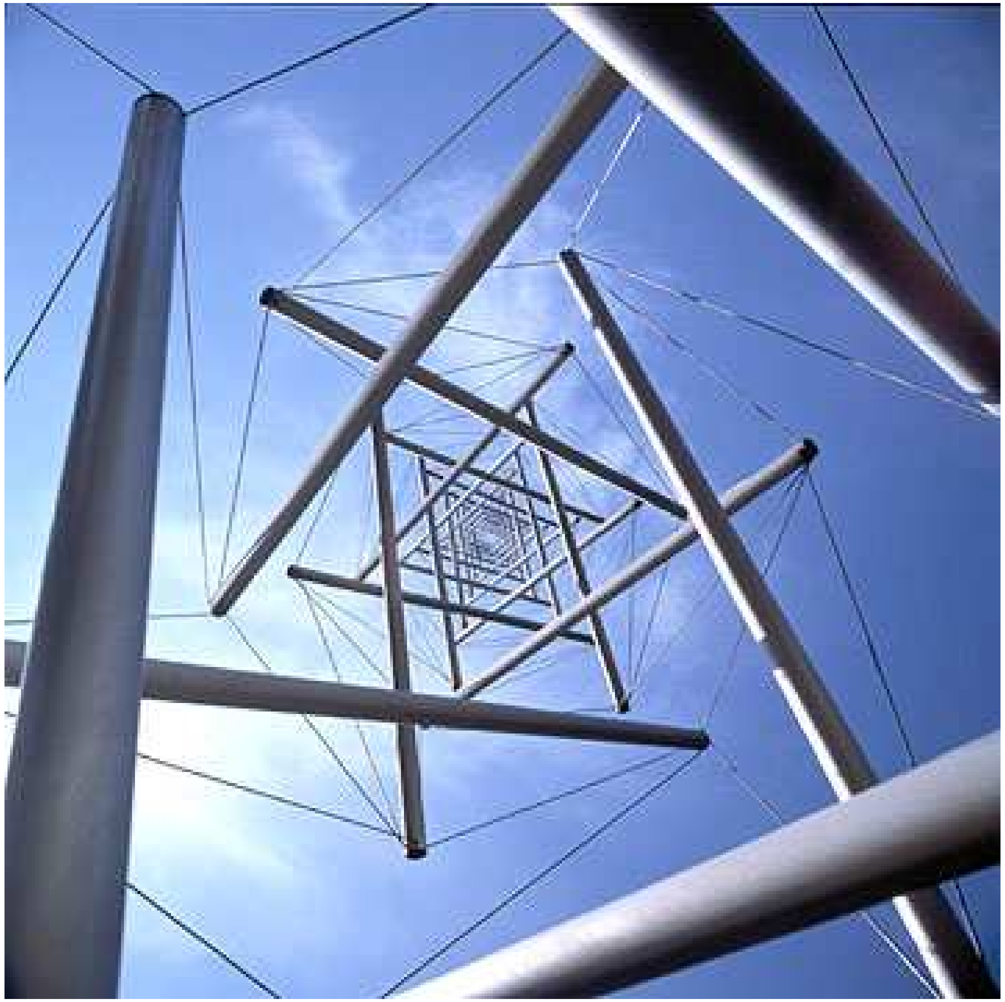


Figure 3: Tensegrity Graph: A Needle Tower; provided by Anstreicher

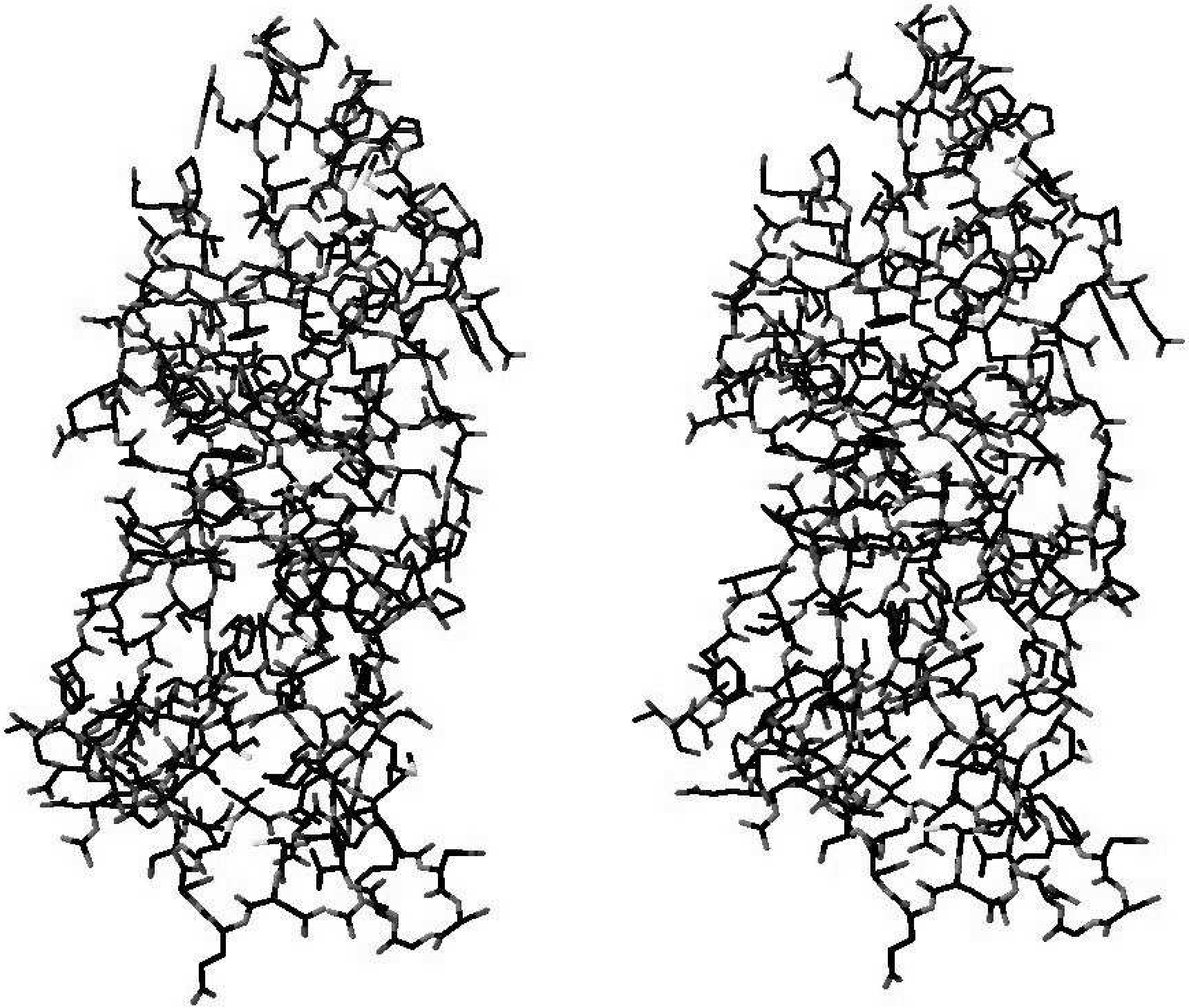


Figure 4: Molecular Conformation: 1F39(1534 atoms) with 85% of distances below 6Å and 10% noise on upper and lower bounds

Sensor Localization Model

Given $\mathbf{a}_k \in \mathbf{R}^d$, $d_{ij} \in N_x$, and $\hat{d}_{kj} \in N_a$, find $\mathbf{x}_i \in \mathbf{R}^d$ such that

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = d_{ij}^2, \ \forall\,(i,j) \in N_x, \ i < j,$$
$$\|\mathbf{a}_k - \mathbf{x}_j\|^2 = \hat{d}_{kj}^2, \ \forall\,(k,j) \in N_a,$$

(ij) ((kj)) connects points $\mathbf{x}_i$ and $\mathbf{x}_j$ ($\mathbf{a}_k$ and $\mathbf{x}_j$) with an edge whose Euclidean length is d_{ij} ($\hat{d}_{kj}$).

Does the system have a localization or realization of all $\mathbf{x}_j$'s? Is the localization unique? Is there a certification for the solution to make it reliable or trustworthy? Is the system partially localizable with certification?

Matrix Representation

Let $X = [\mathbf{x}_1\ \mathbf{x}_2\ \ldots\ \mathbf{x}_n]$ be the $2 \times n$ matrix that needs to be determined. Then

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = (\mathbf{e}_i - \mathbf{e}_j)^T X^T X (\mathbf{e}_i - \mathbf{e}_j) \text{ and}$$

$$\|\mathbf{a}_k - \mathbf{x}_j\|^2 = (\mathbf{a}_k; -\mathbf{e}_j)^T [I\ \ X]^T [I\ \ X] (\mathbf{a}_k; -\mathbf{e}_j),$$

where $\mathbf{e}_j$ is the vector of all zero except 1 at the jth position.

$$
\begin{aligned}
&(\mathbf{e}_i - \mathbf{e}_j)^T Y (\mathbf{e}_i - \mathbf{e}_j) = d_{ij}^2, \ \forall\, i, j \in N_x, \ i < j, \\
&(\mathbf{a}_k; -\mathbf{e}_j)^T \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} (\mathbf{a}_k; -\mathbf{e}_j) = \hat{d}_{kj}^2, \ \forall\, k, j \in N_a, \\
&Y = X^T X.
\end{aligned}
$$

SDP Relaxation

Change

$$Y = X^T X$$

to

$$Y \succeq X^T X.$$

This matrix inequality is equivalent to

$$\begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succeq 0,$$

Biswas and Y 2004; Krislock et al 2007.

This matrix has rank at least 2; if it's 2, then $Y = X^T X$, and the converse is also true.

SDP Standard Form

$$Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix}.$$

Find a symmetric matrix $Z \in \mathbf{R}^{(2+n)\times(2+n)}$ such that

$$Z_{1:2,1:2} = I$$

$$(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)^T \bullet Z = d_{ij}^2, \ \forall\, i, j \in N_x, \ i < j,$$

$$(\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \bullet Z = \hat{d}_{kj}^2, \ \forall\, k, j \in N_a,$$

$$Z \succeq 0.$$

If every sensor point is connected, directly or indirectly, to an anchor point, then the solution set must be bounded.

The Dual of the SDP Relaxation

$$\text{minimize} \quad I \bullet V + \sum_{i<j \in N_x} w_{ij} d_{ij}^2 + \sum_{k,j \in N_a} \hat{w}_{kj} \hat{d}_{kj}^2$$

$$\text{subject to} \quad \begin{pmatrix} V & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \sum_{i<j \in N_x} w_{ij} (\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)^T$$

$$+ \sum_{k,j \in N_a} w_{kj} (\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \succeq 0,$$

where variable matrix $V \in \mathcal{M}^2$, variable w_{ij} is the (stress) weight on edge between $\mathbf{x}_i$ and $\mathbf{x}_j$, and $\hat{w}_{kj}$ is the (stress) weight on edge between $\mathbf{a}_k$ and $\mathbf{x}_j$.

Note that the dual is always feasible since $V = 0$ and all $w_{\cdot}$ equal 0 is a feasible solution.

The rank of any optimal dual (stress) slack matrix is less or equal to n.

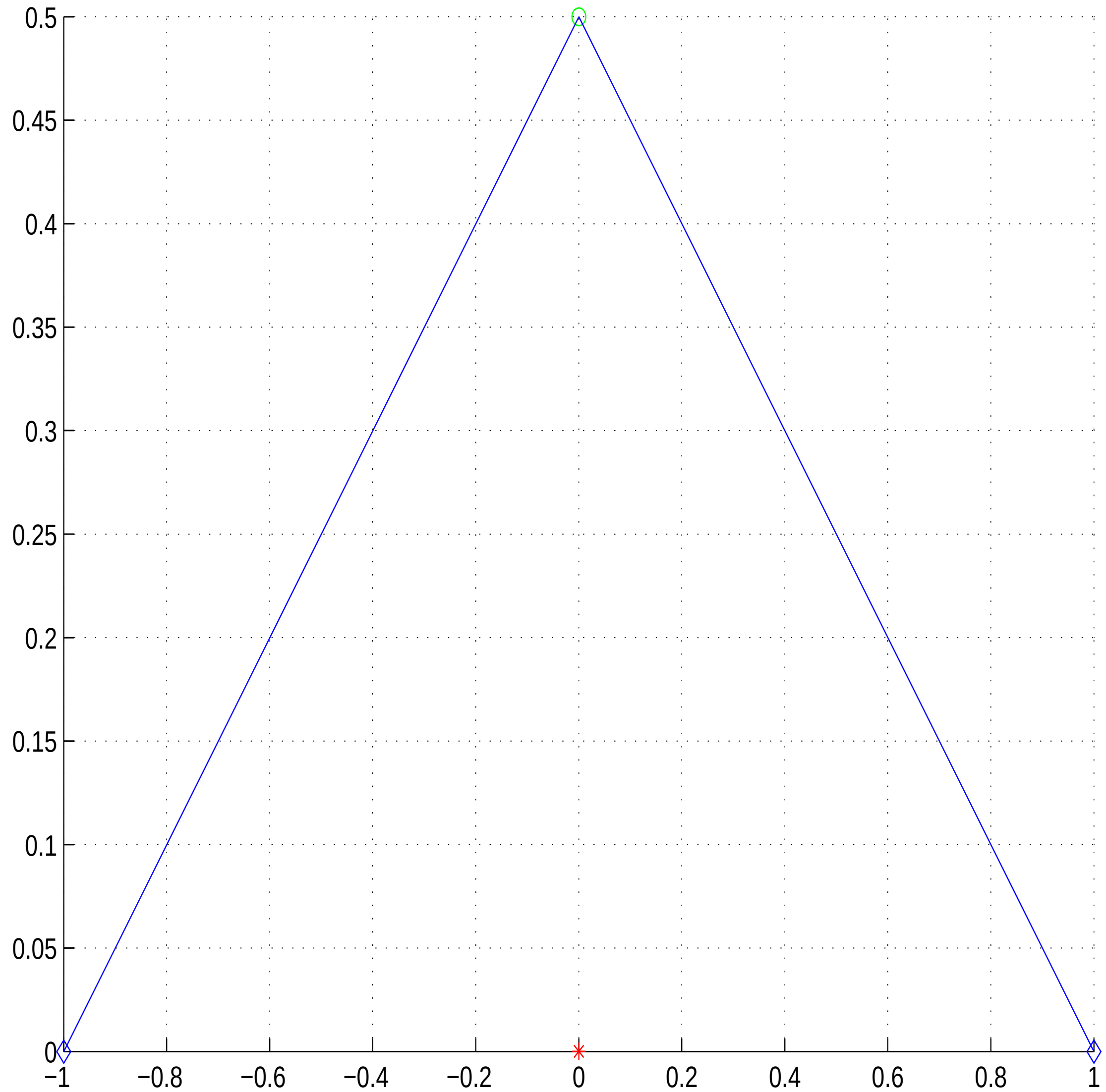
Unique Localizability

A sensor network is 2-uniquely-localizable if there is a unique localization in $\mathbf{R}^2$ and there is no $\mathbf{x}_j \in \mathbf{R}^h,\ j = 1, ..., n$, where $h > 2$, such that

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = d_{ij}^2,\ \forall\, i, j \in N_x,\ i < j,$$
$$\|(\mathbf{a}_k; \mathbf{0}) - \mathbf{x}_j\|^2 = \hat{d}_{kj}^2,\ \forall\, k, j \in N_a.$$

The latter says that the problem cannot be localized in a higher dimension space where anchor points are simply augmented to $(\mathbf{a}_k; \mathbf{0}) \in \mathbf{R}^h$, $k = 1, ..., m$.

Figure 5: One sensor-Two anchors: Not localizable



Uniquely-Localizable Graphs

Theorem 2.
- *If every edge length is specified, then the sensor network is 2-uniquely-localizable (Schoenberg 1942).*
- *There is a sensor network, with $O(n)$ edge lengths specified, that is 2-uniquely-localizable (So 2007).*
- *If one sensor with its edge lengths to at least three anchors (in general positions) specified, then it is 2-uniquely-localizable (So and Y 2005).*

ULPs can be localized in polynomial time

Theorem 3. *(So and Y 2005) The following statements are equivalent:*

1. *The sensor network is 2-uniquely-localizable;*
2. *The max-rank solution of the SDP relaxation has rank 2;*
3. *The solution matrix has $Y = X^T X$ or Trace$(Y - X^T X) = 0$.*

When an optimal dual (stress) slack matrix has rank n, then the problem is 2-strongly-localizable.

If one sensor with its edge lengths to at least three anchors (in general positions) specified, then it is 2-strongly-localizable

Figure 6: Two sensor-Three anchors: Strongly Localizable

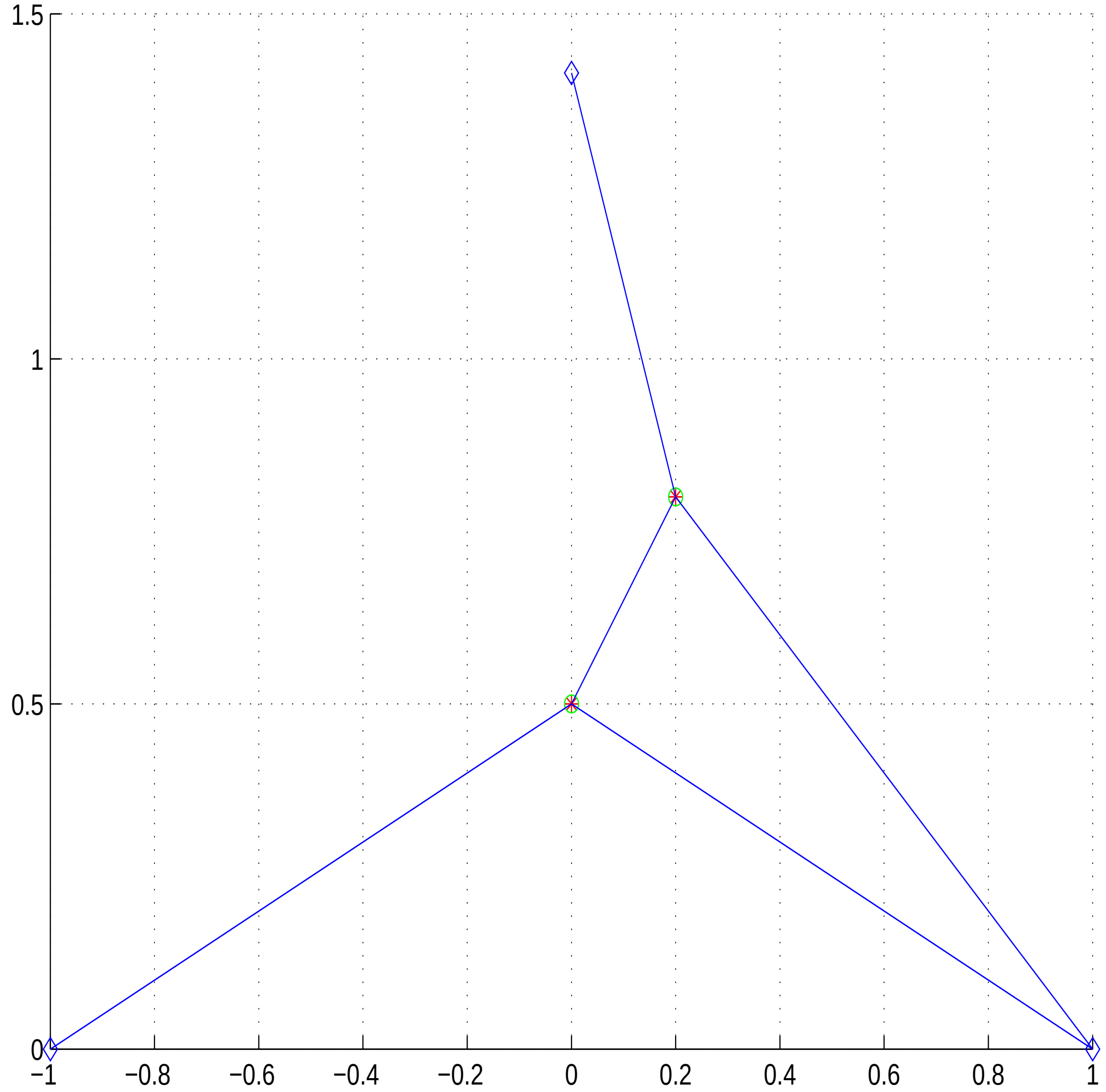


Figure 7: Two sensor-Three anchors: Localizable but not Strongly

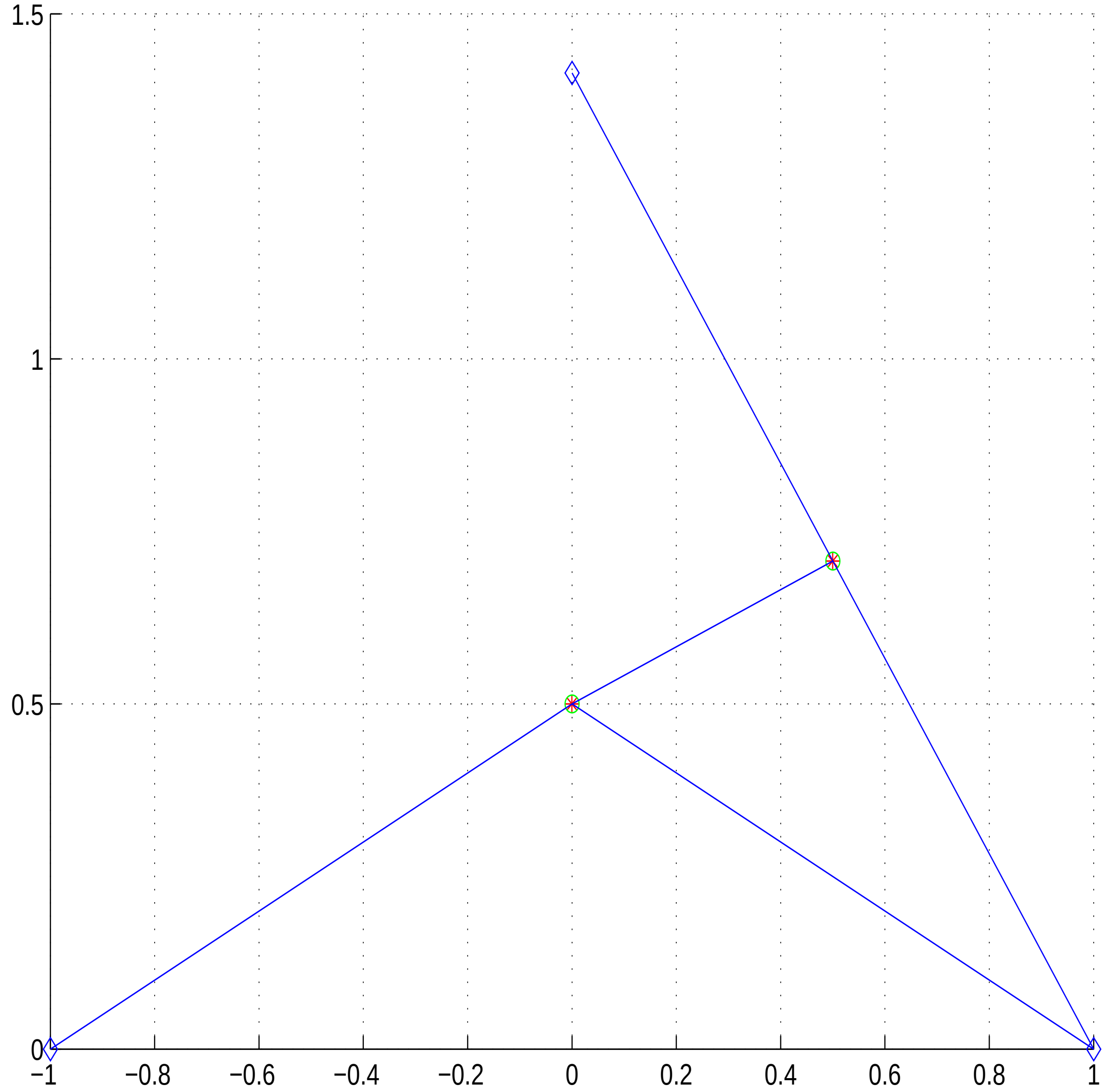


Figure 8: Two sensor-Three anchors: Not localizable

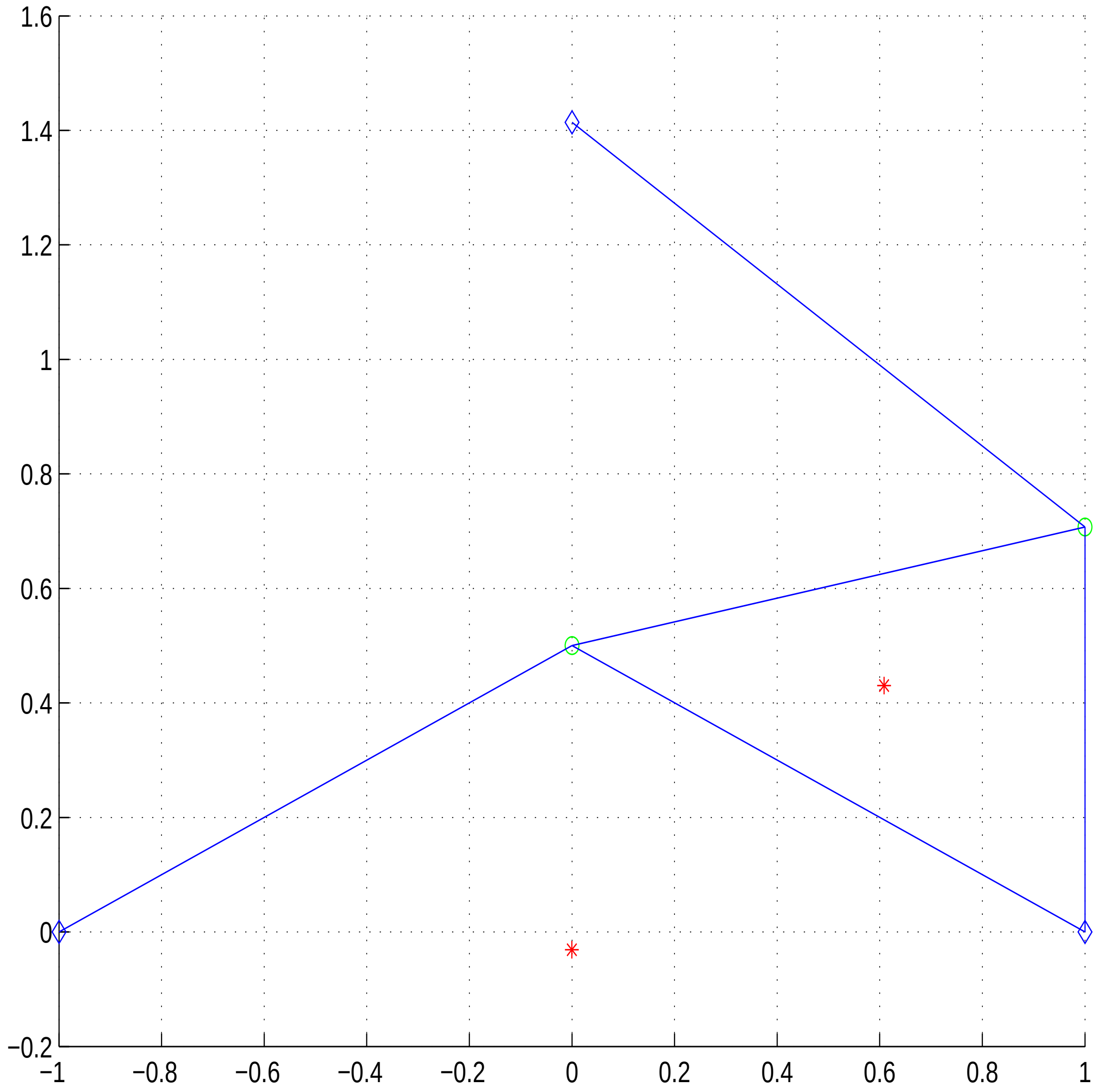
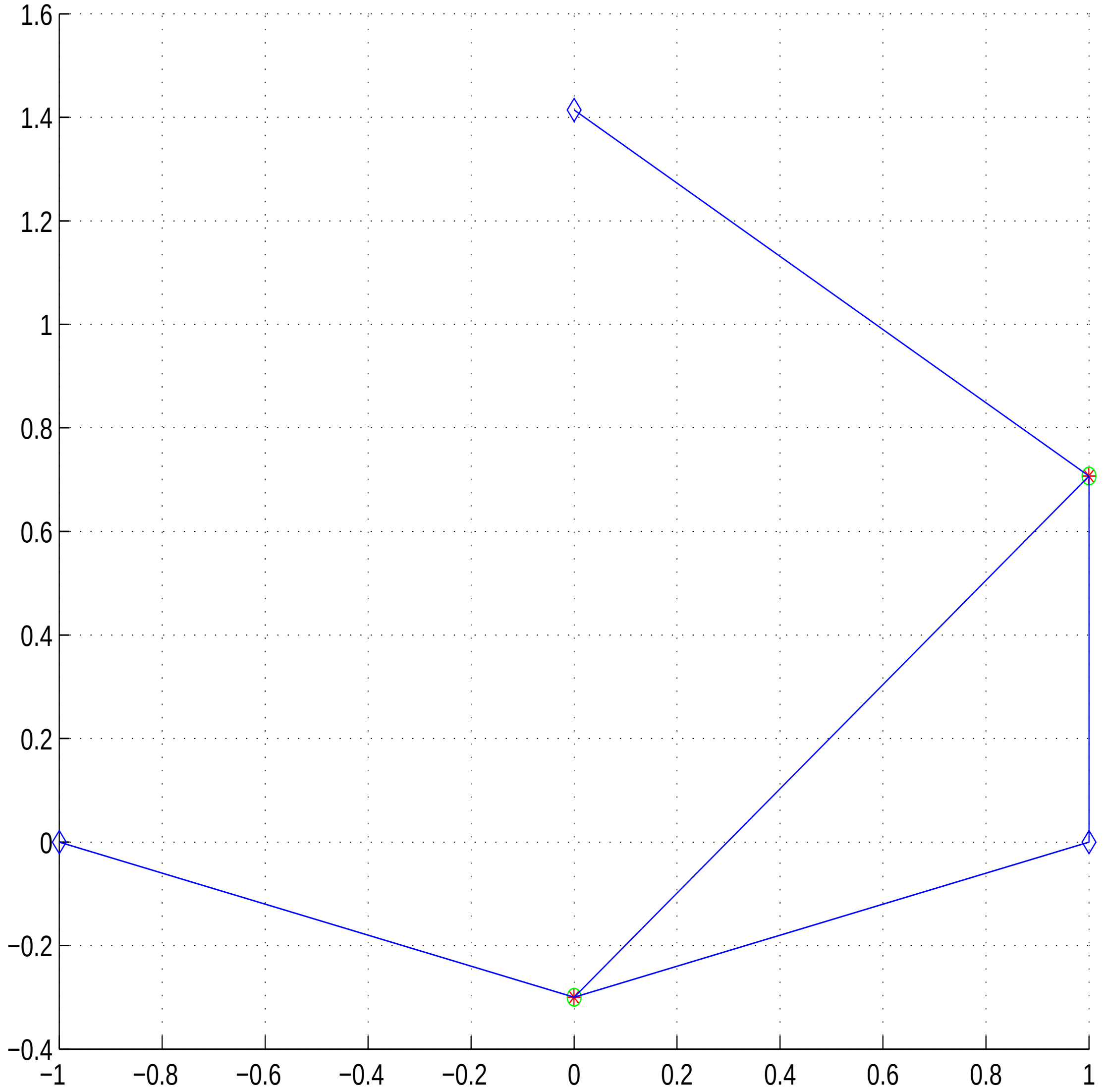


Figure 9: Two sensor-Three anchors: Strongly Localizable



Localize All Localizable Points

Theorem 4. *(So and Y 2005) If a problem (graph) contains a subproblem (subgraph) that is localizable, then the submatrix solution corresponding to the subproblem in the SDP solution has rank 2. That is, the SDP relaxation computes a solution that localize all possibly localizable unknown sensor points.*

Implication: Diagonals of “co-variance” matrix

$$\bar{Y} - \bar{X}^T \bar{X},$$

$\bar{Y}_{jj} - \|\bar{\mathbf{x}}_j\|^2$, can be used as a measure to see whether jth sensor’s estimated position is reliable or not.

Uncertainty Analysis and Confidence Measure

Alternatively, each $\mathbf{x}_j$'s can be viewed as uncertain points from the incomplete distance measures. Then the solution to the SDP problem provides the first and second moment estimation (Bertsimas and Y 1998).

Generally, $\bar{\mathbf{x}}_j$ is a point estimate of $\mathbf{x}_j$ and $\bar{Y}_{ij}$ is a point estimate $\mathbf{x}_i^T \mathbf{x}_j$.

Consequently,

$$\bar{Y}_{jj} - \|\bar{\mathbf{x}}_j\|^2,$$

which is the individual variance estimation of sensor j, gives an interval estimation for its true position (Biswas and Y 2004).

Deterministic Way on Finding a Low-Rank Solution

Add a regularization objective to minimize

$$(SDP) \quad \text{minimize} \quad C \bullet Z$$
$$\text{subject to} \quad A_i \bullet Z = b_i, i = 1, 2, ..., m, \; Z \succeq 0.$$

$$\begin{aligned} \text{minimize} \quad & 2x_1 + x_2 + x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 1, \\ & \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq \mathbf{0}. \end{aligned}$$

For sensor localization problem, we typically choose $C = -I$.

d–Realizable Graphs

A graph is d–realizable if it can always be realized in $\mathbf{R}^d$ whenever it is realizable (the edge weights are Euclidean metric) for every instance of the graph.

- Connelly and Sloughter have recently given a complete characterization of the class of d–realizable graphs, where $d = 1, 2, 3$
- It is trivial to find a realization of an 1–realizable graph, since a graph is 1–realizable iff it is a forest.
- A polynomial time algorithm for realizing 2–realizable graphs exists: since a graph is a partial 2–tree and triangulation works. (The complete graph on k vertices is an k–tree. An k–tree with

$n+1$ vertices (where $n \geq k$) can be constructed from an k–tree with n vertices by adding a vertex adjacent to all vertices of one of its k–vertex complete subgraphs, and only to those vertices. A partial k–tree is a subgraph of an k–tree.)

- Finding realization for 3–realizable graphs is posed as an open question.

3–Realizable Graph

Using the forbidden minor characterization of partial 3–trees, one can show that a graph is 3–realizable if it either

- contains an V_8 or an $C_5 \times C_2$ as a minor

Figure 10: V-8

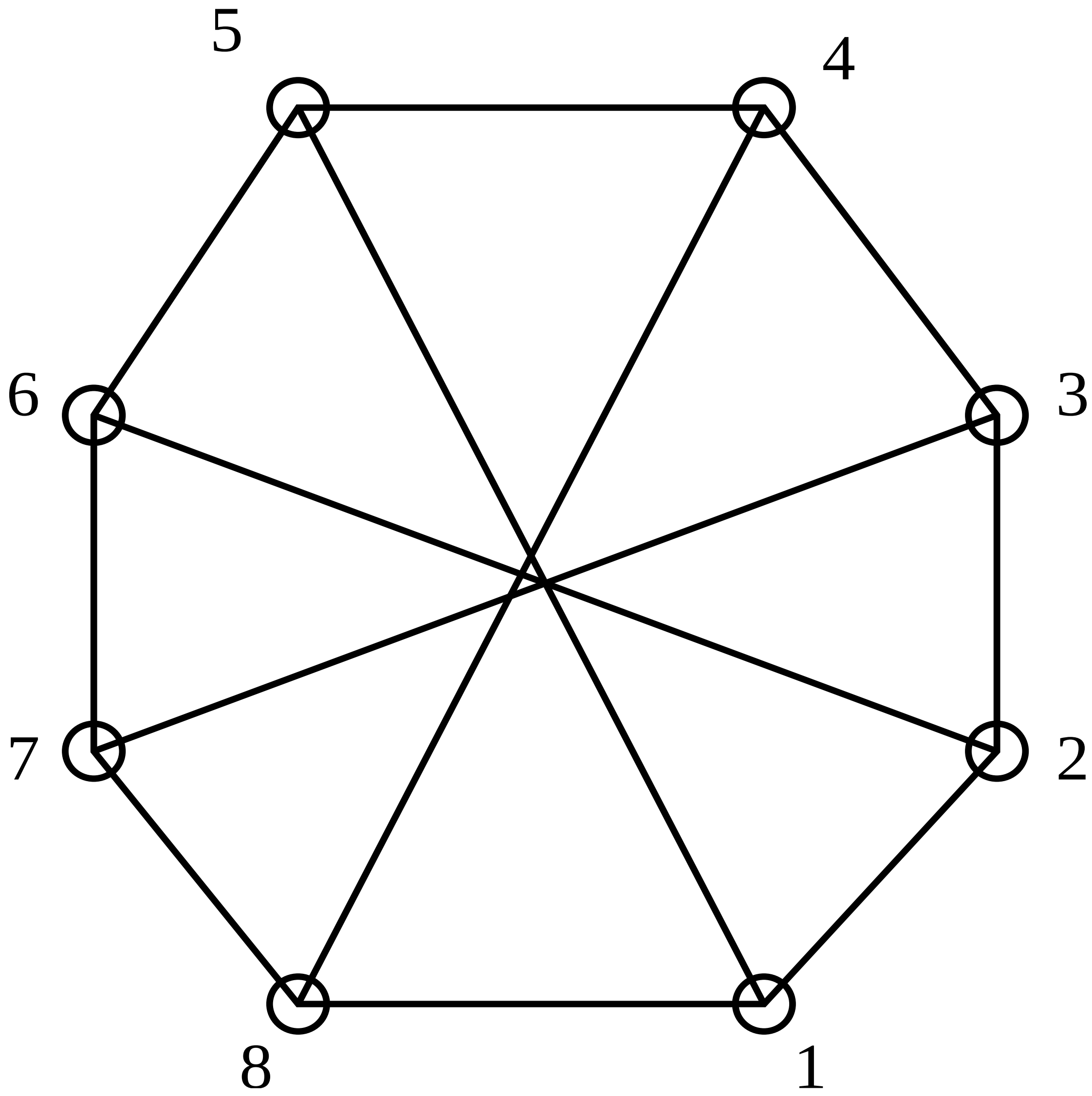
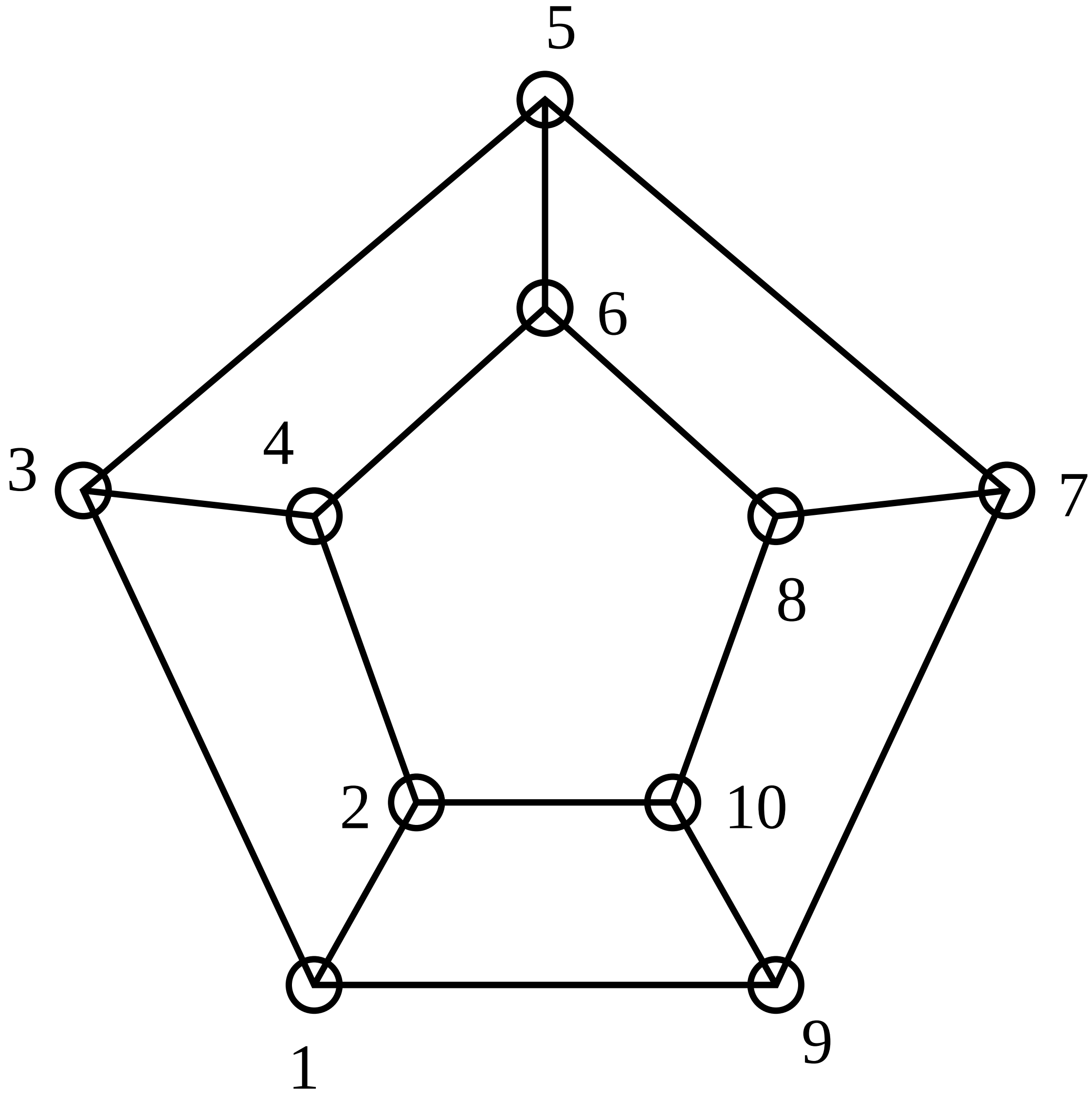


Figure 11: C-5×C-2



- or does not contain either graphs as a minor.

Indeed, if it is the latter, then G is a partial 3–tree.

An k-tree is defined recursively as follows. The complete graph on k vertices is an k–tree. An k–tree with $n+1$ vertices (where $n \geq k$) can be constructed from an k–tree with n vertices by adding a vertex adjacent to all vertices of one of its k–vertex complete subgraphs, and only to those vertices.

A partial k–tree is a subgraph of an k–tree.

So, Y and Zhang (2006) Result

We resolve the above open question by giving a polynomial time algorithm for realizing 3–realizable graphs. The main bottleneck in the proof is to show that two graphs, V_8 and $C_5 \times C_2$, are 3–realizable.

There exists a realization of $H \in \{V_8, C_5 \times C_2\}$ such that the distance between a certain pair of non–adjacent vertices (i, j) is maximized in the SDP relaxation. Such a realization induces a non–zero equilibrium stress, which are the optimal dual multipliers of our SDP relaxation. Then use this equilibrium force to prove that the dual SDP has a rank-$(n-3)$ solution.

More Applications: The Kissing Problem

- Given a unit center sphere, the maximum number of unit spheres, in d dimensions, can touch or kiss the center sphere at same time?
- General Solutions does not exist.
- Delsarte Method uses linear programming to provide an upper bound on the number of spheres.
- K(8) = 240, K(24) = 196650.
- K(4) = 24: proved using Delsarte Method by Oleg Musin only 3 years ago.
- For other dimensions, lower bounds have been provided.

The Kissing Problem as a Graph Realization

Given a unit center sphere in d dimensions, can n unit spheres touch or kiss the center sphere at same time?

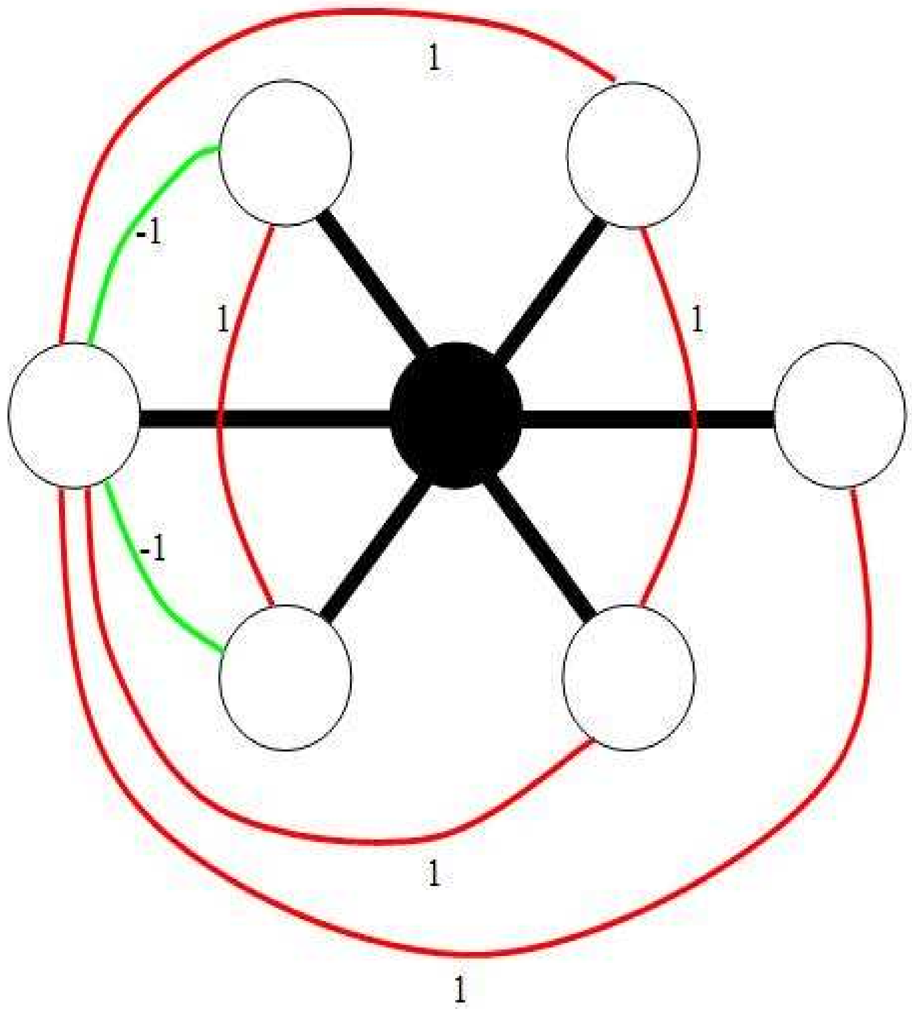
This can be formulated as a SDP feasibility problem with rank constraint.

$$\begin{aligned}(\mathbf{e}_i - \mathbf{e}_j)^T Y (\mathbf{e}_i - \mathbf{e}_j) &\geq 4, \ \forall i \neq j, \\ \mathbf{e}_i^T Y \mathbf{e}_i &= 4, \ \forall i, \\ \text{rank}(Y) &= d.\end{aligned}$$

The objective construction

- Use pull some struts and/or push some cables in order to force SDP solution into low rank.
- For example, for 2D, 6 spheres can be connected as follows (thick lines are bars, red lines are struts, green lines are cables).

Figure 12: 6 Spheres in 2-D

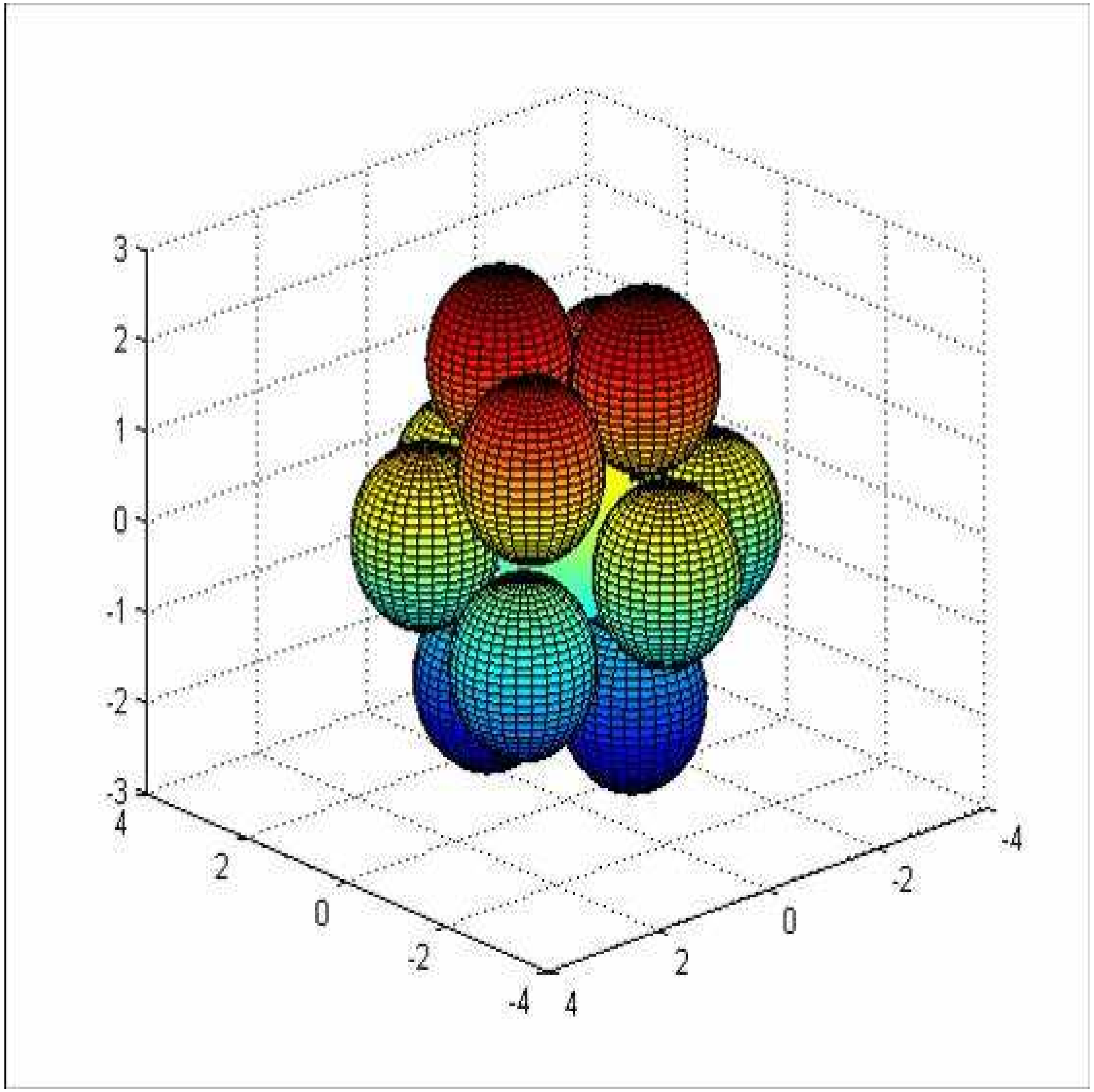


Solving the 3-D Kissing Problem

A regularization objective structure can be extended to dimension 3.

For 12 spheres, SDP method provides the following realization

Figure 13: 12 Spheres in 3-D



More Questions

- Can the distortion upp bound be improved such that it's independent of rank of A_i?
- Is there deterministic algorithm? Choose the largest d eigenvalue component of X?
- In practical applications, we see much smaller distortion, why?
- How to construct a regularization objective to find a low rank SDP solution?