Leontief Economies Encode Nonzero Sum Two-Player Games

Bruno Codenotti* Amin Saberi† Kasturi Varadarajan‡ Yinyu Ye§

Abstract

We consider Leontief exchange economies, i.e., economies where the consumers desire goods in fixed proportions. Unlike bimatrix games, such economies are not guaranteed to have equilibria in general. On the other hand, they include suitable restricted versions which always have equilibria.

We give a reduction from two-player games to a special family of Leontief exchange economies, which are guaranteed to have equilibria, with the property that the Nash equilibria of any game are in one-to-one correspondence with the equilibria of the corresponding economy.

Our reduction exposes a potential hurdle inherent in solving certain families of market equilibrium problems: finding an equilibrium for Leontief economies (where an equilibrium is guaranteed to exist) is at least as hard as finding a Nash equilibrium for two-player nonzero sum games.

As a corollary of the one-to-one correspondence, we obtain a number of hardness results for questions related to the computation of market equilibria, using results already established for games [17]. In particular, among other results, we show that it is NP-hard to say whether a particular family of Leontief exchange economies, that is guaranteed to have at least one equilibrium, has more than one equilibrium.

Perhaps more importantly, we also prove that it is NP-hard to decide whether a Leontief exchange economy has an equilibrium. This fact should be contrasted against the known PPAD-completeness result of [30], which holds when the problem satisfies some standard sufficient conditions that make it equivalent to the computational version of Brouwer's Fixed Point Theorem.

On the algorithmic side, we present an algorithm for finding an approximate equilibrium for some special Leontief economies, which achieves quasi-polynomial time whenever each trader does not demand too much more of any good than some other good.

*IIT-CNR, Pisa, Italy. This work was done while visiting the Toyota Technological Institute at Chicago. Email: bruno.codenotti@iit.cnr.it.

†Department of Management Science and Engineering, Stanford University, Stanford CA 94305. Email: saberi@stanford.edu.

‡Department of Computer Science, The University of Iowa, Iowa City IA 52242. Email: kvaradar@cs.uiowa.edu. Partially supported by NSF CAREER grant CCR-0237431.

§Department of Management Science and Engineering, Stanford University, Stanford CA 94305. Email: yinyu-ye@stanford.edu.

1 Introduction

In the last few years, there has been a lot of interest in the computation of market equilibrium prices in an economy. In a very short time, polynomial-time algorithms have been developed for computing the prices for different special cases of this problem using techniques such as primal-dual [9, 21], auction algorithms [15, 16], and convex programming [29, 20, 32, 5, 4, 3]. However, it seems that all the markets for which these polynomial-time algorithms have been derived share a common property: their equilibrium set is convex.

Roughly speaking, these results take advantage, explicitly or implicitly, of settings where the market's reaction to price changes is *well-behaved* either because the market demand retains some properties of the individual demands or thanks to the special structure of the individual utility functions (e.g., linear, Cobb-Douglas, CES in a certain range of its defining parameter, the elasticity of substitution).

In this paper, we study economies in which the players have *Leontief utility functions*. A Leontief utility function describes the behavior of an *extreme* CES consumer, who desires goods in fixed proportions. These utility functions have a very nice combinatorial description and they come up in different contexts such as modeling congestion control mechanisms like TCP [22].

An economy with Leontief consumers can lead to very "expressive" market demand functions.[1]. The set of equilibria in these markets can be disconnected [18, 6]. Furthermore, no efficient algorithm is known for computing the equilibrium prices in these markets, except in the case of proportional endowments, where the set of equilibria is convex [5]. Our result shows that polynomial time algorithms handling the equilibrium problem in such a scenario where multiple disconnected equilibria can readily appear, would have an extremely important computational consequence for bimatrix games. In particular, we can show that any algorithm which computes an equilibrium price for a (special case of a) market with Leontief utility functions can

[1]For instance, it is known that an economy with Leontief consumers can generate the Jacobian of any market excess demand at a given price (see [25], p.119).

also compute a Nash equilibrium for a bimatrix game. We also establish a one-to-one correspondence between the Nash equilibria in any two-player nonzero sum game and the equilibrium prices in the corresponding special Leontief exchange economy and use this correpsondence to obtain several NP-hardness results.

1.1 The Game-Market correspondence. We consider exchange economies where ℓ, the number of traders, is equal to the number of goods, and the i-th trader has an initial endowment given by one unit of the i-th good. (We call this the *pairing model* [33].) The traders have a Leontief (or fixed-proportion) utility function, which describes their goal of getting a bundle of goods in proportions determined by ℓ given parameters.

Given an arbitrary bimatrix game, specified by a pair of $n \times m$ matrices A and B, with positive entries, we construct a Leontief exchange economy with $n+m$ traders and $n+m$ goods as follows.

Trader i comes to the market with one unit of good i, for $i = 1, \ldots, n+m$. Traders indexed by any $j \in \{1, \ldots, n\}$ receive some utility only from goods $j \in \{n+1, \ldots, n+m\}$, and this utility is specified by parameters corresponding to the entries of the matrix B. More precisely the proportions in which the j-th trader wants the goods are specified by the entries on the jth row of B. Vice versa, traders indexed by any $j \in \{n+1, \ldots, n+m\}$ receive some utility only from goods $j \in \{1, \ldots, n\}$. In this case, the proportions in which the j-th trader wants the goods are specified by the entries on the j-th column of A.

In the economy above, we can partition the traders in two groups, which bring to the market disjoint sets of goods, and are only interested in the goods brought by the group they do not belong to.

We show that the Nash equilibria of any bimatrix game are in one-to-one correspondence with the market equilibria of such an economy.

1.2 Applications. We use this correspondence to show a potential difficulty inherent in computing equilibrium prices: even for families in which equilibrium prices are guaranteed to exist, finding an equilibrium price could be at least as hard as finding a Nash equilibrium for nonzero sum bimatrix games.

Moreover, our one-to-one correspondence allows us to import the results of Gilboa and Zemel [17] on the NP-hardness of some computational problems connected with Nash equilibria, and show, among other results, that saying whether there is more than one equilibrium in an exchange economy is NP-hard. Note that this latter problem is relevant for applied work, where the uniqueness question is of fundamental importance.

It is well known that, under mild assumptions, an equilibrium exists [1]. However, in general, given an economy expressed in terms of traders' utility functions and initial endowments, an equilibrium does not need to exist. For instance, for economies where the traders have linear utility functions, Gale [13] determined necessary and sufficient conditions for the existence of an equilibrium. These conditions boil down to the biconnectivity of a directed graph, which can be verified in polynomial time.

We prove that for Leontief exchange economies testing for existence is instead NP-hard. More precisely, we construct an economy where the traders have Leontief utility functions, and such that saying whether an equilibrium exists is NP-hard. Note that this result does not contradict what is shown in [30], where the market equilibrium problem (both in the version where the input is expressed in terms of utilities and endowments, and in that in terms of excess demand functions) is put in the class PPAD, a subclass of the class TFNP, which is unlikely to coincide with FNP. Indeed such a result assumes standard sufficient conditions which guarantee existence by either Kakutani's or Brouwer's fixed point theorem.

Note that the previous NP-hardness results for market equilibrium problems were in the context of indivisible goods [8].

1.3 Organization of this paper. In Section 2 we define Nash equilibria for bimatrix games as a linear complementarity problem, and introduce the notions of equilibria and quasi-equilibria for certain Leontief economies. In Section 3 we reduce an arbitrary bimatrix game to a special pairing Leontief economy, thus establishing a one-to-one correspondence between the Nash equilibria of the game and the equilibria of the economy. In Section 4 we describe a partial converse of the previous result, by reducing a class of pairing Leontief economies to bimatrix games. In Section 5 we first use the one-to-one correspondence stated in Section 3 to import the hardness results of [17] for Nash equilibria in bimatrix games, and get corresponding hardness results for the market equilibrium problem. We then use one of these hardness results to prove that it is NP-hard to decide whether a Leontief exchange economy has an equilibrium. In Section 6 we use the correspondence with games to obtain a quasi-polynomial time algorithm for pairing Leontief economies where each trader does not want too much more of some good compared to some other good. Our algorithm is inspired by an algorithm of Lipton et al. [24] in the context of bimatrix games.

2 Games, Markets, and LCP

Let us consider the problem of computing the Nash equilibria for any bimatrix game (A, B), where A and B are $n \times m$ matrices, which we assume to be strictly positive without loss of generality. This can be rewritten as the following linear complementarity problem (see pages 91–93 of [28]), which we call LCP1.

Find a nonnegative $w \neq \mathbf{0}$ and a nonnegative z such that

$$\begin{aligned} Hw + z &= \mathbf{1} \\ w^T z &= 0, \end{aligned}$$

where

$$H = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix} \in \Re^{(n+m)\times(n+m)}.$$

Note that the system LCP1 may be equivalently viewed as the problem of finding a nonnegative vector $0 \neq w \in \Re^{n+m}$ such that

$$\sum_j h_{ij} w_j \leq 1 \text{ for all } 1 \leq i \leq n+m,$$

and

$$w_i > 0 \Rightarrow \sum_j h_{ij} w_j = 1 \text{ for all } 1 \leq i \leq n+m.$$

¿From Nash Theorem on the existence of a Nash equilibrium, it follows that LCP1 has at least one solution w. Let $\mathcal{N} = \{j : j \leq n\}$ and $\mathcal{M} = \{j : n < j \leq n+m\}$. It is easy to see that $w_j > 0$ for some action $j \in \mathcal{N}$ as well as some action $j \in \mathcal{M}$, since each of the players is playing a mixed strategy. In other words, if $w_i > 0$ and $i \in \mathcal{N}$, then there must be at least one $j \in \mathcal{M}$ such that $w_j > 0$; otherwise,

$$1 = \sum_j h_{ij} w_j = \sum_{j=n+1}^{n+m} h_{ij} w_j = 0$$

which is a contradiction. Similarly, $w_i > 0$ and $i \in \mathcal{M}$ imply that there must be at least one $j \in \mathcal{N}$ such that $w_j > 0$.

We now describe a special form of a Leontief exchange economy, the *pairing model* [33], in which there are ℓ traders and ℓ goods. The economy is described by a square matrix F of size ℓ. The j-th trader comes in with one unit of the j-th good, and has a Leontief utility function

$$u_j(x) = \min_{i: f_{ij} \neq 0} \left\{ \frac{x_i}{f_{ij}} \right\}.$$

An equilibrium for such an economy is given by a nonnegative price vector $0 \neq \pi \in \Re^\ell$ such that

1. For each $1 \leq j \leq \ell$, $\beta_j = \frac{\pi_j}{\sum_k f_{kj}\pi_k}$ is well-defined, that is, $\sum_k f_{kj}\pi_k > 0$.
2. For each good $1 \leq i \leq \ell$, $\sum_j f_{ij}\beta_j \leq 1$; that is, the total trading volume does not exceed the quantity available.

Note that β_j represents the utility value of the optimal bundle of the trader j at equilibrium, and the optimal bundle itself is $(f_{1j}\beta_j, \ldots, f_{\ell j}\beta_j)$. Standard arguments imply that if $\pi_i > 0$, then in fact $\sum_j f_{ij}\beta_j = 1$. Moreover, we also have that $\pi_j > 0$ if and only if $\beta_j > 0$.

A closely related notion is that of a quasi-equilibrium. This is obtained, in our case, by replacing condition (1) above by

1'. For each $1 \leq j \leq \ell$, there exists β_j such that $\beta_j(\sum_k f_{kj}\pi_k) = \pi_j$.

In a quasi-equilibrium, the zero-bundle, corresponding to $\beta_j = 0$, is a valid bundle when $\pi_j = 0$, even though $\sum_k f_{kj}\pi_k = 0$.

Thus the main difference between an equilibrium and a quasi-equilibrium is that in the latter, a trader with zero income is not required to optimize her utility. The reader is referred to the textbook of Mas-Colell et al. [26] for a more systematic development. One standard way to establish sufficient conditions for the existence of an equilibrium is to first use fixed point theorems to establish the existence of a quasi-equilibrium, and then argue that under the sufficient conditions, every quasi-equilibrium is an equilibrium.

A simple example of a (pairing) Leontief economy that has a quasi-equilibrium but no equilibrium is encoded by the matrix

$$F = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

3 Leontief economies encode bimatrix games

We give a polynomial time computable reduction from any two-player nonzero sum game to a special class of the pairing Leontief economies, which we call the *two-groups* Leontief economies, with the property that the Nash equilibria of the game and the equilibria of the market are in one-to-one correspondence. This shows that the problem of computing Nash equilibria for a bimatrix game is equivalent to that of computing market equilibria for these exchange economies. To prove this result, we exploit ideas developed by Ye [33].

Given an instance of the problem of computing the Nash equilibria for a bimatrix game (A, B), where A and B are positive $n \times m$ matrices, we construct an instance of a (pairing) exchange economy with $(n+m)$ traders and $(n+m)$ goods that is given by setting $F = H$. It is also easy to see that trading needs to occur between some trader $j \in \mathcal{N}$ and some trader $j \in \mathcal{M}$, since trader in $\mathcal{N}$ are only interested in goods that are brought in by traders in $\mathcal{M}$, and viceversa. We call this economy *two-groups Leontief economy.* It easily follows from the definition that at any equilibrium π of the economy, we must have $\pi_i > 0$ for some $i \in \mathcal{N}$ as well as some $i \in \mathcal{M}$.

3.1 From the Market to the Game.

We first prove that any market equilibrium of the two-groups Leontief economy corresponds to a Nash equilibrium in the associated two-player bimatrix game.

LEMMA 3.1. *Let $\beta = (\beta_1, \ldots, \beta_{n+m})$ be the vector of the utility values at equilibrium prices π for the two-groups Leontief economy. Then β solves LCP1, and thus it encodes the Nash equilibria of the game described by LCP1.*

Proof. At any equilibrium of the market, we have $\sum_j h_{ij}\beta_j \le 1$ for each $1 \le i \le n+m$, and $\beta_j > 0$ if and only if $\pi_j > 0$. Moreover, $\beta_i > 0 \Rightarrow \sum_j h_{ij}\beta_j = 1$. Thus the β's from the equilibrium solve the system LCP1 with $w = \beta$. Moreover, π_j, and thus β_j, is positive for some j, so that $w = \beta \neq 0$.

3.2 From the Game to the Market.

We now show that any Nash equilibrium of a bimatrix game corresponds to a market equilibrium of the corresponding two-groups Leontief economy.

LEMMA 3.2. *Let $w \neq 0$, be any solution to LCP1. Then there exists an equilibrium price vector π such that $w = (w_1, \ldots, w_{n+m})$ is the vector of the utility values at these equilibrium prices for the two-groups Leontief economy.*

Proof. Let $w \neq 0$ be any complementarity solution to LCP1. Partition the index set $\{1, \ldots, n+m\}$ into two groups $P = \{j : w_j > 0\}$ and $Z = \{j : w_j = 0\}$. As we showed before, $P \cap \mathcal{N} \neq \emptyset$ and $P \cap \mathcal{M} \neq \emptyset$.

We claim that there exists $\pi_j > 0$ for each $j \in P$ such that $w_j = \frac{\pi_j}{\sum_{k\in P} h_{kj}\pi_k}$, or in a different form, $\sum_{k\in P} h_{kj} w_j \pi_k = \pi_j$. Let H_{PP} be the $|P|\times|P|$ principal submatrix of H induced by the indices in P, and W_P the $|P| \times |P|$ diagonal matrix whose diagonal contains the w's corresponding to P. Our claim is equivalent to saying that the system $C\sigma = \sigma$, where $C = (H_{PP}W_P)^T$, has a solution in which all the entries of σ are positive. Note that each column of C sums to one: this follows because $i \in P \Rightarrow w_i > 0$ and

$$w_i > 0 \Rightarrow \sum_{j\in P} h_{ij} w_j = \sum_j h_{ij} w_j = 1.$$

Moreover,

$$C = \begin{pmatrix} 0 & D \\ E^T & 0 \end{pmatrix},$$

where E and D are $(|P| - l) \times l$ matrices, for some $1 \le l \le |P| - 1$. The bounds on l follow from the fact that $P \cap \mathcal{N} \neq \emptyset$ and $P \cap \mathcal{M} \neq \emptyset$.

The existence of such a positive solution to $C\sigma = \sigma$ follows from Proposition 3.1 below.

We have established our claim that there exists $\pi_j > 0$ for each $j \in P$ such that

$$w_j = \frac{\pi_j}{\sum_{k\in P} h_{kj}\pi_k}.$$

Set $\pi_j = 0$ for $j \in Z$. We now argue that π is an equilibrium.

Note that for $j \in P$, we have

$$w_j = \frac{\pi_j}{\sum_{k\in P} h_{kj}\pi_k} = \frac{\pi_j}{\sum_k h_{kj}\pi_k}.$$

For $j \in Z$, observe that $\sum_k h_{kj}\pi_k > 0$. This is because there exists $k \in P$ such that $h_{kj} > 0$, since P contains elements from both $\mathcal{N}$ and $\mathcal{M}$. For this k, we have $h_{kj}\pi_k > 0$. Therefore,

$$w_j = \frac{\pi_j}{\sum_{k\in P} h_{kj}\pi_k} = \frac{\pi_j}{\sum_k h_{kj}\pi_k} = 0.$$

Moreover, we have, for each good $1 \le i \le n+m$, $\sum_j h_{ij} w_j \le 1$, since w is a solution of LCP1. Thus both the conditions for an equilibrium are fulfilled, with the w_i's playing the role of the β_i's.

PROPOSITION 3.1. *The linear system $C\sigma = \sigma$ has a positive solution.*

Proof. Consider the matrix

$$C^2 = \begin{pmatrix} DE^T & 0 \\ 0 & E^T D \end{pmatrix}.$$

Notice that both DE^T and $E^T D$ are column stochastic, because C and hence D and E^T are column stochastic. Therefore the system $C^2 z = z$ has a positive solution. We can write $(C^2 - I)z = 0$ as $(C - I)(C + I)z = 0$. Consider now the vector $\sigma = (C + I)z$. Clearly σ has all positive components, if z has. Also $(C - I)\sigma = 0$ or $C\sigma = \sigma$.

Note that Proposition 3.1 implies that C is irreducible besides column-stochastic, so that σ is in fact the unique Perron-Frobenius eigenvector of C (see, for example, [23], p. 141). Consequently, we observe that there is precisely one equilibrium price vector π, the one we have constructed above, that corresponds to the utility vector w. This follows because we must have $\pi_j > 0$ if and only if $w_j > 0$. Thus $\pi_j = 0$ for $j \in Z$, $\pi_j > 0$ for $j \in P$, and thus the unique positive solution of $C\sigma = \sigma$ gives the only possible values for the prices of goods in P. From the definition, it follows that there is a unique utility vector corresponding to an equilibrium price vector.

The following theorem summarizes the results of this section.

THEOREM 3.1. *Let (A, B) denote an arbitrary bimatrix game, where we assume, w.l.o.g., that the entries of the matrices A and B are all positive. Let the columns of*

$$H = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix}$$

describe the utility parameters of the traders in a two-groups Leontief economy. There is a one-to-one correspondence between the Nash equilibria of the game (A, B) and the market equilibria of the two-groups Leontief economy. Furthermore, the correspondence has the property that a strategy is played with positive probability at a Nash equilibrium if and only if the good held by the corresponding trader has a positive price at the corresponding market equilibrium.

COROLLARY 3.1. *If there is a polynomial time algorithm to find an equilibrium for a two-groups Leontief economy, then there is a polynomial time algorithm for finding a Nash equilibrium of a bimatrix game.*

4 Bimatrix games encode the (pairing) Leontief economy

In this section, we establish a partial converse to the result of Section 3. We will show that bimatrix games encode a special case of the pairing Leontief economies. In this setting, there are n traders and n goods. The j-th trader comes in with one unit of the j-th good, and has a Leontief utility function

$$u_j(x) = \min_i \left\{ \frac{x_i}{a_{ij}} \right\},$$

where $a_{ij} > 0$. In other words, every trader j is interested in all the goods, and she wants the goods in a fixed proportion determined by the j-th column of a positive matrix $A \in \Re^{n\times n}$.

We will show that finding equilibrium prices for the economy above is equivalent to finding symmetric equilibria of the symmetric game defined by (A, A^T). This problem can be written as the following linear complementarity problem, which we call LCP2. Find a nonnegative $w \neq \mathbf{0}$ and a nonnegative z such that

$$\begin{aligned} Aw + z &= \mathbf{1} \\ w^T z &= 0 \end{aligned}$$

In the program above, any nonzero w defines a symmetric equilibrium strategy of the game. More precisely, if w is a nonzero feasible solution for LCP2 then $w/|w|_1$ is an equilibrium strategy for both players.

We now argue that any nonzero solution w to the complementarity problem LCP2, or equivalently any symmetric Nash equilibrium of the game, corresponds to an equilibrium of the Leontief economy.

THEOREM 4.1. *For any nonzero solution (w, z) of LCP2 with a positive matrix A, there is an equilibrium price π such that the utility value of player i at π is w_i. Moreover, given (w, z), π can be computed in polynomial time.*

Proof. The proof of this theorem is implied by [33]. Let $P = \{j : w_j > 0\}$, and $Z = \{j : w_j = 0\}$. Then consider the stochastic matrix $A_{PP}D(w_P)$, where A_{PP} is $|P| \times |P|$ principal submatrix of A induced by the indices in P, $D(w_P)$ is the diagonal matrix whose entries are w_j, $j \in P$. Since $A_{PP}D(w_P) > 0$, it has a positive left eigenvector $\pi_P > 0$. Let $\pi_j = 0$ for $j \in Z$.

Since for some i, $w_i > 0$, P is non-empty and therefore π is also nonzero. Furthermore, it is very easy to see that:

1. For every $1 \leq i \leq n$, $\sum_{j=1}^{n} a_{ij} w_i \leq 1$
2. $w_i > 0 \implies \sum_{j=1}^{n} a_{ij} w_i = 1$

Therefore, w is an allocation supported by the equilibrium price vector π.

It is straightforward to see that any equilibrium of the pairing Leontief economy yields a symmetric Nash equilibrium of the game (A, A^T).

Now the symmetric Nash equilibria of the game (A, A^T) are in one-to-one correspondence with the Nash equilibria of the game (A, I), and it is possible to go from one to the other in polynomial time. See McLennan and Tourky ([27], Proposition 26) for a proof. Therefore, we have:

COROLLARY 4.1. *If there is a polynomial time algorithm for finding a Nash equilibrium for a bimatrix game, then there is a polynomial time algorithm for finding an equilibrium price in a Leontief pairing economy with a positive utility matrix.*

We should also add that in [14, 27], a similar one-to-one correspondence and polynomial-time reduction is established between finding Nash equilibria of a game, finding symmetric equilibria of a symmetric game and a solution to an instance of LCP2. Using those results, it is possible to give a shorter, but less self-contained, proof for Theorem 3.1.

Note that, while the reduction in Theorem 3.1 is from arbitrary bimatrix games, the reduction in this section is from only a special family of Leontief economies. As in bimatrix games, the equilibrium points of the pairing Leontief economies are rational numbers [33]. However, in the case where the endowments of the buyers are unrestricted, Eaves [10] gives an example showing that equilibrium points could be irrational. This suggests that there is no natural linear complementarity formulation for general Leontief exchange economies, and perhaps even that solving these economies might be strictly harder than finding Nash equilibria of a bimatrix game.

Furthermore, we have assumed that the utility matrix of our market A is positive. This restriction is necessary because if some entries of A are zero, A_{PP} may be reducible and a strictly positive left eigenvector π_P may not exist. This shows a subtle difference in the structure of equilibria in these two settings despite their similar linear complementarity programs. It is easy to see that adding a constant to all the entries of a matrix corresponding to a game does not change its equilibria points, but adding a constant to all entries of the utility matrix of a Leontief economy might change the set of equilibria.

Our result can be generalized to the Leontief economy where all goods are *differentiate*, a case previously also studied in [33].

5 Hardness Results

Well known sufficient conditions guarantee that an equilibrium for an exchange economy does exist (see, e.g., [26] Section 17C). Under such assumptions, its equivalence to fixed point problems follows from the combination of two results: a simple and nice transformation introduced by Uzawa [31], which maps any continuous function into an excess demand function, inducing a one-to-one correspondence between the fixed points of the function and the equilibria, and the SMD Theorem (see [26], pp. 598-606) which states the essentially arbitrary nature of the market excess demand function.

Theorem 3.1 shows that there is a one-one correspondence between two-groups Leontief economies and bimatrix games. Combining this theorem with the NP-hardness results of Giboa and Zemel for some questions related to Nash equilibria [17], we show hardness results for Leontief economies.

One of these hardness results pertains the existence of an equilibrium where the prices of some prescribed goods are positive. This specific hardness result allows us to construct a Leontief exchange economy for which an equilibrium exists if and only if in another Leontief economy there is an equilibrium where the prices of some prescribed goods are positive. This correspondence proves that it is NP-hard to test for existence.

Note that in general the equilibria of Leontief exchange economies can be irrational ([5], Section 3) so that the existential problem does not belong to NP, and we thus talk of NP-hardness as opposed to NP-completeness.

5.1 Uniqueness and Equilibria with additional properties. Gilboa and Zemel [17] proved a number of hardness results related to the computation of Nash equilibria (NE) for finite games in normal form. Since the NE for games with more than two players can be irrational, these results have been formulated in terms of NP-hardness for multi-player games, while they can be expressed in terms of NP-completeness for two-player games.

Given a two-player game G in normal form, i.e., expressed as a bimatrix game, consider the following problems:

1. *NE uniqueness*: Given G, does there exist a unique NE in G?
2. *NE in a subset*: Given G, and a subset of strategies T_i for each player i, is there a NE where all the strategies outside T_i are played with probability zero?
3. *NE containing a subset*: Given G, and a subset of strategies T_i for each player i, is there a NE where all the strategies in T_i are played with positive probability?
4. *NE maximal support*: Given G and an integer $r \geq 1$, does there exist a NE in G such that each player uses at least r strategies with positive probability?
5. *NE minimal support*: Given G and an integer $r \geq 1$, does there exist a NE in G such that each player uses at most r strategies with positive probability?

Gilboa and Zemel showed that

1. *NE uniqueness* is co-NP complete;
2. *NE in a subset*, *NE containing a subset*, *NE maximal support*, and *NE minimal support* are NP-complete.

Combining the above results with Theorem 3.1, we get the following theorem.

THEOREM 5.1. *Given an exchange economy, where each trader is specified by an initial endowment and a Leontief utility function, such that the economy has at least one equilibrium, the following problems are NP-hard:*

1. *Is there more than one equilibrium?*
2. *Is there an equilibrium where the prices of a given set of goods are positive?*

Proof. The results use the reduction of Theorem 3.1, which, together with Nash Theorem on the existence of a Nash equilibrium, tells us that the Leontief economy constructed by the reduction always has an equilibrium.

1. The NP-hardness follows from the coNP-completeness of *NE uniqueness*, and from the one-to-one correspondence of Theorem 3.1. We also note that the construction of Gilboa and Zemel [17] for *NE uniqueness* yields games with a finite number of equilibria.
2. The NP-hardness follows from the NP-completeness of *NE containing a subset*, and from Theorem 3.1.

Additional hardness results can be obtained by working out other reductions from [17], or their refinements in [2, 7, 27].

5.2 Existence of an equilibrium. We now give a reduction from statement (2) of Theorem 5.1 to show that the problem of deciding whether a Leontief exchange economy has an equilibrium is NP-hard.

THEOREM 5.2. *It is NP-hard to decide whether a Leontief exchange economy has an equilibrium.*

Proof. The reduction is from Theorem 5.1 (2). Suppose M is an instance of an economy with n traders and goods, and we want to know if there is an equilibrium with goods $1, \dots, k$ priced positively. We construct an economy M' with k additional traders and goods: for $1 \le j \le k$, the $(n+j)$-th trader brings in one unit of the $(n+j)$-th good and wants just the j-th good.

We argue that M' has an equilibrium if and only if M has an equilibrium with goods $1, \dots, k$ priced positively.

Suppose M has an equilibrium in which goods $1, \dots, k$ are priced positively. Then this can be extended to an equilibrium of M' by setting the prices of goods $n+1, \dots, n+k$ to be 0, and giving the $(n+j)$-th trader 0 utility (and 0 units of good j). It is evident that condition (1) for an equilibrium holds for the $(n+j)$-th trader, since the j-th good is priced positively. Condition (2) also holds.

Consider now an equilibrium for M'. For $1 \le j \le k$, it can be seen from Walras' Law that the price of the $(n+j)$-th good must be zero, since nobody wants this good. For condition (1) to hold for the $(n+j)$-th trader, it must be that the j-th good is priced positively. It follows that the prices of the first n goods, together with the optimal bundles of the first n traders, constitutes an equilibrium for the original economy M in which the prices of goods $1, \dots, k$ are positive.

We have proved that M' has an equilibrium if and only if M has an equilibrium with goods $1, \dots, k$ priced positively. M' can clearly be constructed from M in polynomial time.

Notice that the reduction can be easily modified, if needed, to ensure that each good in M' is desired by some trader. (We simply make the $(n+j)$-th trader want both the $(n+j)$-th good and the j-th good in the ratio 1 : 2.)

6 Computing an Approximate Equilibrium

Let the $\ell \times \ell$ matrix F encode the pairing Leontief exchange economy (Section 2) with ℓ traders and ℓ goods. Let $\kappa \ge 1$ be a number such that for each trader j, $\frac{\max_i f_{ij}}{\min_i f_{ij}} \le \kappa$. In this section, we describe an algorithm that, for any parameter $0 < \varepsilon < 1$, runs in time that is $\ell^{O(\frac{\kappa^2 \log \ell}{\varepsilon^2})}$ times a polynomial in the input size and computes an ε-approximate equilibrium for the economy.

An ε-approximate equilibrium is given by a nonnegative price vector $0 \ne \pi \in \Re^\ell$ such that

1. For each $1 \le j \le \ell$, we have $\sum_k f_{kj}\pi_k > 0$, and there exists β_j such that $(1-\varepsilon)\frac{\pi_j}{\sum_k f_{kj}\pi_k} \le \beta_j \le \frac{\pi_j}{\sum_k f_{kj}\pi_k}$.
2. For each good $1 \le i \le \ell$, $\sum_j f_{ij}\beta_j \le 1$.

The definition says that an approximate equilibrium is a vector of prices at which approximately utility maximizing bundles $(\beta_j f_{1j}, \dots, \beta_j f_{nj})$ for each trader j leads to market clearance. This is now a fairly standard notion [21, 4].

We note that multiple disconnected equilibria can exist in such economies for any $\kappa > 1$. By scaling the columns of the matrix F, we can assume that each entry of F is between 1 and κ, and that the largest entry in each column is precisely κ. It will be convenient to let f_i denote the i'th row of F.

We define an ε-complementarity solution for F to be a vector $\mathbf{0} \ne w \in \Re^n_+$ such that for each i, (1) $f_i \cdot w \le 1$, and (2) $w_i > 0 \Rightarrow f_i \cdot w \ge 1 - \varepsilon$. The following proposition says that every ε-complementarity solution corresponds to approximate utility maximizing bundles at some ε-approximate equilibrium.

Proposition 6.1. *Let w be an ε-complementarity solution for F. Then there is a price vector $\pi \in \Re^n_+$ such that for each j, we have $\sum_k f_{kj}\pi_k > 0$, and $(1-\varepsilon)\frac{\pi_j}{\sum_k f_{kj}\pi_k} \le w_j \le \frac{\pi_j}{\sum_k f_{kj}\pi_k}$.*

Proof. Partition the index set $\{1,\dots,\ell\}$ into two groups $P = \{j : w_j > 0\}$ and $Z = \{j : w_j = 0\}$. Note that $P \neq \emptyset$. For each $i \in P$, let $\eta_i = 1/(f_i \cdot w)$; note that $1 \le \eta_i \le 1/(1-\varepsilon)$. We claim that there exists $\pi_j > 0$ for each $j \in P$ such that for each trader $j \in P$, we have $w_j = \frac{\pi_j}{\sum_{k\in P} f_{kj}\eta_k\pi_k}$, or in a different form $\sum_{k\in P} f_{kj}w_j\eta_k\pi_k = \pi_j$.

Let F_{PP} be the $|P|\times|P|$ principal submatrix of F induced by the indices in P, W_P the $|P|\times|P|$ diagonal matrix whose diagonal contains the w's corresponding to P, and E_P the $|P|\times|P|$ diagonal matrix whose diagonal contains the η's corresponding to P. Our claim is equivalent to saying that the system $C\sigma = \sigma$, where $C = (E_P H_{PP} W_P)^T$, has a solution in which all the entries of σ are positive. Note that each entry of C is positive and each column of C sums to one, because $\eta_i f_i \cdot w = 1$ for $i \in P$. The claim therefore follows from the Perron-Frobenius Theorem.

Since $1 \le \eta_k \le 1/(1-\varepsilon)$, it follows that for each $j \in P$, $(1-\varepsilon)\frac{\pi_j}{\sum_k f_{kj}\pi_k} \le w_j \le \frac{\pi_j}{\sum_k f_{kj}\pi_k}$. Set $\pi_j = 0$ for $j \in Z$. For this vector π, the Proposition is now readily seen to hold.

The following lemma and proof are inspired by a corresponding result for bimatrix games [24].

Lemma 6.1. *For any $0 < \varepsilon < 1$, there exists an ε-complementarity solution w to F with only $O(\frac{\kappa^2\log\ell}{\varepsilon^2})$ non-zeroes.*

Proof. Let $0 \neq \beta \in \Re^\ell$ be a "0-complementarity" solution to F: for each i, we have (1) $f_i\cdot\beta \le 1$, and (2) $\beta_i > 0 \Rightarrow f_i\cdot\beta = 1$. Such a β corresponds to the utilities of the traders at equilibrium, which can be shown to exist via standard arguments [26] using the fact that each entry of F is positive. Note that $\sum_j \beta_j \le f_i\cdot\beta \le 1$.

Let $\delta = c_1\varepsilon$, where $c_1 > 0$ is a small enough constant, and τ be the smallest integer that is at least $\frac{\kappa^2\log\ell}{\delta^2}$. Let Ω be the probability distribution over $\Re^\ell$ where the unit vector e_i has a probability β_i and the origin has a probability $1-\sum_i\beta_i$. Let $x^1,\dots,x^\tau$ be τ independent choices from the distribution Ω. Let $x = \frac{\sum_{1\le t\le\tau} x^t}{\tau}$.

Fix any $1 \le i \le \ell$. The random variable $f_i\cdot x^t$ ranges over $[0,\kappa]$ and $E[f_i\cdot x^t] = f_i\cdot\beta$. Since $f_i\cdot x = \frac{\sum_{1\le t\le\tau} f_i\cdot x^t}{\tau}$, it follows that $E[f_i\cdot x] = f_i\cdot\beta$. Using the Hoeffding bound ([19], Theorem 2), we conclude that

$$\Pr[|f_i\cdot x - f_i\cdot\beta| \ge \delta] \le e^{\frac{-2\tau^2\delta^2}{\tau\kappa^2}} \le e^{-2\log\ell} \le 1/\ell^2.$$

From the union bound, it follows that with positive probability, we have $|f_i\cdot x - f_i\cdot\beta| \le \delta$ for every i. Let w' be an outcome of x for which this good event happens. Clearly, w' has at most $\tau = O(\frac{\kappa^2\log\ell}{\varepsilon^2})$ nonzeroes. For each i, we have $f_i\cdot w' \le f_i\cdot\beta+\delta \le 1+\delta$. Moreover, if $w'_i > 0$, then it must be that $\beta_i > 0$. Thus if $w'_i > 0$, then $f_i\cdot w' \ge f_i\cdot\beta-\delta = 1-\delta$. Setting $w = \frac{1}{1+\delta}w'$, the proof of the lemma is complete.

The algorithm for computing an ε-approximate market equilibrium easily follows from Lemma 6.1 and Proposition 6.1. By solving $\ell^{O(\frac{\kappa^2\log\ell}{\varepsilon^2})}$ linear programs, one for each possible subset of size $O(\frac{\kappa^2\log\ell}{\varepsilon^2})$, we can compute an ε-complementarity solution w to F. Given w, we compute an ε-approximate equilibrium by solving another linear program. Proposition 6.1 guarantees that a corresponding ε-approximate equilibrium price exists.

7 Concluding Remarks

In this paper, we have described certain connections between exchange economies and bimatrix games, and analyzed some related computational consequences. In particular, we showed that any algorithm which computes a Nash equilibrium for a bimatrix game computes a market equilibrium for a special Leontief economy, and, viceversa, any algorithm for the market equilibrium with Leontief utility functions must have the ability to compute a Nash equilibrium for a bimatrix game.

Our reduction uses a formulation of the market equilibrium allocation as a solution to a special type of linear complementarity problems. Prior to this work, Eaves had shown in [10] that the equilibrium in exchange economies with Cobb-Douglas utility functions can be obtained as the solution to a special linear programming problem. Because of the well known equivalence between zero-sum games and linear programming (due to Von Neumann Minimax Theorem), we have that Cobb-Douglas exchange economies can be coded as special two-player zero-sum games.

Acknowledgements. The authors thank Kamal Jain for raising the possibility that Leontief economies may encode bimatrix games, and Mohammad Mahdian and Christos Papadimitriou for very helpful discussions.

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