Approximating quadratic programming with bound and quadratic constraints

Yinyu Ye*
Department of Management Sciences
The University of Iowa
Iowa City, Iowa 52242, U.S.A.

April 21, 1997, Revised March 6, 1998

Abstract

We consider the problem of approximating the global maximum of a quadratic program (QP) subject to bound and (simple) quadratic constraints. Based on several early results, we show that a 4/7-approximate solution can be obtained in polynomial time.

Key words. Quadratic programming, global maximizer, approximation algorithm

*This author is supported in part by NSF grants DMI-9522507 and DMS-9703490.

1 Introduction

Consider the quadratic programming (QP) problem with bound and (simple) quadratic constraints

$$
\text{(QP)} \qquad \begin{array}{rll} \bar{q}(Q) := & \text{Maximize} & q(x) := x^T Q x \\ & \text{Subject to} & \sum_{j=1}^n a_{ij} x_j^2 = b_i, \;\; i = 1, \ldots, m, \\ & & -e \leq x \leq e, \end{array}
$$

where symmetric matrix $Q \in \Re^{n \times n}$, $A = \{a_{ij}\} \in \Re^{m \times n}$ and $b \in \Re^m$ are given and $e \in \Re^n$ is the vector of all ones. We assume that the QP problem is feasible (this can be checked by a linear program considering x_j^2 as nonnegative variables). Let $\bar{x}(Q)$ be a maximizer of the problem. The (QP) problem has applications in combinatorial and global optimization problems, see, e.g., Gibbons, Hearn and Pardalos [6].

Normally, there is a linear term in the objective function:

$$
\begin{array}{ll} \text{Maximize} & x^T Q x + c^T x \\ \text{Subject to} & \sum_{j=1}^n a_{ij} x_j^2 = b_i, \;\; i = 1, \ldots, m, \\ & -e \leq x \leq e, \end{array}
$$

However, the problem can be homogenized as

$$
\begin{array}{ll} \text{Maximize} & x^T Q x + t c^T x \\ \text{Subject to} & \sum_{j=1}^n a_{ij} x_j^2 = b_i, \;\; i = 1, \ldots, m, \;\; t^2 = 1, \\ & -e \leq x \leq e, \\ & -1 \leq t \leq 1 \end{array}
$$

by adding a scalar variable t. There always is an optimal solution $(\bar{x}, \bar{t})$ for this problem in which $\bar{t} = 1$ or $\bar{t} = -1$. If $\bar{t} = 1$, then $\bar{x}$ is also optimal for the non-homogeneous problem; if $\bar{t} = -1$, then $-\bar{x}$ is optimal for the non-homogeneous problem. Thus, without loss of generality, we can let $q(x) = x^T Q x$ throughout this paper.

The function $q(x)$ has a minimizer and a maximizer over the bounded feasible set

$$
\mathcal{F} := \{x \in \Re^n : \sum_{j=1}^n a_{ij} x_j^2 = b_i, \;\; i = 1, \ldots, m, \;\; -e \leq x \leq e\}.
$$

Let $\underline{q} := -\bar{q}(-Q)$ and $\bar{q} := \bar{q}(Q)$ denote their minimal and maximal objective values, respectively. An ϵ-maximal solution or ϵ-maximizer, $\epsilon \in [0, 1]$, for (QP) is defined as an $x \in \mathcal{F}$ such that

$$
\frac{\bar{q} - q(x)}{\bar{q} - \underline{q}} \leq \epsilon.
$$

Recently, there were several significant results on approximating specific quadratic problems. Goemans and Williamson [7] (also see Frieze and Jerrum [3]) proved an approximation result for the Maxcut problem where $\epsilon \leq 1-0.878$ when all arc weights are nonnegative. Nesterov [9] generalized their result to approximating a boolean QP problem

$$\begin{array}{ll} \text{Maximize} & q(x) = x^T Q x \\ \\ \text{Subject to} & |x_j| = 1, \ j = 1, \ldots, n, \end{array}$$

where $\epsilon \leq 4/7$. Ye [14] extended the 4/7 result to solving the continuous QP problem

$$\begin{array}{ll} \text{Maximize} & q(x) = x^T Q x \\ \\ \text{Subject to} & |x_j| \leq 1, \ j = 1, \ldots, n. \end{array}$$

Some negative results on this problem were given by Bellare and Rogaway [2]. Other results can be found in Fu, Luo and Ye [4], Pardalos and Rosen [11], Vavasis [13], and Ye [15].

In this short paper, we further generalize the 4/7 result to approximating (QP) containing (simple) quadratic constraints. These constraints have added a few difficulties in analyzing the problem, and they frequently appear in some practical applications.

2 Positive Semi-Definite Relaxation

The approximation algorithm for (QP) is to solve a positive semi-definite programming (SDP) relaxation problem

$$\text{(SDP)} \qquad \begin{array}{rll} \bar{p}(Q) := & \text{Maximize} & \langle Q, X \rangle \\ \\ & \text{Subject to} & \langle a_i, d(X) \rangle = b_i, \ i = 1, \ldots, m, \\ \\ & & d(X) \leq e, \ X \succeq 0. \end{array} \tag{1}$$

Here, $a_i = (a_{i1}, \ldots, a_{in}) \in \Re^n$ and unknown $X \in \Re^{n \times n}$ is a symmetric matrix. Furthermore, $\langle \cdot, \cdot \rangle$ is the matrix inner product $\langle Q, X \rangle = \text{trace}(Q^T X)$, $d(X)$ is a vector containing the diagonal components of X, and $X \succeq Z$ means that $X - Z$ is positive semi-definite. Since the original QP problem is feasible, so is the SDP relaxation.

The dual of the problem (see, e.g., [10]) is

$$\begin{array}{rll} \bar{p}(Q) = & \text{Minimize} & e^T z + b^T y \\ \\ & \text{Subject to} & D(z) + \sum_{i=1}^m y_i D(a_i) \succeq Q, \ z \geq 0, \end{array} \tag{2}$$

where $D(z)$ is the diagonal matrix such that $d(D(z)) = z \in \Re^n$. Note that the primal is feasible and the dual has an interior so that there is no duality gap between the primal and dual. Denote by $\bar{X}(Q)$ and $(\bar{y}(Q), \bar{z}(Q))$ an optimal solution pair for the primal (1) and dual (2).

The positive semi-definite relaxation was first proposed by Lovász and Shrijver [8], also see recent papers by Alizadeh [1], Fujie and Kojima [5] and Polijak, Rendl and Wolkowicz [12]. This relaxation problem can be solved in polynomial time, e.g., see Nesterov and Nemirovskii [10].

We have the following relations between (QP) and (SDP).

Proposition 1 *Let $\bar{q} = \bar{q}(Q)$, $\underline{q} = -\bar{q}(-Q)$, $\bar{p} = \bar{p}(Q)$, $\underline{p} = -\bar{p}(-Q)$, and $(\underline{y}, \underline{z}) = (-\bar{y}(-Q), -\bar{z}(-Q))$. Then, $\underline{q}$ is the minimal objective value of $x^T Qx$ in the feasible set of (QP) and $\underline{p} = e^T \underline{z} + b^T \underline{y}$ is the minimal objective value of $\langle Q, X\rangle$ in the feasible set of (SDP). Furthermore,*

$$\underline{p} = -\bar{p}(-Q) \leq \underline{q} = -\bar{q}(-Q) \leq \bar{q}(Q) = \bar{q} \leq \bar{p}(Q) = \bar{p}.$$

Proof. The first statement is straightforward to verify. Let $X = \bar{x}(Q)\bar{x}(Q)^T \in \Re^{n\times n}$. Then $X \succeq 0$, $d(X) \leq e$,

$$\langle a_i, d(X)\rangle = \sum_{j=1}^{n} a_{ij}(\bar{x}(Q)_j)^2 = b_i, \; i = 1, \ldots, m,$$

and $\langle Q, X\rangle = \bar{x}(Q)^T Q \bar{x}(Q) = q(\bar{x}(Q)) = \bar{q}(Q)$. Thus, we have $\bar{q}(Q) = \langle Q, X\rangle \leq \bar{p}(Q)$. Similarly, we can prove $\bar{q}(-Q) \leq \bar{p}(-Q)$, or $-\bar{p}(-Q) \leq -\bar{q}(-Q)$. ■

In what follows, we let $\bar{x} = \bar{x}(Q)$, $\bar{X} = \bar{X}(Q)$. Since $\bar{X}$ is positive semi-definite, there is a factorization matrix $\bar{V} = (\bar{v}_1, \ldots, \bar{v}_n) \in \Re^{n\times n}$, i.e., $\bar{v}_j$ is the jth column of $\bar{V}$, such that $\bar{X} = \bar{V}^T\bar{V}$. The algorithm (Goemans and Williamson [7], Nesterov [9], and Ye [14]) generates a random vector u uniformly distributed on the n-dimensional unit ball and then assigns

$$\hat{x} = \bar{D}\sigma(\bar{V}^T u), \tag{3}$$

where

$$\bar{D} = \mathrm{diag}(\|\bar{v}_1\|, \ldots, \|\bar{v}_n\|) = \mathrm{diag}(\sqrt{\bar{x}_{11}}, \ldots, \sqrt{\bar{x}_{nn}}),$$

and for any $x \in \Re^n$, $\sigma(x)$ is the vector whose components are $\mathrm{sign}(x_j)$, $j = 1, \ldots, n$, that is,

$$\mathrm{sign}(x_j) = \begin{cases} 1 & \text{if } x_j \geq 0 \\ -1 & \text{otherwise.} \end{cases}$$

It is easily see that $\hat{x}$ is a feasible point for (QP) and we will show later that the expected objective value, $\mathrm{E}_u q(\hat{x})$, satisfies

$$\frac{\bar{q} - \mathrm{E}_u q(\hat{x})}{\bar{q} - \underline{q}} \leq \frac{\pi}{2} - 1 \leq \frac{4}{7}.$$

3 Approximation Analysis

The following lemma is an analogue to the lemma of Nesterov [9] and Ye [14].

Lemma 1 *Let u be uniformly distributed on the n-dimensional unit ball. Then,*

$$\begin{aligned} \bar{q}(Q) = \quad & \textit{Maximize} \quad \mathrm{E}_u(\sigma(V^T u)^T DQD\sigma(V^T u)) \\ & \textit{Subject to} \quad \langle a_i, d(V^T V)\rangle = b_i, \; i = 1, \ldots, m, \\ & \qquad\qquad\quad \|v_j\| \le 1, \; j = 1, \ldots, n, \end{aligned}$$

where

$$D = diag(\|v_1\|, \ldots, \|v_n\|).$$

Proof. Since, for any feasible V, $D\sigma(V^T u)$ is a feasible point for (QP), we have

$$\bar{q}(Q) \ge \mathrm{E}_u(\sigma(V^T u)^T DQD\sigma(V^T u)).$$

On the other hand, for any fixed u with $\|u\| = 1$, we have

$$\mathrm{E}_u(\sigma(V^T u)^T DQD\sigma(V^T u)) = \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}\|v_i\|\|v_j\|\mathrm{E}_u(\sigma(v_i^T u)\sigma(v_j^T u)). \tag{4}$$

Let us choose $v_i = \frac{\bar{x}_i}{\|\bar{x}\|}\bar{x}$, $i = 1, \ldots, n$. (Note that V is feasible for the problem above.) Then

$$\mathrm{E}_u(\sigma(v_i^T u)\sigma(v_j^T u)) = \begin{cases} 1 & \text{if } \sigma(\bar{x}_i) = \sigma(\bar{x}_j) \\ -1 & \text{otherwise.} \end{cases}$$

Thus,

$$\|v_i\|\|v_j\|\mathrm{E}_u(\sigma(v_i^T u)\sigma(v_j^T u)) = \bar{x}_i\bar{x}_j$$

which implies that for this particular feasible V

$$\bar{q}(Q) = q(\bar{x}) \le \mathrm{E}_u(\sigma(V^T u)^T DQD\sigma(V^T u)).$$

These two relations give the desired result. ∎

For any function of one variable $f(t)$ and $X \in \Re^{n\times n}$, let $f[X] \in \Re^{n\times n}$ be the matrix with the components $f(x_{ij})$. Nesterov [9] has proved the next technical lemma.

Lemma 2 *Let $X \succeq 0$ and $d(X) \le 1$. Then* $\arcsin[X] \succeq X$. ∎

Now we are ready to prove the following theorem.

Theorem 1

$$\begin{aligned} \bar{q}(Q) = \quad & \textit{Supremum} \quad \tfrac{2}{\pi}\langle Q, D\arcsin[D^{-1}XD^{-1}]D\rangle \\ & \textit{Subject to} \quad \langle a_i, d(X)\rangle = b_i, \; i = 1, \ldots, m, \\ & \qquad\qquad\quad d(X) \le e, \; X \succ 0, \end{aligned}$$

where

$$D = diag(\sqrt{x_{11}}, \ldots, \sqrt{x_{nn}}).$$

Proof. For any $X = V^T V \succ 0$, $d(X) \le e$, we have

$$\mathrm{E}_u(\sigma(v_i^T u)\sigma(v_j^T u)) = 1 - 2\Pr\{\sigma(v_i^T u) \ne \sigma(v_j^T u)\} = 1 - 2\Pr\{\sigma(\frac{v_i^T u}{\|v_i\|}) \ne \sigma(\frac{v_j^T u}{\|v_j\|})\}.$$

From Lemma 1.2 of Goemans and Williamson [7], we have

$$\Pr\{\sigma(\frac{v_i^T u}{\|v_i\|}) \ne \sigma(\frac{v_j^T u}{\|v_j\|})\} = \frac{1}{\pi}\arccos(\frac{v_i^T v_j}{\|v_i\|\|v_j\|}).$$

Using the above lemma and equality (4) and noting $\arcsin(t)+\arccos(t) = \frac{\pi}{2}$ give the desired result. ■

We have used *Supremum* and $X \succ 0$ in the problem above merely for the technical presentation of D^{-1}. The feasible set of this problem can be closed if we rewrite it in terms of variable $Y = D^{-1}XD^{-1}$

Theorem 1 leads to our main result.

Theorem 2 *We have*

1.

$$\bar{q} - \underline{p} \ge \frac{2}{\pi}(\bar{p} - \underline{p}).$$

2.

$$\bar{p} - \underline{q} \ge \frac{2}{\pi}(\bar{p} - \underline{p}).$$

3.

$$\bar{p} - \underline{p} \ge \bar{q} - \underline{q} \ge \frac{4-\pi}{\pi}(\bar{p} - \underline{p}).$$

Proof. Recall $\underline{z} = -\bar{z}(-Q) \le 0$, $\underline{p} = -\bar{p}(-Q) = e^T\underline{z} + b^T\underline{y}$, and $Q - D(\underline{z}) - \sum_{i=1}^m \underline{y}_i D(a_i) \succeq 0$. Thus, for any $X \succ 0$, $d(X) \le e$ and $D = \mathrm{diag}(\sqrt{x_{11}}, \ldots, \sqrt{x_{nn}})$, we have from Theorem 1

$$\begin{aligned}
\frac{\pi}{2}\bar{q} &= \frac{\pi}{2}\bar{q}(Q) \\
&\ge \langle Q, D\arcsin[D^{-1}XD^{-1}]D\rangle \\
&= \langle Q - D(\underline{z}) - \sum_{i=1}^m \underline{y}_i D(a_i) + D(\underline{z}) + \sum_{i=1}^m \underline{y}_i D(a_i), D\arcsin[D^{-1}XD^{-1}]D\rangle \\
&= \langle Q - D(\underline{z}) - \sum_{i=1}^m \underline{y}_i D(a_i), D\arcsin[D^{-1}XD^{-1}]D\rangle + \langle D(\underline{z}) + \sum_{i=1}^m \underline{y}_i D(a_i), D\arcsin[D^{-1}XD^{-1}]D\rangle \\
&\ge \langle Q - D(\underline{z}) - \sum_{i=1}^m \underline{y}_i D(a_i), DD^{-1}XD^{-1}D\rangle + \langle D(\underline{z}) + \sum_{i=1}^m \underline{y}_i D(a_i), D\arcsin[D^{-1}XD^{-1}]D\rangle
\end{aligned}$$

$$
\begin{aligned}
& \text{(since } Q - D(\underline{z}) - \sum_{i=1}^m \underline{y}_i D(a_i) \succeq 0 \text{ and } \arcsin[D^{-1}XD^{-1}] \succeq D^{-1}XD^{-1}) \\
= \; & \langle Q - D(\underline{z}) - \sum_{i=1}^m \underline{y}_i D(a_i), X\rangle + \langle D(\underline{z}) + \sum_{i=1}^m \underline{y}_i D(a_i), D \arcsin[D^{-1}XD^{-1}]D\rangle \\
= \; & \langle Q, X\rangle - \langle D(\underline{z}) + \sum_{i=1}^m \underline{y}_i D(a_i), X\rangle + \langle D(\underline{z}) + \sum_{i=1}^m \underline{y}_i D(a_i), D \arcsin[D^{-1}XD^{-1}]D\rangle \\
= \; & \langle Q, X\rangle - \underline{z}^T d(X) - \sum_{i=1}^m \underline{y}_i a_i^T d(X) \\
& + \underline{z}^T d(D \arcsin[D^{-1}XD^{-1}]D) + \sum_{i=1}^m \underline{y}_i a_i^T d(D \arcsin[D^{-1}XD^{-1}]D) \\
= \; & \langle Q, X\rangle - \underline{z}^T d(X) - \underline{y}^T b + \underline{z}^T (\frac{\pi}{2} d(X)) + \underline{y}^T (\frac{\pi}{2} b) \\
& \text{(since } d(D \arcsin[D^{-1}XD^{-1}]D) = \frac{\pi}{2} d(X) \text{ and } a_i^T d(X) = b_i) \\
= \; & \langle Q, X\rangle + (\frac{\pi}{2} - 1)\underline{z}^T d(X) + (\frac{\pi}{2} - 1)\underline{y}^T b \\
\geq \; & \langle Q, X\rangle + (\frac{\pi}{2} - 1)(\underline{z}^T e + \underline{y}^T b)) \\
& \text{(since } 0 \leq d(X) \leq e \text{ and } \underline{z} \leq 0) \\
= \; & \langle Q, X\rangle + (\frac{\pi}{2} - 1)\underline{p}.
\end{aligned}
$$

Let X converge to $\bar{X}$, then $\langle Q, X\rangle \to \bar{p}$ and we have the desired first inequality.

Replacing Q with $-Q$ proves the second inequality in the theorem.

Adding the first two inequalities gives the third statement in the theorem. ■

The result indicates that the positive semi-definite relaxation value $\bar{p} - \underline{p}$ is a constant approximation of $\bar{q} - \underline{q}$.

Similarly, the following corollary can be devised.

Corollary 1 *Let $X = V^T V \succ 0$, $d(X) \leq e$, $\langle a_i, d(X)\rangle = b_i$ $(i = 1, \ldots, m)$, $D = diag(\sqrt{x_{11}}, \ldots, \sqrt{x_{nn}})$, and $\hat{x} = D\sigma(V^T u)$ where u with $\|u\| = 1$ is a random vector uniformly distributed on the unit ball. Moreover, let $X \to \bar{X}$. Then,*

$$
\lim_{X \to \bar{X}} \mathrm{E}_u(q(\hat{x})) = \lim_{X \to \bar{X}} \frac{2}{\pi} \langle Q, D \arcsin[D^{-1}XD^{-1}]D\rangle \geq \frac{2}{\pi}\bar{p} + (1 - \frac{2}{\pi})\underline{p}.
$$

Finally, we have

Theorem 3 *Let $\hat{x}$ be generated above from $X = \bar{X}$. Then*

$$
\frac{\bar{q} - \mathrm{E}_u q(\hat{x})}{\bar{q} - \underline{q}} \leq \frac{\pi}{2} - 1.
$$

Proof. The proof is similar to that in Nesterov [9] and Ye [14]. We include it here for completeness. Since

$$\bar{p} \geq \bar{q} \geq \frac{2}{\pi}\bar{p} + (1 - \frac{2}{\pi})\underline{p} \geq (1 - \frac{2}{\pi})\bar{p} + \frac{2}{\pi}\underline{p} \geq \underline{q} \geq \underline{p}$$

we have

$$\begin{aligned}
\frac{\bar{q} - \mathrm{E}_u q(\hat{x})}{\bar{q} - \underline{q}} &\leq \frac{\bar{q} - \frac{2}{\pi}\bar{p} - (1 - \frac{2}{\pi})\underline{p}}{\bar{q} - \underline{q}} \\
&\leq \frac{\bar{q} - \frac{2}{\pi}\bar{p} - (1 - \frac{2}{\pi})\underline{p}}{\bar{q} - (1 - \frac{2}{\pi})\bar{p} - \frac{2}{\pi}\underline{p}} \\
&\leq \frac{\bar{p} - \frac{2}{\pi}\bar{p} - (1 - \frac{2}{\pi})\underline{p}}{\bar{p} - (1 - \frac{2}{\pi})\bar{p} - \frac{2}{\pi}\underline{p}} \\
&= \frac{(1 - \frac{2}{\pi})(\bar{p} - \underline{p})}{\frac{2}{\pi}(\bar{p} - \underline{p})} \\
&= \frac{(1 - \frac{2}{\pi})}{\frac{2}{\pi}} = \frac{\pi}{2} - 1.
\end{aligned}$$

■

References

[1] F. Alizadeh, *Combinatorial Optimization with Interior Point Methods and Semi–Definite Matrices*, Ph.D. thesis, University of Minnesota, Minneapolis, MN, 1991.

[2] M. Bellare and P. Rogaway, The complexity of approximating a nonlinear program, *Mathematical Programming* 69 (1995) 429-442.

[3] A. Frieze and M. Jerrum, Improved approximation algorithms for max k-cut and max bisection, *Proc. 4th IPCO Conference* (1995) 1-13.

[4] M. Fu, Z.-Q. Luo and Y. Ye Approximation algorithms for quadratic programming , manuscript, Dept. of Electrical and Computer Engineering, McMaster University, Hamilton, Ontario, CANADA L8S 4K1, 1996. To appear in *Journal of Combinatorial Optimization.*

[5] T. Fujie and M. Kojima, Semidefinite programming relaxation for nonconvex quadratic programs, Research Report B-298, Dept. of Mathematical and Computing Sciences, Tokyo Institute of Technology, Meguro, Tokyo 152, May 1995. To appear in *Journal of Global Optimization.*

[6] L. E. Gibbons, D. W. Hearn and P. M. Pardalos, A continuous based heuristic for the maximum clique problem, *DIMACS Series in Discrete Mathematics and Theoretical Computer Science* 26 (1996) 103-124.

[7] M. X. Goemans and D. P. Williamson, Improved approximation algorithms for Maximum Cut and Satisfiability problems using semidefinite programming, *Journal of ACM* 42 (1995) 1115–1145.

[8] L. Lovász and A. Shrijver, Cones of matrices and setfunctions, and $0-1$ optimization, *SIAM Journal on Optimization* 1 (1990) 166-190.

[9] Yu. E. Nesterov, Quality of semidefinite relaxation for nonconvex quadratic optimization, CORE Discussion Paper #9719, Belgium, March 1997.

[10] Yu. E. Nesterov and A. S. Nemirovskii, *Interior Point Polynomial Methods in Convex Programming : Theory and Algorithms* (SIAM Publications, SIAM, Philadelphia, 1993).

[11] P. M. Pardalos and J. B. Rosen, *Constrained Global Optimization: Algorithms and Applications* (Springer-Verlag, Lecture Notes in Computer Sciences 268, 1987).

[12] S. Polijak, F. Rendl and H. Wolkowicz, A recipe for semidefinite relaxation for 0-1 quadratic programming, *Journal of Global Optimization* 7 (1995) 51-73.

[13] S. A. Vavasis, *Nonlinear Optimization: Complexity Issues* (Oxford Science, New York, 1991).

[14] Y. Ye, Approximating quadratic programming with bound constraints, Working Paper, Department of Management Science, The University of Iowa, Iowa City, IA 52242, 1997.

[15] Y. Ye, On affine scaling algorithms for nonconvex quadratic programming, *Mathematical Programming* 56 (1992) 285-300.