Constrained Logarithmic Least Squares in Parameter Estimation [1]

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Abstract

The contribution of this paper is twofolds. First, it is shown that while robust in terms of the average output error, the least squares estimate is sensitive to outliers with respect to the maximum output error. In fact the worst-case output error of the least squares can go unbounded. Then, a constrained logarithmic least squares for system identification is proposed. Analytic center algorithms are presented to solve this constrained logarithmic least squares problem.

1 Introduction

The least squares method has been extensively studied for identification: see for example [1], [2], [3], [5], [6] and [8]. By now, its divergence property in terms of the operator norm in the worst-case scenario is well known. In fact, it was shown in [2] that the estimated transfer function by the least squares either in H_∞ or l_1 norm diverges in the worst case scenario as the order of the system goes to infinite. This implies that by taking the estimated transfer function as a system, its output could be unbounded for *some bounded input.* It was not clear, however, whether the same bounded input sequence that produces the estimated transfer function would give rise to a unbounded output error. This paper reports that the very same input sequence that provides the estimated transfer function by the least squares method would give rise to a unbounded output error. The implication is that while the least squares

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estimate is robust to outliers with respect to the average output error, it can be very sensitive with respect to the maximum output error.

To overcome this problem, the constrained least squares is proposed in this paper along with its solution, analytic center. It is shown that the analytic center not only is easily computable allowing a sequential algorithm, but also has a clear geometrical interpretation of maximizing the product of the complementary errors (recall that the least squares maximizes the summation of the complementary errors).

Consider a single input-single output discrete-time system

$$y_i = q_i^T \theta + v_i, \quad i = 1, 2, \ldots, N \tag{1.1}$$

where $y_i \in R$ is the system output, $q_i \in R^n$ the measurable regressor, $\theta \in R^n$ the unknown parameter vector to be identified, and $v_i \in R$ the noise. In vector form, the system can be re-written more compactly as

$$Y = Q\theta + V \tag{1.2}$$

where

$$Y = (y_1, y_2, ..., y_N)^T, \;\; Q = (q_1, q_2, , ..., q_N)^T, V = (v_1, v_2, ..., v_N)^T.$$

The purpose of parametric system identification is to find an estimate $\hat{\theta}$ of the unknown parameter vector θ from available input-output measurements Y and Q. For any given estimate $\hat{\theta}$, the output error is defined as

$$e_i(\hat{\theta}) = \hat{y}_i - y_i = q_i^T \hat{\theta} - y_i, \quad i = 1, 2, ..., N.$$

The quality of the estimate $\hat{\theta}$ may be measured by the "size" of the corresponding output error using some norm in R^N. There are two types of norms used in literature. One is the minimax type

$$\hat{\theta} = \arg\min_{\hat{\theta}} \max_{1 \le i \le N} |e_i(\hat{\theta})|$$

which minimizes the maximum output error. In general, this type of norm is sensitive to outliers [1, 7]. Occasional large noises influence the estimate $\hat{\theta}$ considerably and therefore this type of norm is not statistically robust. The other norm commonly used is of average type

$$\hat{\theta} = \arg\min_{\hat{\theta}} \frac{1}{N} \sum_{i=1}^{N} \ell(e_i^2(\hat{\theta})) \tag{1.3}$$

where $\ell(\cdot)$ is a smooth non-decreasing function. This type of norm minimizes the average output error and is less sensitive to outliers in terms of the estimate $\hat{\theta}$ and is consequently statistically robust. The celebrated least squares estimate

$$\theta_{ls} = \arg\min_{\hat{\theta}} \frac{1}{N} \sum_{i=1}^{N} e_i^2 = (\frac{1}{N} Q^T Q)^{-1} \frac{1}{N} Q^T Y$$

belongs to this class by taking $\ell(e_i^2) = e_i^2$. It is important to point out, however, that while the average output error is minimized, the maximum output error $\max_{1 \le i \le N} |e_i|$ generated by the average type norm can be very sensitive to noises. A single bad measurement can result in a large maximum output error. Take the least squares estimate as an example. Let $N = 100$ and consider the system

$$y_i = q_i^T \theta + v_i = (u_{i-1}, u_{i-2}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + v_i, \tag{1.4}$$

with the input sequence

$$u_i = \begin{cases} 1 & i = odd \\ -0.9 & i = even \end{cases}.$$

Consider two noise sequences

$$v_i = 0.5, \quad i = 1, 2, ..., 100 \tag{1.5}$$

and

$$v_i = \begin{cases} 0.5 & i \neq 51 \\ -0.5 & i = 51 \end{cases} \tag{1.6}$$

respectively. Note that these two noise sequences only differ by a single point value at $i = 51$.

The top diagram of Figure 1 shows the output error $|e_i|$, $i = 1, 2, ..., 100$, generated by θ_{ls} with the noise sequence (1.5) and the bottom diagram shows the output error with the noise sequence (1.6). We see that a single value change of the noise sequence results in a considerable variation in the maximum output error. The sensitive nature of the least squares can be in fact proved rigorously by showing that the maximum output error $\max_{1 \le i \le N} |e_i|$ generated by the least squares estimate θ_{ls} is unbounded in the worst case scenario as the order of the system n goes to infinite, even if the noise sequence is bounded and the regressor q_i is persistently exciting. To be more specific, let (1.1) be a FIR system with $N = 2^p + n$ and $q_i = (u_{i-1}, u_{i-2}, ..., u_{i-n})^T$. Let s_k be a pseudo-random binary signal (PRBS), for instance the *Galois sequence* see [2] or [9], satisfying

- binary, assuming the values ± 1;
- approximately uncorrelated, i.e.,

$$\sum_{k=1}^{2^p-1} s_k s_{k+m} = \begin{cases} 2^p - 1 & m = 0 \bmod (2^p - 1) \\ -1 & \text{otherwise} \end{cases} \tag{1.7}$$

- contains all possible n-tuples of ± 1.

Figure 1:

Let the input u_i and noise v_i, respectively, be

$$u_i = \begin{cases} 1 & i = 1, 2, ..., n \\ s_{i-n} & i = n+1, ..., N \end{cases} \qquad v_i = sign(\sum_{j=0}^{n-1} u_{i-j}).$$

Then, we have

Theorem 1: Suppose (1.1) is a FIR system with $N = 2^p + n$, $p >> n \geq 11$. Consider the above input sequence u_i and noise sequence v_i respectively. Then, for large p, the maximum output error generated by the least squares estimate θ_{ls} is bounded from below by

$$\max_{1\le i\le N} |e_i(\theta_{ls})| = \max_{1\le i\le N} |q_i^T\theta_{ls} - y_i| \ge \sqrt{\frac{2n}{\pi}} e^{\frac{-1}{3n}} - O(\frac{n^2}{2^p}) - 1.$$

Clearly, for large p, $\max_{1\le i\le N} |e_i(\theta_{ls})| = O(\sqrt{n}) \to \infty$, as $n \to \infty$.

Proof: Since each element of Q^TQ is of the form $\sum s_k s_{k+m}$ for some m, $\frac{1}{N}Q^TQ \to diag(\frac{2^p-1}{2^p+n}, \frac{2^p-1}{2^p+n}, ..., \frac{2^p-1}{2^p+n})$ from the property (1.7). This implies

$$(\frac{1}{N}Q^TQ)^{-1} \to diag(\frac{2^p-1+n+1}{2^p-1}, ..., \frac{2^p-1+n+1}{2^p-1}) = I + O(\frac{n}{2^p})$$

for large p. Now,

$$e_i = q_i^T\theta_{ls} - y_i = q_i^T(\frac{1}{N}Q^TQ)^{-1}\frac{1}{N}Q^TY - q_i^T\theta - v_i = q_i^T(\frac{1}{N}Q^TQ)^{-1}\frac{1}{N}Q^TV - v_i.$$

This implies that

$$\begin{aligned}
\max_{1\le i\le N} |e_i| \ge |e_n| \quad &\ge |q_n^T(\tfrac{1}{N}Q^TQ)^{-1}\tfrac{1}{N}Q^TV| - 1 \\
&= |q_n^T\tfrac{1}{N}Q^TV + O(\tfrac{n}{2^p})q_n^T\tfrac{1}{N}Q^TV| - 1 \\
&\ge |\tfrac{1}{N}\textstyle\sum_{i=n}^N v_i \sum_{j=0}^{n-1} u_{i-j}| - O(\tfrac{n^2}{2^p}) - 1 \\
&\ge \sqrt{\tfrac{2n}{\pi}} e^{\frac{-1}{3n}} - O(\tfrac{n^2}{2^p}) - 1.
\end{aligned}$$

The last inequality comes from the assumptions that $N = 2^p+n$, $v_i = sign(\sum_{j=0}^{n-1} u_{i-j})$ and the fact [2]

$$\frac{1}{N}\sum_{i=n}^{N}|\sum_{j=0}^{n-1} u_{i-j}| \ge \sqrt{\frac{2n}{\pi}} e^{\frac{-1}{3n}} - (p-n)n2^{-p}.$$

This completes the proof.

2 Constrained average norm

The maximum output error of the average type norm (1.3) is sensitive to outliers. If both the average and the maximum output errors are of concern, (1.3) can be modified to add a constraint on the maximum allowable output error ϵ, i.e.,

$$\hat{\theta} = \arg\min_{\hat{\theta}} \frac{1}{N}\sum_{i=1}^{N} \ell(e_i^2), \tag{2.1}$$

$$\text{s.t.} \max_{1\le i\le N} |e_i| = \max_{1\le i\le N} |q_i^T\hat{\theta} - y_i| \le \epsilon.$$

In a sense, the objective is to minimize the average output error, but at the same time to limit the maximum sensitivity on the worst case output error. Notice that the performance in terms of the average output error $min\frac{1}{N}\sum_{i=1}^{N}\ell(e_i^2)$ depends on the maximum allowable error ϵ. For large ϵ, the average output error is reduced but the maximum error increases. Notice that ϵ can not be made arbitrarily small. In fact, if ϵ is too small the solution of the minimization problem (2.1) may not exist. A sufficient condition on ϵ to have a solution is that $\epsilon \ge \max_i |v_i|$ because in this case the parameter set under the constraints $|e_i| \le \epsilon$, $i = 1, ..., N$, is not empty containing at least one parameter, the true but unknown θ.

It is interesting to note that if $\epsilon \ge \max_i |v_i|$, the constrained average norm problem (2.1) can be re-formulated. Take the least squares as an example again. Define

$$\epsilon^2 - e_i^2 = \epsilon^2 - (q_i^T\theta_{ls} - y_i)^2$$

as the complementary output error. Because of the constraint in (2.1), $|e_i| \le \epsilon$ implies $\epsilon^2 - e_i^2 \ge 0$. In other words, the output error $(y_i - q_i^T\hat{\theta})^2$ is minimized within the set $|e_i| \le \epsilon$ if and only if the complementary error $\epsilon^2 - (y_i - q_i^T\hat{\theta})^2$ is maximized within the set $|e_i| \le \epsilon$, i.e.,

$$\arg\min_{\hat{\theta}, |e_i(\hat{\theta})|\le\epsilon} \frac{1}{N}\sum_{i=1}^{N}(y_i - q_i^T\hat{\theta})^2 = \arg\max_{\hat{\theta}, |e_i(\hat{\theta})|\le\epsilon} \frac{1}{N}\sum_{i=1}^{N}(\epsilon^2 - (y_i - q_i^T\hat{\theta})^2).$$

Therefore, for a given maximum allowable error ϵ, the constrained average norm problem of (2.1) may be re-formulated as

$$\hat{\theta} = \arg\max_{\hat{\theta}, |e_i(\hat{\theta})|\le\epsilon} \frac{1}{N}\sum_{i=1}^{N} \ell(\epsilon^2 - e_i^2). \tag{2.2}$$

Notice that the formulation of the constraint average norm problem (2.1) or (2.2) is reminiscent to the formulation of the membership set identification [10]. The philosophy is, however, completely different. In the membership set identification, the goal is to determine the membership set that requires a tight upper bound ϵ on the noise, as tight as possible. In the constrained average norm case here, the choice of ϵ depends on the allowable maximum output error. In fact, the larger ϵ results in the smaller average output error.

3 Analytic center

Notice again $\epsilon \geq |e_i| = |q_i^T \hat{\theta} - y_i|$, $i = 1, 2, ..., N$ if and only if

$$q_i^T \hat{\theta} \leq \epsilon + y_i, \quad - q_i^T \hat{\theta} \leq \epsilon - y_i, \quad i = 1, 2, ..., N.$$

Define

$$A = (a_1, a_2, ..., a_{2N}) := (q_1, -q_1, q_2, -q_2, ..., q_N, -q_N) \in R^{n \times 2N},$$

$$c = (c_1, c_2, ..., c_{2n-1}, c_{2N})^T := (\epsilon + y_1, \epsilon - y_1, ..., \epsilon + y_N, \epsilon - y_N)^T \in R^{2N}$$

and

$$\Omega = \{\hat{\theta} \in R^n \ : \ A^T \hat{\theta} \leq c\} \tag{3.1}$$

where the vector inequality applies element-wise. Then, the constrained average norm problem (2.2) becomes

$$\hat{\theta} = \arg \max_{\hat{\theta} \in \Omega} \frac{1}{N} \sum_{i=1}^{N} \ell(\epsilon^2 - e_i^2). \tag{3.2}$$

Depending on the function $\ell(\cdot)$, algorithms and their corresponding complexities for finding the optimal $\hat{\theta}$ are different. In this paper, we propose to use the logarithmic function $\ell(\epsilon^2 - e_i^2) = \ln(\epsilon^2 - e_i^2)$, called the logarithmic least squares, which allows an easy-to-compute sequential algorithm with low computational complexity. To the best of our knowledge, there does not exist any sequential algorithm for calculating the optimal $\hat{\theta}$ of (3.2) besides the one reported in this paper when $\ell(\cdot) = \ln(\cdot)$.

Before presenting the algorithms, we need to introduce the concept of the analytic center [4]: Let $\phi_d(\theta)$ denote the potential function

$$\phi_d(\theta) := \sum_{i=1}^{2N} \ln(c_i - a_i^T \theta). \tag{3.3}$$

The analytic center θ_a of Ω is an interior point of Ω maximizing the potential function $\phi_d(\theta)$

$$\theta_a = \arg\max_{\theta\in\Omega} \phi_d(\theta) = \arg\max_{\theta\in\Omega} \sum_{i=1}^{2N} \ln(c_i - a_i^T\theta).$$

It is easily verified that

$$\theta_a = \arg\max_{\hat\theta\in\Omega} \sum_{i=1}^{2N} \ln(c_i - a_i^T\hat\theta) = \arg\max_{\hat\theta\in\Omega} \sum_{i=1}^{N} \ln[(c_{2i-1} - a_{2i-1}^T\hat\theta)(c_{2i} - a_{2i}^T\hat\theta)]$$

$$= \arg\max_{\hat\theta\in\Omega} \sum_{i=1}^{N} \ln(\epsilon + y_i - q_i^T\hat\theta)(\epsilon - y_i + q_i^T\hat\theta)$$

$$= \arg\max_{\hat\theta\in\Omega} \sum_{i=1}^{N} \ln[\epsilon^2 - (y_i - q_i^T\hat\theta)^2] = \arg\max_{\hat\theta\in\Omega} \Pi_{i=1}^{N}[\epsilon^2 - (y_i - q_i^T\hat\theta)^2],$$

i.e., θ_a is the solution of constrained average norm problem (3.2) when $\ell(\cdot) = \ln(\cdot)$.

In the rest of the paper, we will present two algorithms, one non-sequential and one sequential, for calculating a δ-approximate analytic center θ_δ. θ_δ is called the δ-approximate center of Ω if it is in Ω and δ-close to the analytic center θ_a in the sense that

$$0 \le \phi_d(\theta_a) - \phi_d(\theta_\delta) \le \frac{\delta^2}{2(1-\delta)}, \ \delta > 0.$$

Non-sequential algorithm for computing θ_δ:

Consider the system (1.1). Suppose the data set Q, Y, the maximum allowable output error $\epsilon > 0$ and an arbitrarily small tolerance $\delta > 0$ are given. The goal is to compute a δ-approximate center θ_δ.

Initial: Set $j = 0$ and $x(0) = \frac{1}{\epsilon}w$ where $w = (1, ..., 1)^T \in R^{2N}$.

Phase I:

Step 1: For any j, define $X(j) = \text{diag}(x(j))$,

$$\theta(j) = (AX(j)^2A^T)^{-1}AX(j)^2(c - X(j)^{-1}w),$$

$$s(j) = c - A^T\theta(j),$$

$$S(j) = \text{diag}(s(j)),$$

$$p(j) = X(j)(c - A^T\theta(j)) - w = X(j)s(j)w$$

If $\|p(j)\| \leq \delta \leq 1/2$, set $\theta_\delta = \theta(j), x_\delta = x(j), s_\delta = s(j)$ and stop; else if $\delta < \|p(j)\| \leq 1/2$, then $\theta(j)$ is a 1/2-approximate center and go to Step 3 in Phase II; otherwise $\|p(j)\| > 1/2$ and go to Step 2.

Step 2: Define

$$\hat{x}(j+1) = x(j) - \frac{1}{4}X(j)p(j)/\|p(j)\|,$$
$$x(j+1) = 2N\hat{x}(j+1)/(c^T\hat{x}(j+1)).$$

Set $j = j+1$ and go to Step 1.

Phase II: After Phase I, we have a 1/2-approximate center pair.

Step 3: Solve for $d_x(j), d_\theta(j)$ and $d_s(j)$ from the system of linear equations (Newton's method)

$$S(j)d_x(j) + X(j)d_s(j) = w - X(j)s(j),$$
$$Ad_x(j) = 0,$$
$$A^T d_\theta(j) + d_s(j) = 0$$

and define

$$x(j+1) = x(j) + d_x(j),\ \ \theta(j+1) = \theta(j) + d_\theta(j),\ \ s(j+1) = s(j) + d_s(j).$$

Go to Step 4.

Step 4: If $\|X(j+1)(c - A^T\theta(j+1)) - w\| \leq \delta$, set $\theta_\delta = \theta(j+1), x_\delta = x(j+1), s_\delta = s(j+1)$ and stop; otherwise, set $j = j+1$ and go to Step 3.

The non-sequential algorithm has two phases. The purpose of Phase I is to find a 1/2-approximate center from the initial $x(0)$ and that of Phase II is to find a δ-approximate center from a 1/2-approximate center. This algorithm can be carried out only if all the data from $i = 1$ to $i = N$ are available. Due to the nature of identification, the input-output data y_i, q_i are observed sequentially in time i and in many cases it is required to calculate a sequence of δ approximate centers θ_δ^n, θ_δ^{n+1},...,θ_δ^N, preferable in a sequential way, i.e., the calculation of θ_δ^i is based on θ_δ^{i-1} and the new data y_i and q_i. To this end, with a slight abuse of notation, we use the upper i to emphasize the dependence on time i. At each i,

$$\Omega^i = \{\hat{\theta} \in R^n \ : \ (A^i)^T\hat{\theta} \leq c^i\} \tag{3.4}$$

where

$$A^i = (a_1, a_2, ..., a_{2i}) := (q_1, -q_1, q_2, -q_2, ..., q_i, -q_i) \in R^{n\times 2i},$$

$$\text{and} \;\; c^i = (c_1, c_2, ..., c_{2i-1}, c_{2i})^T := (\epsilon + y_1, \epsilon - y_1, ..., \epsilon + y_i, \epsilon - y_i)^T \in R^{2i}.$$

Then, the sequential algorithm can be described as follows:

Sequential Algorithm:

Consider the system (1.1), the maximum allowable output error $\epsilon > 0$ and the tolerance $\delta > 0$. Suppose the data set is given sequentially, i.e., at time i, only $y_k, q_k, k = 1, 2, ..., i$ are available. The goal is to compute a sequence of δ-approximate center θ^i_δ, $i = n, n+1, ..., N$ sequentially.

Step 1: Start at $i = n$. Let

$$x^n(0) = (1/\epsilon, ..., 1/\epsilon)^T \in R^{2n}.$$

Apply the non-sequential algorithm with $x(0)$ replaced by $x^n(0)$ to calculate a 1/2-approximate center and a δ-approximate center θ^n_δ for Ω^n. Let T^n denotes the number of Newton iterations to find the 1/2-approximate center $\theta^n(T^n) = \theta^n_{\delta=1/2}$. Go to Step 2.

Step 2: When new data y_{i+1}, q_{i+1} become available for $n \le i < N$, let

$$x^{i+1}(0) = (x^i(T^i), 1/\epsilon, ..., 1/\epsilon)^T \in R^{2(i+1)}.$$

Apply the non-sequential algorithm, with $x(0)$ replaced by $x^{i+1}(0)$, to calculate a 1/2-approximate center and a δ-approximate center θ^{i+1}_δ for Ω^{i+1}. Let T^{i+1} denotes the number of Newton iterations to find the 1/2-approximate center $\theta^{i+1}(T^{i+1}) = \theta^{i+1}_{\delta=1/2}$ at $(i+1)th$ iteration. If $i = n - 1$, stop. Otherwise, set $i = i + 1$ and repeat Step 2.

It should be pointed out that both the non-sequential and the sequential algorithms were developed in our previous work [4] for a different purpose, the membership set identification. However, these two algorithms can be applied equally well here to solve the constrained logarithmic least squares problem (3.2) and all the convergence results remain the same as shown in the following theorem. For detailed proofs, see [4].

Theorem 2: Assume that the rank of the matrix A^i is n for $i \ge n$ and for given maximum allowable output error $\epsilon > 0$, the set Ω has a non-empty interior, i.e., there exists some $\rho > 0$ and some $\hat{\theta} \in \Omega$ so that

$$-\epsilon + \rho \leq y_i - q_i^T \hat{\theta} \leq \epsilon - \rho, \quad i = 1, 2, ..., N. \tag{3.5}$$

Then,

1. The non-sequential algorithm terminates with a δ approximate center θ_δ in a finite number of Newton iterations and this number is bounded by $O(N \cdot \ln \frac{\epsilon}{\rho} + \ln \ln \frac{2}{\delta})$.

2. The sequential algorithm terminates, with a sequence of δ-approximate centers θ_δ^n, θ_δ^{n+1},...,θ_δ^N of $\Omega^n, \Omega^{n+1}, ..., \Omega^N$, in a finite number of Newton iterations and this number is bounded by $O(N \cdot (\ln \frac{\epsilon}{\rho} + \ln \ln \frac{2}{\delta}))$.

We end this paper by giving simulation results of the analytic center. Consider the example (1.4) and let $\epsilon = 0.6$. The top diagram of Figure 2 shows the output error generated by the approximate analytic center θ_δ ($\delta = 0.1$) with the noise sequence (1.5) and the bottom diagram shows the output error with the noise sequence (1.6). We see from the figure that the maximum output error is less sensitive to noise compared to the least squares estimate shown in Figure 1. In fact, no large output error (> 0.6) exists.

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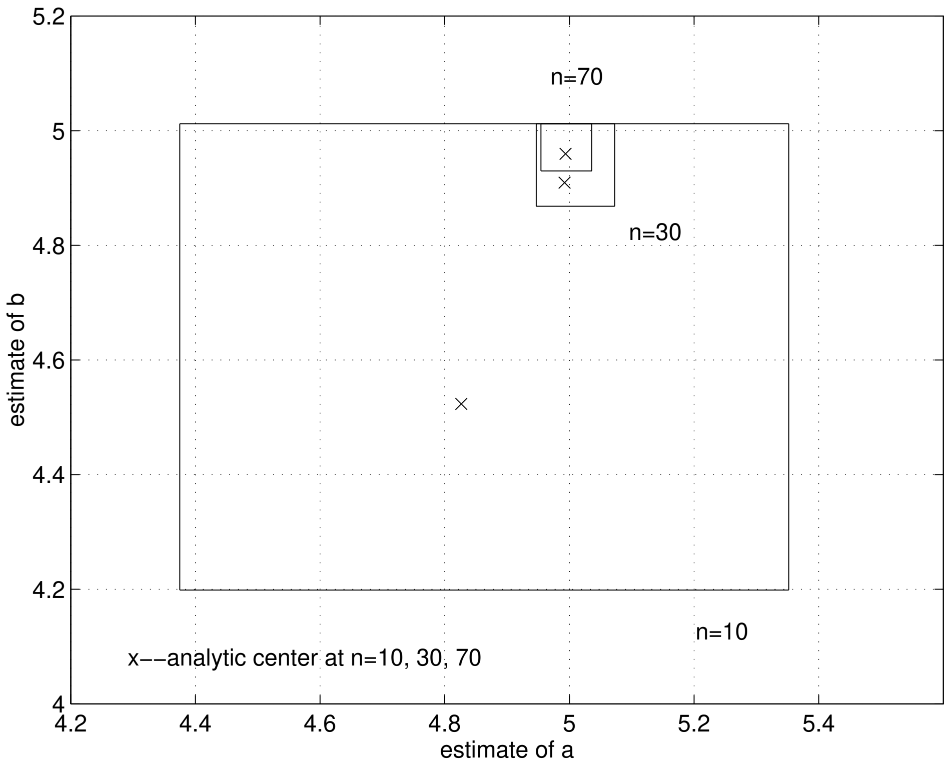

Figure 2:

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