# Improved Analyses of Facility Location Algorithms *

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This short note gives an improved approximation result for the (uncapacitated) metric facility location problem (UFLP), where we are given a set of cities $D$ and a set of facilities $F$. The *facility cost* of opening facility $i \in F$ is denoted by $f_i \geq 0$, and the *connection cost* of connecting city $j \in D$ to facility $i$ is $c_{ij} \geq 0$. Here the connection costs $c_{ij}$ are assumed to be symmetric and satisfy the triangle inequality. The objective is to find a set of facilities to open and then to assign every city to one of the opened facilities, so as to minimize the sum of facility costs and connection costs. We call $(\{c_{ij}\}, \{f_i\})$ the input of UFLP. Combining a number of previous results, we show that UFLP can be approximated by a guaranteed factor of 1.57, compared with the previously best result 1.61 by Jain et al. [3].

Let SOL be any (fractional) feasible solution of UFLP with facility cost $F_{SOL}$ and connection cost $C_{SOL}$. Then, Jain et al. proved the following.

**Lemma 1** *(Theorem 13 of [3]) There is an algorithm that finds an (integer) solution $z$ of UFLP with facility cost $F_z$ and connection cost $C_z$ such that*

$$F_z + C_z \leq F_{SOL} + 2C_{SOL}.$$

Charikar, Guha and Khuller proved:

**Lemma 2** *([1, 2]) Given a solution $z$ of UFLP with cost $F_z$ and $C_z$, a greedy augmentation finds a new (integer) solution of UFLP with a total cost at most*

$$F_z + F_{SOL} \max\{0, \ln(\frac{C_z - C_{SOL}}{F_{SOL}})\} + F_{SOL} + C_{SOL}.$$

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Let $\delta = .8815$. Then using Lemma 1 we find an (integer) solution, $SOL_1$, of UFLP with input $(\{c_{ij}\}, \{2\delta f_i\})$, such that its cost $F'_{SOL_1}$ and $C'_{SOL_1}$ satisfy

$$F'_{SOL_1} + C'_{SOL_1} \leq 2\delta F_{SOL} + 2C_{SOL}.$$

(Note that the facility-cost coefficients are all scaled by $2\delta$ in this modified problem. This technique is due to Charikar and Guha [1].) This is because that there must be a (fractional) feasible solution for the scaled UFLP with connection cost $C_{SOL}$ and facility cost $2\delta F_{SOL}$.

Consider $SOL_1$ for the original UFLP with input $(\{c_{ij}\}, \{f_i\})$. Its connection cost $C_{SOL_1} = C'_{SOL_1}$ and facility cost $F_{SOL_1} = F'_{SOL_1}/(2\delta)$. Thus, we have

$$2\delta F_{SOL_1} + C_{SOL_1} \leq 2\delta F_{SOL} + 2C_{SOL}. \tag{1}$$

Next, given $SOL_1$ as a solution to the original UFLP, using Lemma 2 we can find an (integer) solution $SOL_2$ such that its total cost is no more than

$$F_{SOL_1} + F_{SOL} \max\{0, \ln(\frac{C_{SOL_1} - C_{SOL}}{F_{SOL}})\} + F_{SOL} + C_{SOL}.$$

Then, we choose the better solution between $SOL_1$ and $SOL_2$ as the final solution for the original UFLP. We now ready to see:

**Theorem 1** *UFLP can be approximated within factor 1.57.*

**Proof.** Consider the case $C_{SOL_1} \leq F_{SOL} + C_{SOL}$. Then by (1)

$$\begin{aligned} F_{SOL_1} + C_{SOL_1} &= \frac{2\delta F_{SOL_1} + C_{SOL_1}}{2\delta} + (1 - \frac{1}{2\delta})C_{SOL_1} \\ &\leq \frac{2\delta F_{SOL} + 2C_{SOL}}{2\delta} + (1 - \frac{1}{2\delta})C_{SOL_1} \\ &= (2 - \frac{1}{2\delta})F_{SOL} + (1 + \frac{1}{2\delta})C_{SOL} < 1.5673(F_{SOL} + C_{SOL}). \end{aligned}$$

Otherwise, $C_{SOL_1} > F_{SOL} + C_{SOL}$, and $\ln(\frac{C_{SOL_1} - C_{SOL}}{F_{SOL}}) > 0$. Then we have again by (1)

$$C_{SOL_1} \leq 2(\delta F_{SOL} + C_{SOL}) - 2\delta F_{SOL_1}.$$

By Lemma 2, the cost of $SOL_2$ is at most

$$\begin{aligned} & F_{SOL_1} + F_{SOL} \ln(\frac{C_{SOL_1} - C_{SOL}}{F_{SOL}}) + F_{SOL} + C_{SOL} \\ \leq\ & F_{SOL_1} + F_{SOL} \ln(\frac{2(\delta F_{SOL} + C_{SOL}) - 2\delta F_{SOL_1} - C_{SOL}}{F_{SOL}}) + F_{SOL} + C_{SOL}. \end{aligned}$$

Note that the above term is maximized at $F_{SOL_1} = \frac{1}{2\delta} C_{SOL}$, so it is bounded by

$$(1 + \ln(2\delta))F_{SOL} + (1 + \frac{1}{2\delta})C_{SOL} \leq 1.5673(F_{SOL} + C_{SOL}).$$

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## References

[1] M. Charikar and S. Guha, "Improved combinatorial algorithms for facility location and $k$-median problems," in *Proceedings of IEEE Foundations of Computer Science*, 1999.

[2] S. Guha and S. Khuller, "Greedy strikes back: improved facility location algorithms," in *Proceeding of the 9th Annual ACM-SIAM Symposium on Discrete Algorithms*, 649-657, 1998.

[3] K. Jain, M. Mahdian and A. Saberi, "A new greedy approach for facility location problems," *Manuscript*, 2001.