On Homogeneous and Self-Dual Algorithms for LCP *

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Abstract: We present some generalizations of a homogeneous and self-dual linear programming (LP) algorithm to solving the monotone linear complementarity problem (LCP). Again, while it achieves the best known interior-point iteration complexity, the algorithm does not need to use any "big-M" number, and it detects LCP infeasibility by generating a certificate. To our knowledge, this is the first interior-point and infeasible-starting algorithm for the LCP with these desired features.

Key words: Linear complementarity problem, homogeneous and self-dual model, infeasible-starting algorithm.

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1 Introduction

Consider the linear complementarity problem (LCP) in the standard form:

$$
\begin{array}{lll}
LCP(M,q) & \text{minimize} & x^T s \\
 & \text{subject to} & s = Mx + q, \ (x,s) \geq 0,
\end{array}
$$

where data $M \in R^{n\times n}$ and $q \in R^n$ are given, $x, s \in R^n$, and ^{T} denotes transpose. $LCP(M,q)$ is said to be feasible if and only if its constraints are consistent; it has a feasible interior if there is a feasible point $(x,s) > 0$; it has an "optimal" or "complementary" solution if there is a feasible (x,s) such that $x^T s = 0$. Note that the objective function $x^T s$ is always nonnegative for any feasible (x,s).

Denote the feasible set of $LCP(M,q)$ by $\mathcal{F}$. If $LCP(M,q)$ is feasible but not necessarily has a feasible interior, then a point $(x,s) \in \mathcal{F}$ is maximal if and only if the number of positive components in (x,s) is maximal (see Güler and Ye [4]). Note that the indices of those positive components are invariant among all maximal feasible points, and an feasible interior point is a maximal feasible point where all components are positive.

In this paper, we consider $LCP(M,q)$ where M is a monotone matrix, i.e., $M + M^T$ is positive semi-definite. In this case, that $LCP(M,q)$ is feasible implies that it has a complementarity solution. Moreover, for any two complementarity solutions (x^1, s^1) and (x^2, s^2),

$$q^T x^1 = q^T x^2 \quad \text{and} \quad (M + M^T)x^1 = (M + M^T)x^2.$$

These results were first established by Adler and Gale [1], see an excellent book by Cottle, Pang, and Stone [2]. Denote the solution set of $LCP(M,q)$ by $\mathcal{S}$. Then, a solution (x,s) is maximal if and only if the number of positive components in (x,s) is maximal. Also note that the indices of those positive components are invariant among all maximal solutions for $LCP(M,q)$ (again see [4]). Furthermore, let us call $q^T x$, $(x,s) \in \mathcal{S}$, the q-value of $LCP(M,q)$, denoted by $q(M,q)$, which is a fixed nonpositive number for all solutions.

Recently Ye, Todd, and Mizuno [12] developed a homogeneous and self-dual linear programming algorithm based on the construction of a homogeneous and self-dual LP model, in which the dimension of the problem is increased by 2 (see Xu, Hung and Ye [13] for a simplification of the model). It has been an open question whether the model can be applied to solving the monotone LCP problem. In this paper, we give a positive answer. To our knowledge, our result is the first LCP algorithm possessing the following desired features:

- It achieves $O(\sqrt{n}L)$-iteration complexity, the best known theoretical result up to date (see Kojima, Mizuno, and Yoshise [3]), where L is the binary data length of rational (M,q).
- It solves the problem without any regularity assumption concerning the existence of optimal, feasible, or feasible interior points.

- It can start at a positive point, feasible or infeasible, near the central ray of the positive orthant (cone), and it does not need to use any Big-M penalty parameter or lower bound.
- If the LCP has a solution, the algorithm generates a sequence that approaches feasibility and optimality (nor necessarily simultaneously); if the problem is infeasible, the algorithm correctly detects infeasibility during the iterative process.

For other interior-point infeasible starting algorithms, see Mizuno, Jarre and Stoer [7] and Potra [9], and references therein.

2 A homogeneous and self-dual LCP model

Let $x^0 = e$, the vector of all ones, and $s^0 = \sigma e$ such that

$$\hat{e} := s^0 - (M + M^T)x^0 \geq e \quad \text{with} \quad \sigma = \max\{0, \max([M + M^T]e)\} + 1 \geq 1. \tag{1}$$

Furthermore, let $\tau^0 = 1$ and $\kappa^0 = \sigma$. Then, consider an artificial "homogeneous and self-dual" LCP model related to $LCP(M, q)$:

$$(HLCP) \quad \text{minimize} \quad x^T s + \tau\kappa$$

$$\text{subject to} \quad \begin{pmatrix} s \\ \kappa \\ 0 \end{pmatrix} = \begin{pmatrix} M & q & \bar{r} \\ -q^T & 0 & \bar{z} \\ -\bar{r}^T & -\bar{z} & 0 \end{pmatrix} \begin{pmatrix} x \\ \tau \\ \theta \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \bar{n} \end{pmatrix},$$

$$(x, \tau, s, \kappa) \geq 0, \ \theta \text{ free},$$

where

$$\bar{r} = s^0 - Mx^0 - q\tau^0, \quad \bar{z} = q^T x^0 + \kappa^0,$$

and

$$\bar{n} = (x^0)^T s^0 + \tau^0\kappa^0 - (x^0)^T M x^0 \leq (n+1)\sigma.$$

For (HLCP), denote the M-matrix by M_h, the q-vector by q_h, and the set of all feasible $(x, \tau, s, \kappa, \theta)$ by $\mathcal{F}_h$. Note that for the choice of x^0 and s^0 given in (1),

$$\bar{n} = (x^0)^T(s^0 - (M + M^T)x^0) + \tau^0\kappa^0 + (x^0)^T M x^0 \geq e^T\hat{e} + \sigma \geq n + \sigma \geq n + 1.$$

Similar to the homogeneous and self-dual LP model, the last equality in (HLCP) can be rewritten as

$$(s^0 - (M + M^T)x^0)^T x + (x^0)^T s + \kappa^0\tau + \tau^0\kappa - \bar{n}\theta = \bar{n}.$$

or

$$\hat{e}^T x + e^T s + \sigma\tau + \kappa - \bar{n}\theta = \bar{n}. \tag{2}$$

Thus, at $\theta = 0$ this equality represents a simplex constraint:

$$\hat{e}^T x + e^T s + \sigma\tau + \kappa = \bar{n}.$$

We have the following result regarding (HLCP):

Theorem 1 .

i. *(HLCP) is monotone, that is*

$$\begin{pmatrix} d_s \\ d_\kappa \\ 0 \end{pmatrix} = \begin{pmatrix} M & q & \bar{r} \\ -q^T & 0 & \bar{z} \\ -\bar{r}^T & -\bar{z} & 0 \end{pmatrix} \begin{pmatrix} d_x \\ d_\tau \\ d_\theta \end{pmatrix} \tag{3}$$

implies that

$$d_x^T d_s + d_\tau d_\kappa = d_x^T M d_x \geq 0.$$

Moreover, if M is skew-symmetric (self-dual), then M_h of (HLCP) is also skew-symmetric.

ii. *(HLCP) is feasible and it has a feasible interior point*

$$x^0 = e,\ \tau^0 = 1,\ s^0 = \sigma e,\ \kappa^0 = \sigma,\ \theta^0 = 1,$$

with

$$X^0 s^0 = \sigma e \quad \text{and} \quad \tau^0 \kappa^0 = \sigma,$$

where X^0 is the diagonal matrix of x^0.

iii. *(HLCP) has a complementarity solution $(x^*, \tau^*, s^*, \kappa^*, \theta^*)$ such that*

$$(x^*)^T s^* + \tau^* \kappa^* = (x^*)^T M x^* + \bar{n}\theta^* = 0.$$

Denote its solution set by $\mathcal{S}_h$.

iv. *There is a continuous path, starting from $(x^0, \tau^0, s^0, \kappa^0, 1)$, toward a maximal solution of $\mathcal{S}_h$:*

$$\mathcal{C}(\mu) := \left\{ (x, \tau, s, \kappa, \theta) \in \mathcal{F}_h : \begin{pmatrix} Xz \\ \tau\kappa \end{pmatrix} = \mu e, \quad \mu = \frac{x^T s + \tau\kappa}{n+1}, \quad 0 < \mu \leq \sigma \right\}.$$

v. *The q-value of (HLCP) is*

$$q(M_h, q_h) = \bar{n}\theta^*,$$

and that $LCP(M, q)$ is feasible and its q-value $q(M, q) = 0$ implies $q(M_h, q_h) = 0$; that $LCP(M, q)$ is infeasible also implies $q(M_h, q_h) = 0$.

Proof. Statements (i), (ii) and (iii) are true by the construction of the model. Statement (iv) is true due to the standard interior-point central-path theory. To prove (v), we note that if (x, s) is a solution of $LCP(M, q)$ with $q^T x = 0$, then $(\tau x, \tau, \tau s, 0, 0)$ is a solution for (HLCP) for some $\tau > 0$ (τ is determined by meeting the simplex constraint (2)). Thus, we have $q(M_h, q_h) = 0$.

If $LCP(M, q)$ is infeasible, then there is a certificate x satisfying conditions

$$-M^T x \geq 0, \quad x \geq 0, \quad \text{and} \quad -q^T x = 1. \tag{4}$$

The first two inequalities imply that

$$x^T(-M^T x) \geq 0 \quad \text{or} \quad x^T M^T x = x^T M x \leq 0.$$

Since M is monotone, we must have

$$x^T M^T x = x^T M x = x^T (M + M^T) x = 0,$$

which further implies that

$$(M + M^T)x = 0 \quad \text{or} \quad Mx = -M^T x.$$

Thus, we must have an (x, s)

$$s := Mx \geq 0, \quad s^T x = 0, \quad -q^T x = 1, \quad \text{and} \quad x \geq 0.$$

Thus, $(\kappa x, 0, \kappa s, \kappa, 0)$ is a solution for (HLCP) for some $\kappa > 0$ (κ is determined by meeting the simplex constraint (2)). Thus, we also have $q(M_h, q_h) = 0$.

□

3 The interior-point algorithm

Now, we can apply an *existing* $O(\sqrt{n}L)$-iteration algorithm to solving (HLCP), which generates iterates $(x, \tau, s, \kappa, \theta)$ within a neighborhood of $\mathcal{C}(\mu)$ (e.g., Kojima, Mizuno, and Yoshise [3]). In each iteration, the algorithm solves a system of linear equations for direction $(d_x, d_\tau, d_s, d_\kappa, d_\theta)$ from equation (3) and

$$\begin{aligned} Xd_s + Sd_x &= \gamma\mu e - Xs, \\ \tau d_\kappa + \kappa d_\tau &= \gamma\mu - \tau\kappa, \end{aligned} \tag{5}$$

where γ is chosen as

$$\gamma = 1 - \frac{1}{2\sqrt{n+1}}.$$

For a step size α, the next iterate is given by

$$x^+(\alpha) := x + \alpha d_x, \;\; \tau^+(\alpha) := \tau + \alpha d_\tau,$$

$$s^+(\alpha) := s + \alpha d_s, \;\; \kappa^+(\alpha) := \kappa + \alpha d_\kappa, \;\text{ and }\; \theta^+(\alpha) := \theta + \alpha d_\theta.$$

The step size α is chosen as the longest step such that $(x^+(\alpha), \tau^+(\alpha), s^+(\alpha), \kappa^+(\alpha), \theta^+(\alpha))$ remains in the neighborhood of $\mathcal{C}(\mu^+(\alpha))$ where

$$\mu^+(\alpha) = \frac{(x^+(\alpha))^T s^+(\alpha) + \tau^+(\alpha)\kappa^+(\alpha)}{n+1}.$$

It has been shown by Kojima et al. [3] that

$$\mu^+(\alpha) \le (1 - \frac{1}{2\sqrt{n+1}})\mu. \tag{6}$$

Our only modification to the algorithm is that we require step size α to meet

$$\theta^+(\alpha) \ge 0,$$

and we stop the process momentarily if $\theta^+(\alpha) = 0$.

Based on Theorem 1 we have following limiting behavior for the iteration sequence generated by the algorithm.

Theorem 2 . *The algorithm will generate a sequence of $(x^k, \tau^k, s^k, \kappa^k, \theta^k)$ in a neighborhood of the central path $\mathcal{C}(\mu)$, where*

$$\mu^{k+1} \le (1 - \frac{1}{2\sqrt{n+1}})\mu^k,$$

and

$$\theta^k \le \frac{(n+1)\mu^k}{\bar{n}} \le \mu^k.$$

Thus, we will either have (A): $\theta^k = 0$ before $\mu^k = 0$, i.e., $\theta^k = 0$ but $\mu^k > 0$, or (B): $\theta^k \to 0$ and $\mu^k \to 0$ (or $\theta^k = 0$ and $\mu^k = 0$) simultaneously, where $(x^k, \tau^k, s^k, \kappa^k)$ is a maximal complementarity solution for (HLCP) [4].

A. *Case (A) holds if and only if $LCP(M,q)$ has a feasible interior and its $q(M,q) < 0$. Note that $(x^k/\tau^k, s^k/\tau^k) > 0$ is a feasible interior-point for $LCP(M,q)$. Thus, we can continue the same algorithm from this point to directly solving $LCP(M,q)$ or $LCP(M,\tau^k q)$. The whole process is bounded by $O(\sqrt{n}L)$ iterations.*

B. *In Case (B), for all k:*

a) $\kappa^k \ge \kappa(M,q,n) > 0$ if and only if $LCP(M,q)$ is infeasible, where $\kappa(M,q,n)$ is fixed, and $\lim(x^k/\kappa^k, s^k/\kappa^k)$ is a certificate to prove infeasibility of $LCP(M,q)$;

b) $\tau^k \ge \tau(M,q,n) > 0$ if and only if $LCP(M,q)$ is feasible and its $q(M,q) = 0$, where $\tau(M,q,n)$ is fixed, and $\lim(x^k/\tau^k, s^k/\tau^k)$ is a solution for $LCP(M,q)$;

c) both $\kappa^k \to 0$ and $\tau^k \to 0$ if and only if $LCP(M,q)$ is feasible but it does not have a feasible interior, and its $q(M,q) < 0$.

Before prove the theorem, we make few comments. First, due to the "if and only if" conditions in the theorem, the classification of cases (A), (B.a), (B,b), and (B.c) are mutually exclusive and complete. For example, we cannot have a situation that a subsequence of κ^k tends to zero and another subsequence of κ^k tends to a positive number. Also, from equality (2) $(x^k, \tau^k, s^k, \kappa^k, \theta^k)$ is bounded, and, therefore, it has a limit. Second, if $\theta^k = 0$ and $\mu^k = 0$ at a same finite time, then we must have a feasible and complementary solution, where the results for case (B) still hold. Third, the fixed numbers $\kappa(M, q, n)$ and $\tau(M, q, n)$ in cases (B.a) and (B.b) are bounded from below by 2^{-L}. Thus, to generate a solution or an infeasibility certificate for $LCP(M, q)$, the worst-case iteration complexity is $O(\sqrt{n}L\log(1/\tau(M.q.n)))$ or $O(\sqrt{n}L\log(1/\kappa(M.q.n)))$, which is still bounded by $O(\sqrt{n}L)$, see Todd and Ye [11]. Finally, if case (A) occurs, the algorithm is a two-phase algorithm, that is, it first generates a feasible point, then finds a complementarity solution. We will consider a one-phase modification later.

We now prove Theorem 2.

Proof. The convergence rate for μ^k of the theorem is from (6), while the bound for θ^k is due to

$$(n+1)\mu^k = (x^k)^T s^k + \tau^k \kappa^k = (x^k)^T M x^k + \bar{n}\theta^k \geq \bar{n}\theta^k$$

and $\bar{n} \geq n+1$.

Proof of (A): In case (A), we clearly have a feasible point $(x, s) := (x^k/\tau^k, s^k/\tau^k) \geq 0$ and $-q^T x = -q^T x^k/\tau^k = \kappa^k/\tau^k > 0$ for $LCP(M, q)$. We need to prove that this fact implies that the q-value of $LCP(M, q)$ is negative. Suppose not, i.e., there is a feasible $(\bar{x}, \bar{s})$ such that

$$\bar{x}^T \bar{s} = 0 \quad \text{and} \quad q^T \bar{x} = 0.$$

Then, we must have

$$\bar{x}^T M \bar{x} = 0 \quad \text{or} \quad (M + M^T)\bar{x} = 0.$$

Take a convex combination of (x, s) and $(\bar{x}, \bar{s})$, we must have

$$\begin{aligned} 0 &\leq (\alpha\bar{x} + \beta x)^T(\alpha\bar{s} + \beta s) \\ &= (\alpha\beta + \beta^2)q^T x + \beta^2 x^T M x \\ &= \beta(q^T x + \beta x^T M x), \end{aligned}$$

for all $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$. Choose β small enough but positive, we must have

$$q^T x + \beta x^T M x < 0$$

which is a contradiction.

Conversely, if $LCP(M, q)$ has a feasible interior and its q-value is negative, then it has a feasible interior point $(x, s) > 0$ and $q^T x < 0$. Hence, (HLCP) has a feasible interior point $(x, \tau, s, \kappa, 0)$ where $(x, \tau, s, \kappa) > 0$.

For a small perturbation, (HLCP) has a feasible point $(x, \tau, s, \kappa, \theta < 0)$. From the proof above, the q-value of (HLCP) must be negative, that is, we will have $\theta^k = 0$ before $\mu^k = 0$.

In the rest of the proof, we may simply use (x^k, s^k) to represent a limit of (x^k, s^k).

Proof of (B.a): Since $\kappa^k > 0$, then we must have $\tau^k = 0$, that is, we have

$$s^k = Mx^k \geq 0, \quad (s^k)^T x^k = 0, \quad -q^T x^k = \kappa^k > 0, \quad \text{and} \quad (x^k, s^k) \geq 0.$$

Since $(s^k)^T x^k = 0$ can be written as $(x^k)^T M x^k = 0$ and M is monotone, we have $(M + M^T)x^k = 0$ or $-M^T x^k = Mx^k \geq 0$. Thus, x^k/κ^k precisely satisfies conditions (4) for $LCP(M, q)$ being infeasible.

Conversely, if $LCP(M, q)$ is infeasible, then from the proof of (v) in Theorem 1, the q-value of (HLCP) is zero and there is a certificate $(\bar{x}, 0, \bar{s}, \bar{\kappa}, 0)$ feasible to (HLCP) satisfying

$$\bar{s} = M\bar{x} \geq 0, \ \bar{s}^T \bar{x} = 0, \ \bar{\kappa} = -q^T \bar{x} > 0, \ \bar{x} \geq 0 \text{ and } \hat{e}^T \bar{x} + e^T \bar{s} + \bar{\kappa} = \bar{n}.$$

Since the interior-point iterates are in the neighborhood of the central path $\mathcal{C}(\mu)$, we must have κ^k bounded below by $\Omega(\bar{\kappa})/(n+1)$ (see [4]). Thus, we may define $\kappa(M, q, n)$ to be the largest $\bar{\kappa}$ among these certificates, which depends only on the data (M, q, n). In this case we must have $\theta^k \to 0$ and $\mu^k \to 0$ simultaneously.

Proof of (B.b): In this case, we must have $\kappa^k = 0$, which implies that

$$-q^T x^k = \kappa^k = 0 \quad \text{or} \quad -q^T (x^k/\tau^k) = 0.$$

Also note that we have

$$s^k = Mx^k + q\tau^k, \quad (x^k)^T s^k = 0, \quad (x^k, s^k) \geq 0.$$

Thus, $(x^k/\tau^k, s^k/\tau^k)$ is a solution of $LCP(M, q)$ with zero q-value.

Conversely, if $LCP(M, q)$ is feasible and its q-value is zero, then from (v) of Theorem 1 the q-value of (HLCP) must be zero and there is a solution $(\bar{x}, \bar{\tau}, \bar{s}, 0, 0)$ feasible to (HLCP) satisfying

$$\bar{s} = M\bar{x} + q\bar{\tau} \geq 0, \ \bar{s}^T \bar{x} = 0, \ \bar{\tau} > 0, \ \bar{x} \geq 0, \text{ and } \hat{e}^T \bar{x} + e^T \bar{s} + \sigma\bar{\tau} = \bar{n}.$$

Since the interior-point iterates are in the neighborhood of the central path $\mathcal{C}(\mu)$, we must have τ^k bounded below by $\Omega(\bar{\tau})/(n+1)$ (see [4]). Thus, we may define $\tau(M, q, n)$ to be the largest $\bar{\tau}$ among these solutions, which depends only on the data (M, q, n). In this case we must have $\theta^k \to 0$ and $\mu^k \to 0$ simultaneously.

Proof of (B.c): This is the only remaining alternative case for $LCP(M, q)$: it is feasible but it does not have a feasible interior, and its q-value is negative.

□

From Theorem 2 we have a corollary.

Corollary 3 . *In Case (B.c), i.e., $LCP(M, q)$ is feasible but it does not have a feasible interior, and its q-value is negative, we have*

i. *the solution set $\mathcal{S}$ of $LCP(M,q)$ is unbounded;*

ii. *there exist a $\hat{q}$ that is arbitrarily close to q such that $LCP(M,\hat{q})$ is infeasible.*

Proof. Since $LCP(M,q)$ is feasible, it has a solution $(\bar{x},\bar{s})$. Note that in this case the algorithm for solving (HLCP) will generate a limit point (x,s) such that

$$s = Mx, \quad s^T x = 0, \quad q^T x = 0, \quad (x,s) \geq 0, \quad \hat{e}^T x + e^T s = \bar{n} \geq n+1.$$

We can verify that $(\bar{x}+\alpha x, \bar{s}+\alpha s)$ is a complementarity solution of $LCP(M,q)$ for any $\alpha \geq 0$, which implies that its solution set $\mathcal{S}$ is unbounded.

Since the limit point has $q^T x = 0$ and $x \neq 0$, we must have an index i such that $x_i > 0$ and $q_i \leq 0$. Decrease q_i by any $\epsilon > 0$ and keep others unchanged, and call it $\hat{q}$. Then, we must have $\hat{q}^T x < 0$, which implies that $LCP(M,\hat{q})$ is infeasible from conditions (4).

□

Corollary 3 indicates that $LCP(M,q)$, characterized in case (B.c), is ill-posed (Renegar [10]). In other words, it is theoretically "unsolvable" in practical computation due to existing rounding error in problem data. However, in what follows we construct a possible approximate solution for $LCP(M,q)$.

Theorem 4 . *In Case (B.c),*

$$(\tilde{x}^k, \tilde{s}^k) := (x^k/\tau^k, s^k/\tau^k) \geq 0$$

is a point such that

i.

$$\hat{e}^T \tilde{x}^k + e^T \tilde{s}^k = O(\frac{1}{\sqrt{\mu^k}}).$$

ii.

$$\|\tilde{s}^k - M\tilde{x}^k - q\| \leq O(\sqrt{\mu^k}).$$

iii.

$$\frac{(\tilde{x}^k)^T \tilde{s}^k}{\hat{e}^T \tilde{x}^k + e^T \tilde{s}^k} \leq O(\sqrt{\mu^k}).$$

Here big O represents some fixed positive numbers that are independent of k.

Proof. The proof is based on the error bounds result for the monotone LCP, described by Mangasarian and Shiau [6] (also see [5] and [2]). The result implies that for case (B.c)

$$\tau^k = O(\sqrt{\mu^k}) \quad \text{and} \quad \kappa^k = O(\sqrt{\mu^k})$$

in the neighborhood of the central path $\mathcal{C}(\mu^k)$ (see Monteiro and Wright [8]). On the other hand, we have

$$\hat{e}^T x^k + e^T s^k + \sigma\tau^k + \kappa^k - \bar{n}\theta^k = \bar{n}$$

and

$$\theta^k \leq O(\mu^k).$$

These equalities and/or inequalities imply (i) and (ii).

Note that $(x^k)s^k \leq (n+1)\mu^k$. Hence,

$$(\tilde{x}^k)^T \tilde{s}^k \leq O(1),$$

which, together with (i), implies (iii).

□

Theorem 4 tells that $(\tilde{x}^k, \tilde{s}^k)$ is a feasible point as μ^k approaches to zero. In fact, it approaches to a maximal feasible point of $LCP(M,q)$ (see [4]). Thus, when $\mu^k \leq 2^{-L}$, we may round it to a maximal feasible point for $LCP(M,q)$, assuming (M,q) are rational and their binary data length is L. Eliminate those variables that are zero at this feasible point, then we continue to solve the reduced $LCP(M,q)$ with a feasible interior point. Again, the whole process is bounded by $O(\sqrt{n}L)$ iterations.

Theorem 4 (iii) also indicates that $(\tilde{x}^k, \tilde{s}^k)$ is already a relatively approximate complementarity solution for $LCP(M,q)$ as μ^k decreases. This relative criterion, which may not be accurate, is widely used to terminate interior-point algorithms.

4 A one-phase LCP model

For feasible $LCP(M,q)$ with its q-value negative, the homogeneous model constructed in Section 2 leads to a two-phase algorithm: first generate a feasible point, then find a complementarity solution. In this section, we construct a one-phase model and prove its $O(\sqrt{n}L)$-iteration complexity, where, however, we need to make an assumption that either a solution of $LCP(M,q)$ or a certificate proving infeasibility of $LCP(M,q)$ is not too big. This assumption is used by all other infeasible-starting algorithms to guarantee $O(nL)$-iteration complexity. (In general, big solution implies that $LCP(M,q)$ is near infeasible; big certificate for infeasibility implies that $LCP(M,q)$ is near feasible, see Todd and Ye [11].)

Given any $0 < \epsilon < 1$, let $x^0 = e$ and $s^0 = \sigma e$ such that

$$\hat{e} := s^0 - (M + M^T)x^0 \geq e \quad \text{with} \quad \sigma \geq \max\{2, \max([M+M^T]e)\} + 1 \geq 3.$$

Furthermore, choose

$$\tau^0 = 1,\ \kappa^0 = \sigma,\ \theta^0 = 1,\ \eta^0 = \sigma.$$

Then, consider an augmented LCP model related to $LCP(M,q)$:

$$(ALCP) \quad \text{minimize} \quad x^T s + \tau\kappa + \theta\eta$$

$$\text{subject to} \quad \begin{pmatrix} s \\ \kappa \\ \eta \end{pmatrix} = \begin{pmatrix} M & q & \bar{r} \\ -q^T & 1 & \bar{z} \\ -\bar{r}^T & -\bar{z} & 0 \end{pmatrix} \begin{pmatrix} x \\ \tau \\ \theta \end{pmatrix} + \begin{pmatrix} 0 \\ -\epsilon \\ \bar{n} \end{pmatrix},$$

$$(x, \tau, \theta, s, \kappa, \eta) \geq 0,$$

where

$$\bar{r} = s^0 - Mx^0 - q\tau^0, \quad \bar{z} = q^T x^0 + \kappa^0 + \epsilon - \tau^0,$$

and

$$\bar{n} = (x^0)^T s^0 + \tau^0(\kappa^0 + \epsilon) - (x^0)^T M x^0 - (\tau^0)^2 + \sigma \leq (n+2)\sigma + \epsilon - 1.$$

Note that for this initial point,

$$\bar{n} = (x^0)^T (s^0 - (M + M^T)x^0) + \tau^0(\kappa^0 + \epsilon) + (x^0)^T M x^0 - (\tau^0)^2 + \sigma \geq n + 2\sigma + \epsilon - 1.$$

Similar to (HLCP), the last equality of (ALCP) can be rewritten as a simplex constraint

$$\hat{e}^T x + e^T s + (\sigma + \epsilon - 2)\tau + \kappa + \eta - (\bar{n} - \sigma)\theta = \bar{n} - \epsilon.$$

We have the following result regarding (ALCP):

Theorem 5 .

i. *(ALCP) is monotone.*

ii. *(ALCP) is feasible and it has a feasible interior point*

$$x^0 = e, \ \tau^0 = 1, \ \theta^0 = 1,$$

and

$$s^0 = \sigma e, \ \kappa^0 = \sigma, \ \eta^0 = \sigma$$

with

$$X^0 s^0 = \sigma e, \ \tau^0\kappa^0 = \sigma, \ \textit{and} \ \ \theta^0\eta^0 = \sigma.$$

iii. *(ALCP) has a complementarity solution.*

iv. *Assume that either $LCP(M,q)$ is feasible and it has a solution (x,s) satisfying*

$$\hat{e}^T x + e^T s < \frac{\bar{n}}{\epsilon} - 1 - (\sigma + \epsilon - 2),$$

or $LCP(M,q)$ is infeasible and there is a certificate $(x, s := Mx)$ in conditions (4) satisfying

$$\hat{e}^T x + e^T s < \frac{\bar{n}}{\epsilon} - 1.$$

Then, $\theta = 0$ in every complementarity solution of (ALCP). Moreover, a maximal solution of (ALCP) has

$$\tau\kappa = 0 \quad \textit{and} \quad \tau + \kappa \geq c(M,q,n,\epsilon) > 0,$$

where $c(M,q,n,\epsilon)$ is a fixed positive number determined by (M,q), ϵ and n, and if $\tau \geq c(M,q,n,\epsilon) > 0$, then $(x/\tau, s/\tau)$ is a solution for $LCP(M,q)$; otherwise, $LCP(M,q)$ is infeasible.

Proof. Statements (i), (ii) and (iii) are simple observations. To prove (iv), we are basically able to construct a complementarity solution for (ALCP) in each of these two cases, where $\theta = 0$ in both cases, and $\tau \geq c(M,q,n,\epsilon) > 0$ in the first case and $\kappa \geq c(M,q,n,\epsilon) > 0$ in the second case similar to the proof of (B.a) and (B.b) in Theorem 2.

□

The theorem indicates that if ϵ is not small enough, the solution of (ALCP) generated by interior-point algorithms may be inconclusive for $LCP(M,q)$. However, we can, in the worst case, choose $\epsilon = 2^{-L}$. This is because $LCP(M,q)$ has either a solution or an infeasibility certificate (x,s) with $\hat{e}^T x + e^T s < 2^L$. Then, we apply an *existing* polynomial interior-point algorithm to solving (ALCP), which generates a maximal solution in $O(\sqrt{n}L)$ iterations, and either proves infeasibility or provides a solution for $LCP(M,q)$. Also note that the selection of ϵ does not change the convergence rate of the complementarity gap for solving (ALCP), which is in contrast to the selection of the size of an initial point in other infeasible-starting and interior-point algorithms.

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