On a Homogeneous Algorithm for a Monotone Complementarity Problem with Nonlinear Equality Constraints

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Abstract: We present a homogeneous formulation and an interior-point algorithm to solve a monotone complementarity problem (MCP) with nonlinear equality constraints and "free" variables. If the problem is solvable, then the algorithm generates a solution converging towards feasibility and complementarity simultaneously. Moreover, we analyze the complexity of the algorithm when the problem satisfies certain Lipschitz conditions.

Key words: Monotone complementarity problem, interior-point algorithms, equality constraints.

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1 Introduction

Consider a monotone complementarity problem (MCP) with nonlinear equality constraints and "free" variables in the following form:

$$
\begin{array}{lll}
(MCP) & \text{minimize} & x^T s \\
& \text{subject to} & \begin{pmatrix} 0 \\ s \end{pmatrix} = f(y,x) := \begin{pmatrix} h(y,x) \\ g(y,x) \end{pmatrix}, \ (x,s) \geq 0,
\end{array}
$$

where $f(y,x)$ is a continuous *monotone* mapping from $R^m \times R^n_{++} := \{x \in R^n : \ x > 0\}$ to R^{m+n}, $x, s \in R^n$, and ^{T} denotes transpose. In other words, for every $(y^1, x^1 > 0), (y^2, x^2 > 0) \in R^m \times R^n_{++}$, we have (e.g., [4])

$$
((y^1; x^1) - (y^2; x^2))^T (f(y^1, x^1) - f(y^2, x^2)) =
$$

$$
(y^1 - y^2)^T (h(y^1, x^1) - h(y^2, x^2)) + (x^1 - x^2)^T (g(y^1, x^1) - g(y^2, x^2)) \geq 0.
$$

Denote by ∇f the Jacobian matrix of f, which is positive semi-definite in $R^m \times R^n_{++}$. For simplicity, we also assume that ∇h has a full row rank.

A related (MCP) problem with nonlinear equations was considered by Wang, Monteiro and Pang [9], who developed a potential reduction algorithm for solving the problem. Many other complementarity problems can be formulated as (MCP). For example, the mixed complementarity problem, studied by Chen and Mangasarian [3], Dirkse and Ferris [5], and Simantiraki and Shanno [8],

$$
\begin{array}{ll}
\text{minimize} & v^T(u - x) + w^T x \\
\text{subject to} & \hat{f}(x) = w - v, \\
& x \leq d, \ (x, v, w) \geq 0
\end{array}
$$

can be written as

$$
\begin{array}{ll}
\text{minimize} & w^T x + s^T v \\
\text{subject to} & \begin{pmatrix} w \\ s \end{pmatrix} = f(x,v) := \begin{pmatrix} \hat{f}(x) + v \\ d - x \end{pmatrix}, \ (x, v, w, s) \geq 0,
\end{array}
$$

Note that $\hat{f}$ being monotone implies that f is monotone and the problem is in the form of (MCP), where neither h nor y is present, and $s := (w; s)$ and $x = (x; v)$. Here we use a MATLAB-type notation, $(w; s)$, to represent the column vector where w is on the top of s.

(MCP) is said to be (asymptotically) feasible if and only if there is a *bounded* sequence (y^t, x^t, s^t), $t = 1, 2, \ldots$, such that

$$
\lim_{t \to \infty} \quad \begin{pmatrix} 0 \\ s^t \end{pmatrix} - f(y^t, x^t) \to 0,
$$

where any limit point $(\hat{y}, \hat{x}, \hat{s})$ of the sequence is called an (asymptotically) feasible point for (MCP). (MCP) has an interior feasible point if it has an (asymptotically) feasible point $(\hat{y}, \hat{x} > 0, \hat{s} > 0)$. (MCP) is said to be (asymptotically) solvable if there is an (asymptotically) feasible $(\hat{y}, \hat{x}, \hat{s})$ such that $\hat{x}^T \hat{s} = 0$, where $(\hat{y}, \hat{x}, \hat{s})$ is called the "optimal" or "complementary" solution for (MCP).

Denote the feasible set of (MCP) by $\mathcal{F}$ and the solution set by $\mathcal{S}$. Note that (MCP) is feasible does not imply that (MCP) has a solution. If (MCP) has a solution, then it has a maximal solution (y^*, x^*, s^*) where the number positive components in (x^*, s^*) is maximal. Note that the indices of those positive components are invariant among all maximal solutions for (MCP) (see Güler [6]).

Consider a class of (MCP) where f satisfies the following condition. Let

$$\upsilon : (0,1) \times (0, \infty) \to (1, \infty)$$

be a monotone increasing function such that

$$\left\| \begin{pmatrix} \|y\| I & 0 \\ 0 & X \end{pmatrix} \left(f(y + d_y, x + d_x) - f(y, x) - \nabla f(y, x) \begin{pmatrix} d_y \\ d_x \end{pmatrix} \right) \right\|_1 \leq \upsilon(\alpha, \zeta)(d_y; d_x)^T \nabla f(y, x)(d_y; d_x) \tag{1}$$

whenever

$$d_x \in R^n,\ (y, x) \in R^m \times R^n_{++},\ \left\| X^{-1} d_x \right\|_\infty \leq \alpha, \text{ and } \|d_y\| \leq \zeta \|y\| \quad \text{for some fixed} \quad 0 < \alpha < 1,\ 0 < \zeta < \infty.$$

Then, f is said to be υ-scaled Lipschitz in $R^m \times R^n_{++}$.

2 A homogeneous MCP model

Consider an augmented homogeneous model related to (MCP):

$$(HMCP) \quad \text{minimize} \quad x^T s + \tau \kappa$$

$$\text{subject to} \quad \begin{pmatrix} 0 \\ s \\ \kappa \end{pmatrix} = \begin{pmatrix} \tau h(y/\tau, x/\tau) \\ \tau g(y/\tau, x/\tau) \\ -y^T h(y/\tau, x/\tau) - x^T g(y/\tau, x/\tau) \end{pmatrix}, \ (x, \tau, s, \kappa) \geq 0.$$

Let $\psi(y, x, \tau) :\ R^m \times R^{n+1}_{++} \to R^m \times R^{n+1}$ be

$$\psi(y, x, \tau) = \begin{pmatrix} \tau f(y/\tau, x/\tau) \\ -(y; x)^T f(y/\tau, x/\tau) \end{pmatrix} = \begin{pmatrix} \tau h(y/\tau, x/\tau) \\ \tau g(y/\tau, x/\tau) \\ -y^T h(y/\tau, x/\tau) - x^T g(y/\tau, x/\tau) \end{pmatrix}. \tag{2}$$

Then, it is easy to verify that $\nabla\psi$ is positive semi-definite as shown in the following lemma.

Lemma 1 . *Let ∇f be positive semi-definite in $R^m \times R^n_+$. Then $\nabla\psi$ is positive semi-definite in $R^m \times R^{n+1}_{++}$, i.e. given $(y, x > 0, \tau > 0)$*

$$(d_y; d_x; d_\tau)^T \nabla\psi(y, x, \tau)(d_y; d_x; d_\tau) = ((d_y; d_x) - d_\tau(y; x)/\tau)^T \nabla f(y/\tau, x/\tau)((d_y; d_x) - d_\tau(y; x)/\tau) \geq 0$$

for any $(d_y; d_x; d_\tau) \in R^m \times R^{n+1}$, where

$$\nabla\psi(y, x, \tau) =$$

$$\begin{pmatrix} \nabla f(y/\tau, x/\tau) & f(y/\tau, x/\tau) - \nabla f(y/\tau, x/\tau)(y/\tau, x/\tau) \\ -f(y/\tau, x/\tau)^T - (y/\tau, x/\tau)^T \nabla f(y/\tau, x/\tau)^T & (y/\tau; x/\tau)^T \nabla f(y/\tau, x/\tau)(y/\tau; x/\tau) \end{pmatrix}. \tag{3}$$

Furthermore, we have the following theorem.

Theorem 2 . *Let ψ be given by (2). Then,*

i. *ψ is a continuous homogeneous function in $R^m \times R^{n+1}_{++}$ with degree 1 and, for any $(y; x; \tau) \in R^m \times R^{n+1}_{++}$,*

$$(y; x; \tau)^T \psi(y, x, \tau) = 0$$

and

$$(y; x; \tau)^T \nabla\psi(y, x, \tau) = -\psi(y, x, \tau)^T$$

ii. *If f is a continuous monotone mapping from $R^m \times R^n_{++}$ to R^{m+n}, then ψ is a continuous monotone mapping from $R^m \times R^{n+1}_{++}$ to R^{m+n+1}.*

iii. *If f is scaled Lipschitz with $\upsilon = \upsilon_f$, then ψ is scaled Lipschitz, that is, it satisfies condition (1) with*

$$\upsilon = \upsilon_\psi(\alpha, \zeta) = \left(1 + \frac{(2 + \zeta - \alpha)\upsilon_f(2\alpha/(1-\alpha), (\zeta + \alpha)/(1-\alpha))}{1 - \alpha}\right)\left(\frac{1}{1-\alpha}\right).$$

iv. *$(HMCP)$ is (asymptotically) feasible and every (asymptotically) feasible point is an (asymptotically) complementary solution.*

Now, let $(y^*, x^*, \tau^*, s^*, \kappa^*)$ be a maximal complementary solution for $(HMCP)$. Then

v. *(MCP) has a solution if and only if $\tau^* > 0$. In this case, $(y^*/\tau^*, x^*/\tau^*, s^*/\tau^*)$ is a complementary solution for (MCP).*

Proof. The proof of (i) is straightforward. The proof of (ii) is similar to that in our earlier paper [1].

We now prove (iii). Assume $(y; x; \tau) \in R^m \times R^{n+1}_{++}$ and let $(d_y; d_x; d_\tau)$ be given such that

$$\left\|(X^{-1}d_x; \tau^{-1}d_\tau)\right\|_\infty \leq \alpha < 1 \quad \text{and} \quad \|d_y\| \leq \zeta \|y\|.$$

To prove ψ is scaled Lipschitz we must bound

$$\left\| \begin{pmatrix} \|y\| I & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & \tau \end{pmatrix} \left(\psi(y+d_y, x+d_x, \tau+d_\tau) - \psi(y,x,\tau) - \nabla\psi(y,x,\tau) \begin{pmatrix} d_y \\ d_x \\ d_\tau \end{pmatrix} \right) \right\|_1 . \tag{4}$$

From (2) and (3), the upper part in (4) is identical to

$$\begin{aligned}
&\begin{pmatrix} \|y\| I & 0 \\ 0 & X \end{pmatrix} \\
&(f(y'+d_{y'}, x'+d_{x'})(\tau+d_\tau) - f(y',x')\tau - (\nabla f(y',x')(d_y;d_x) + f(y',x')d_\tau - \nabla f(y',x')(y;x)d_\tau/\tau)) \\
&= (\tau+d_\tau) \begin{pmatrix} \|y\| I & 0 \\ 0 & X \end{pmatrix} (f(y'+d_{y'}, x'+d_{x'}) - f(y',x') - \nabla f(y',x')(d_{y'};d_{x'})) \\
&= (\tau+d_\tau) \begin{pmatrix} \tau\|y'\| I & 0 \\ 0 & \tau X' \end{pmatrix} (f(y'+d_{y'}, x'+d_{x'}) - f(y',x') - \nabla f(y',x')(d_{y'};d_{x'})) \qquad (5)
\end{aligned}$$

where

$$(y';x') = (y;x)/\tau \quad \text{and} \quad (y'+d_{y'}; x'+d_{x'}) = \frac{(y+d_y; x+d_x)}{\tau+d_\tau}, \tag{6}$$

that is,

$$(d_{y'};d_{x'}) = \frac{\tau(d_y;d_x) - (y;x)d_\tau}{\tau(\tau+d_\tau)} = \frac{(d_y;d_x) - (d_\tau/\tau)(y;x)}{\tau+d_\tau}. \tag{7}$$

Note that

$$\begin{aligned}
\left\|X'^{-1}d_{x'}\right\|_\infty &= \left\|\tau X^{-1}(\tau d_x - d_\tau x)/(\tau(\tau+d_\tau))\right\|_\infty \\
&= \left\|(\tau X^{-1}d_x - d_\tau e)/(\tau+d_\tau)\right\|_\infty \\
&\le (\left\|X^{-1}d_x\right\|_\infty + \alpha)/(1-\alpha) \\
&\le \tfrac{2\alpha}{1-\alpha}
\end{aligned} \tag{8}$$

and, similarly,

$$\|d_{y'}\| \le \frac{\zeta+\alpha}{1-\alpha}\|y'\|.$$

Thus, similar to the proof in our earlier paper [1],

$$\begin{aligned}
&(\tau+d_\tau)\left\| \begin{pmatrix} \tau\|y'\| I & 0 \\ 0 & \tau X' \end{pmatrix} (f(y'+d_{y'}, x'+d_{x'}) - f(y',x') - \nabla f(y',x')(d_{y'};d_{x'})) \right\|_1 \\
&= \tau(\tau+d_\tau)\left\| \begin{pmatrix} \|y'\| I & 0 \\ 0 & X' \end{pmatrix} (f(y'+d_{y'}, x'+d_{x'}) - f(y',x') - \nabla f(y',x')(d_{y'};d_{x'})) \right\|_1 \\
&\le \tau(\tau+d_\tau)\upsilon_f(2\alpha/(1-\alpha), (\zeta+\alpha)/(1-\alpha))(d_{y'};d_{x'})^T\nabla f(y',x')(d_{y'};d_{x'}) \\
&= \frac{\upsilon_f(2\alpha/(1-\alpha), (\zeta+\alpha)/(1-\alpha))}{1+d_\tau/\tau}(d_y;d_x;d_\tau)^T\nabla\psi(y,x,\tau)(d_y;d_x;d_\tau) \\
&\le \frac{\upsilon_f(2\alpha/(1-\alpha), (\zeta+\alpha)/(1-\alpha))}{1-\alpha}(d_y;d_x;d_\tau)^T\nabla\psi(y,x,\tau)(d_y;d_x;d_\tau).
\end{aligned}$$

Very similar to our earlier paper [1], we can also bound the bottom component in (4) by

$$\left(\frac{(1+\zeta)\upsilon_f(2\alpha/(1-\alpha),(\zeta+\alpha)/(1-\alpha))}{(1-\alpha)^2} + \frac{1}{1-\alpha} \right) (d_y;d_x;d_\tau)^T \nabla\psi(y,x,\tau)(d_y;d_x;d_\tau).$$

These two inequalities prove (iii).

We now prove (iv). Take $(y^t;x^t) = (1/2)^t e$, $\tau^t = (1/2)^t$, $s^t = (1/2)^t e$, and $\kappa^t = (1/2)^t$. Then we see, as $t \to \infty$,

$$0 - \tau^t h(y^t/\tau^t, x^t/\tau^t) = (1/2)^t (e - h(e,e)) \to 0$$

$$s^t - \tau^t g(y^t/\tau^t, x^t/\tau^t) = (1/2)^t (e - g(e,e)) \to 0$$

and

$$\kappa^t + (y^t;x^t)^T f(y^t/\tau^t, x^t/\tau^t) = (1/2)^t (1 + e^T f(e,e)) \to 0.$$

Thus, the system is (asymptotically) feasible. Let $(\hat{y},\hat{x},\hat{\tau},\hat{s},\hat{\kappa}) \geq 0$ be any (asymptotically) feasible point for $(HMCP)$, then we clearly see from (i)

$$\hat{x}^T\hat{s} + \hat{\tau}\hat{\kappa} = (\hat{y};\hat{x};\hat{\tau})^T\psi(\hat{y},\hat{x},\hat{\tau}) = 0.$$

We now prove (v). If $(y^*,x^*,\tau^*,s^*,\kappa^*)$ is a solution for $(HMCP)$ and $\tau^* > 0$. Then we have

$$(0;s^*)/\tau^* = f(y^*/\tau^*, x^*/\tau^*) \quad \text{and} \quad (x^*)^T s^*/(\tau^*)^2 = 0,$$

that is, $(x^*/\tau^*, s^*/\tau^*)$ is a solution for (MCP). Let $(\hat{y},\hat{x},\hat{s})$ be a solution to (MCP). Then for any $\tau = 1$, $(y;x) = (\hat{y};\hat{x})$, $s = \hat{s}$, $\kappa = 0$ is a solution for $(HMCP)$. Thus, every maximal solution of $(HMCP)$ must have $\tau^* > 0$.

$\square$

3 The Central path of the homogeneous model

Due to Theorem 2, we can solve (MCP) by finding a maximal complementary solution of $(HMCP)$. Select y^0, $x^0 > 0$, $s^0 > 0$, $\tau^0 > 0$ and $\kappa^0 > 0$ and let the residual vectors

$$r^0 = (r_h^0; r_g^0) := (0;s^0) - \tau^0 f(y^0/\tau^0, x^0/\tau^0), \quad z^0 = \kappa^0 + (y^0;x^0)^T f(y^0/\tau^0, x^0/\tau^0).$$

Also let

$$\bar{n} = (r^0)^T(y^0;x^0) + z^0\tau^0 = (y^0;x^0)^T(0;s^0) + \tau^0\kappa^0.$$

For simplicity, we set

$$y^0 = e,\ x^0 = e,\ \tau^0 = 1,\ s^0 = e,\ \kappa^0 = 1,\ \theta^0 = 1,$$

with

$$X^0 s^0 = e \quad \text{and} \quad \tau^0 \kappa^0 = 1,$$

where X^0 is the diagonal matrix of x^0. Note that $\bar{n} = n + 1$ in this setting.

We present the next theorem, whose proof is similar to that in [1] and is based on the fact that the set

$$H_{++} := \left\{ \begin{pmatrix} (0; s) \\ \kappa \end{pmatrix} - \psi(y, x, \tau) : \quad (x, \tau, s, \kappa) > 0. \right\}$$

is open and convex (e.g., [7]). We will omit the proof here.

Theorem 3 . *Consider* $(HMCP)$.

i. *For any* $0 < \theta \leq 1$*, there exists a strictly positive point* $(x > 0, \tau > 0, s > 0, \kappa > 0)$ *such that*

$$\begin{pmatrix} (0; s) \\ \kappa \end{pmatrix} - \psi(y, x, \tau) = \begin{pmatrix} (0; s) - \tau f(y/\tau, x/\tau) \\ \kappa + (y; x)^T f(y/\tau, x/\tau) \end{pmatrix} = \theta \begin{pmatrix} r^0 \\ z^0 \end{pmatrix}. \tag{9}$$

ii. *Starting from* $(y^0; x^0 = e, \tau^0 = 1, s^0 = e, \kappa^0 = 1)$*, for any* $0 < \theta \leq 1$ *there is a unique strictly positive point* $(y(\theta), x(\theta), \tau(\theta), s(\theta), \kappa(\theta))$ *that satisfies equation (9) and*

$$\begin{pmatrix} Xs \\ \tau\kappa \end{pmatrix} = \theta e. \tag{10}$$

iii. *For any* $0 < \theta \leq 1$*, the solution* $(y(\theta), x(\theta), \tau(\theta), s(\theta), \kappa(\theta))$ *in [ii] is bounded. Thus,*

$$\mathcal{C}(\theta) := \left\{ (y, x, \tau, s, \kappa) : \begin{pmatrix} (0; s) \\ \kappa \end{pmatrix} - \psi(y, x, \tau) = \theta \begin{pmatrix} r^0 \\ z^0 \end{pmatrix}, \quad \begin{pmatrix} Xs \\ \tau\kappa \end{pmatrix} = \theta e, \quad 0 < \theta \leq 1 \right\} \tag{11}$$

is a continuous bounded trajectory.

iv. *The limit point* $(y(0), x(0), \tau(0), s(0), \kappa(0))$ *is a maximal complementary solution for* $(HMCP)$.

We now present an interior-point algorithm that generates iterates within a neighborhood of $\mathcal{C}(\theta)$. For simplicity, in what follows we let $x := (x; \tau) \in R^{n+1}$, $s := (s; \kappa) \in R^{n+1}$, and $r_g^0 := (r_g^0; z^0)$. Recall that, for any $x, s > 0$

$$(y; x)^T \psi(y, x) = 0 \quad \text{and} \quad (y; x)^T \nabla\psi(y, x) = -\psi(y, x)^T. \tag{12}$$

Furthermore, ψ is monotone and satisfies the scaled Lipschitz. We will use these facts frequently in our analyses.

4 The homogeneous interior-point algorithm

At iteration k with iterate $(y^k, x^k > 0, s^k > 0)$, the algorithm solves a system of linear equations for direction (d_y, d_x, d_s) from

$$(0; d_s) - \nabla\psi(y^k, x^k)(d_y; d_x) \quad = \quad -\eta(r_h^k; r_g^k) \tag{13}$$

and

$$X^k d_s + S^k d_x \quad = \quad \gamma\mu^k e - X^k s^k, \tag{14}$$

where η and γ are proper given parameters between 0 and 1, and

$$r^k = (r_h^k; r_g^k) := (0; s^k) - \psi(y^k, x^k) \quad \text{and} \quad \mu^k = \frac{(x^k)^T s^k}{n+1}.$$

It can be shown that the above linear system always has a solution as long as ψ is a monotone function ([2]).

First we prove the following lemma.

Lemma 4 . *The direction (d_x, d_s) satisfies*

$$d_x^T d_s = (d_y; d_x)^T \nabla\psi(y^k, x^k)(d_y; d_x) + \eta(1 - \eta - \gamma)(n+1)\mu^k.$$

Proof: Premultiplying each side of (13) by $(d_y; d_x)^T$ gives

$$d_x^T d_s - (d_y, d_x)^T \nabla\psi(y^k, x^k)(d_y; d_x) = -\eta(d_y^T(0 - \psi_h(y^k, x^k)) + d_x^T(s^k - \psi_g(y^k, x^k))) \tag{15}$$

Multiplying each side of (13) by $(y^k; x^k)$ and using (12) give

$$\begin{aligned} (x^k)^T d_s + \psi(y^k; x^k)(d_y; d_x) &= -\eta(y^k; x^k)^T r^k \\ &= -\eta((y^k)^T(0 - \psi_h(y^k, x^k)) + (x^k)^T(s^k - \psi_g(x^k))) \\ &= -\eta(x^k)^T s^k \\ &= -\eta(n+1)\mu^k \end{aligned} \tag{16}$$

These two equalities in combination with (14) implies

$$\begin{aligned} d_x^T d_s &= (d_y; d_x)^T \nabla\psi(y^k, x^k)(d_y; d_x) - \eta(d_x^T s^k + d_s^T x^k + \eta(n+1)\mu^k) \\ &= (d_y; d_x)^T \nabla\psi(y^k, x^k)(d_y; d_x) - \eta(-(n+1)\mu^k + \gamma(n+1)\mu^k + \eta(n+1)\mu^k) \\ &= (d_y; d_x)^T \nabla\psi(y^k, x^k)(d_y; d_x) + \eta(1 - \gamma - \eta)(n+1)\mu^k \end{aligned}$$

Q.E.D.

For a step-size $\alpha > 0$, let the new iterate

$$y^+ := y^k + \alpha d_y, \; x^+ := x^k + \alpha d_x > 0, \tag{17}$$

and

$$\begin{aligned} s^+ &:= s^k + \alpha d_s + \psi_g(y^+, x^+) - \psi_g(y^k, x^k) - \alpha \nabla \psi_g(y^k, x^k)(d_y; d_x) \\ &= \psi_g(y^+, x^+) + (s^k - \psi_g(y^k, x^k)) + \alpha(d_s - \nabla \psi_g(y^k, x^k)(d_y; d_x)) \\ &= \psi_g(y^+, x^+) + (s^k - \psi_g(y^k, x^k)) - \alpha \eta r_g^k \\ &= \psi_g(y^+, x^+) + (1 - \alpha \eta) r_g^k. \end{aligned} \tag{18}$$

The last two equalities come from (13) and the definition of r^k. Also let

$$r^+ = (r_h^+; r_g^+) := (0; s^+) - \psi(y^+, x^+).$$

Then, we have

Lemma 5 . *Consider the new iterate $(y^+; x^+, s^+)$ given by (17) and (18).*

i. $r_h^+ = (1 - \eta\alpha) r_h^k + \psi_h(y^k, x^k) + \alpha \nabla \psi_h(y, x)(d_y; d_x) - \psi_h(x^+, y^+)$.

ii. $r_g^+ = (1 - \alpha\eta) r_g^k$.

Proof: (i) follows from

$$\begin{aligned} r_h^+ - (1 - \alpha\eta) r_h^k &= -\psi_h(y^+, x^+) + (1 - \alpha\eta)\psi_h(y, x) \\ &= \psi_h(y, x) + \alpha \nabla \psi_h(y, x)(d_y; d_x) - \psi_h(y^+, x^+). \end{aligned}$$

Next we prove (ii). From (18) it follows

$$\begin{aligned} r_g^+ &= s^+ - \psi_g(y^+, x^+) \\ &= (1 - \alpha\eta) r_g^k. \end{aligned}$$

Q.E.D.

Theorem 6 . *Assume that ψ is scaled Lipschitz with $\upsilon = \upsilon_\psi$ and at the beginning of iteration k*

$$\left\| X^k s^k - \mu^k e \right\| \le \beta \mu^k, \quad \mu^k = \frac{(x^k)^T s^k}{n+1}$$

where

$$\beta = \frac{1}{5 + 8\upsilon_\psi(\sqrt{2}/2, \zeta)} \le 1/5.$$

Let $\eta = \beta/\sqrt{n+1}$, $\gamma = 1 - \eta$, and $\alpha = 1$ in the algorithm. Furthermore, let the problem has a nontrivial solution y^* and let us start at an initial point such that $\|y^k\| \ge 1$ and $\|d_y\| \le \zeta \left\| y^k \right\|$ in the neighborhood of the central path. Then,

i. $y^+ = y^k + d_y, \quad x^+ = x^k + d_x > 0, \quad s^+ = \psi_g(y^+, x^+) + (1-\eta) r_g^k > 0.$

ii. $\mu^+ \leq (1 - 3\beta/(4\sqrt{n+1}))\mu^k$, *where* $\mu^+ = \frac{(x^+)^T s^+}{n+1}$.

iii. $\|X^+ s^+ - \mu^+ e\| \leq \beta \mu^+$

iv. $\|r_g^+\| \leq (1 - \beta/\sqrt{n+1}) \|r_g^k\|$.

v. $\|r_h^+\| + \sqrt{n+1}\mu^+ \leq (1 - \beta/(2\sqrt{n+1}))(\|r_h^k\| + \sqrt{n+1}\mu^k)$.

Proof: From (14) we have

$$S^k d_x + X^k d_s = -X^k s^k + \gamma \mu e.$$

Hence,

$$D^{-1} d_x + D d_s = -(X^k S^k)^{-1/2}(X^k s^k - \gamma \mu^k e)$$

where $D = (X^k)^{1/2}(S^k)^{-1/2}$. Note

$$d_x^T d_s = (d_y; d_x)^T \nabla \psi(y^k, x^k)(d_y; d_x) \geq 0$$

from Lemma 4 and $\gamma = 1 - \eta$. This together with the assumption of the theorem imply

$$\|D^{-1} d_x\|^2 + \|D d_s\|^2 \leq \|(X^k S^k)^{-1/2}(X^k s^k - \gamma \mu^k e)\|^2 \leq \frac{\|X^k s^k - \gamma \mu^k e\|^2}{(1-\beta)\mu^k}.$$

Also note

$$\begin{aligned}
\|X^k s^k - \gamma \mu^k e\|^2 &= \|X^k s^k - \mu^k e + (1-\gamma)\mu^k e\|^2 \\
&= \|X^k s - \mu^k e\|^2 + ((1-\gamma)\mu^k)^2 \|e\|^2 \\
&\leq (\beta^2 + \eta^2(n+1))(\mu^k)^2 \\
&= 2\beta^2 (\mu^k)^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|(X^k)^{-1} d_x\| = \left\|(X^k S^k)^{-1/2} D^{-1} d_x\right\| &\leq \frac{\|D^{-1} d_x\|}{\sqrt{(1-\beta)\mu^k}} \\
&\leq \frac{\sqrt{2}\beta\mu^k}{(1-\beta)\mu^k} = \frac{\sqrt{2}\beta}{1-\beta} \leq \frac{\sqrt{2}}{2},
\end{aligned}$$

since $\beta \leq 1/5$. This implies that $x^+ = x^k + d_x > 0$. Furthermore, we have

$$\begin{aligned}
\|D_x d_s\| &= \|D^{-1} D_x D d_s\| \\
&\leq \|D^{-1} d_x\| \|D d_s\| \\
&\leq (\|D^{-1} d_x\|^2 + \|D d_s\|^2)/2 \\
&\leq \left\|(X^k S^k)^{-1/2}(X^k s^k - \gamma \mu^k)\right\|^2 /2
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\left\|X^k s^k - \gamma\mu^k e\right\|^2}{2(1-\beta)\mu^k} \\
&\leq \frac{2\beta^2(\mu^k)^2}{2(1-\beta)\mu^k} \\
&= \frac{\beta^2\mu^k}{1-\beta},
\end{aligned}
$$

and

$$
d_x^T d_s = d_x^T D^{-1} D d_s \leq \left\|D^{-1} d_x\right\| \|D d_s\| \leq \frac{\beta^2\mu^k}{1-\beta}.
$$

Consider

$$
\begin{aligned}
X^+ s^+ - \gamma\mu^k e &= X^+(s^k + d_s + \psi_g(y^+, x^+) - \psi_g(y^k, x^k) - \nabla\psi_g(y^k, x^k)(d_y; d_x)) - \gamma\mu^k e \\
&= (X^k + D_x)(s^k + d_s) - \gamma\mu^k e + X^+(\psi_g(y^+, x^+) - \psi_g(y^k, x^k) - \nabla\psi_g(y^k, x^k)(d_y; d_x)) \\
&= D_x d_s + X^+(\psi_g(y^+, x^+) - \psi_g(y^k, x^k) - \nabla\psi_g(y^k, x^k)(d_y; d_x)).
\end{aligned}
$$

Using that ψ is the scaled Lipschitz, $(d_y; d_x)^T \nabla\psi(y^k, x^k)(d_y, d_x) = d_x^T d_s$, Lemma 5 and the above four relations we obtain

$$
\begin{aligned}
\left\| \begin{array}{c} -\|y^k\|(r_h^+ - \gamma r_h^k) \\ X^+ s^+ - \gamma\mu^k e \end{array} \right\| &= \left\| \begin{array}{c} \|y^k\| \left(\psi_h(y^+, x^+) - \psi_h(y^k, x^k) - \nabla\psi_h(y^k, x^k)(d_y; d_x)\right) \\ D_x d_s + X^+(\psi_g(y^+, x^+) - \psi_g(y^k, x^k) - \nabla\psi_g(y^k, x^k)(d_y; d_x)) \end{array} \right\| \\
&\leq \|D_x d_s\| + \left\| \begin{pmatrix} \|y^k\| I & 0 \\ 0 & X^+ \end{pmatrix} (\psi(y^+, x^+) - \psi(y^k, x^k) - \nabla\psi(y^k, x^k)(d_y; d_x)) \right\| \\
&\leq \|D_x d_s\| + \max(1, \left\|(X^k)^{-1} X^+\right\|_\infty) \\
&\qquad \left\| \begin{pmatrix} \|y^k\| I & 0 \\ 0 & X^k \end{pmatrix} (\psi(y^+, x^+) - \psi(y^k, x^k) - \nabla\psi(y^k, x^k)(d_y; d_x)) \right\| \\
&\leq \|D_x d_s\| \\
&\qquad +2 \left\| \begin{pmatrix} \|y^k\| I & 0 \\ 0 & X^k \end{pmatrix} (\psi(y^+, x^+) - \psi(y^k, x^k) - \nabla\psi(y^k, x^k)(d_y; d_x)) \right\|_1 \\
&\leq \|D_x d_s\| + 2\upsilon_\psi(\sqrt{2}/2, \zeta)(d_y; d_x)^T \nabla\psi(y^k, x^k)(d_y; d_x) \\
&= \|D_x d_s\| + 2\upsilon_\psi(\sqrt{2}/2, \zeta) d_x^T d_s \\
&\leq \tfrac{\beta^2\mu^k}{1-\beta} + 2\upsilon_\psi(\sqrt{2}/2, \zeta)\tfrac{\beta^2\mu^k}{1-\beta} \\
&\leq \tfrac{(1+2\upsilon_\psi(\sqrt{2}/2),\zeta)\beta^2\mu^k}{1-\beta} \\
&= \delta\mu^k,
\end{aligned}
\tag{19}
$$

where $\delta = (1 + 2\upsilon_\psi(\sqrt{2}/2, \zeta))\frac{\beta^2}{1-\beta}$. Note the choice of β implies that

$$
\delta = \frac{\beta}{4}. \tag{20}
$$

Moreover,

$$\begin{aligned}(\delta\mu^k)^2 &\geq \left\|X^+s^+ - \gamma\mu^k e\right\|^2 \\ &= \left\|X^+s^+ + (-\mu^+ + \mu^+ - \gamma\mu^k)e\right\|^2 \\ &= \left\|X^+s^+ - \mu^+ e\right\|^2 + (\mu^+ - \gamma\mu^k)^2(n+1) \\ &\geq (\mu^+ - \gamma\mu^k)^2(n+1).\end{aligned}$$

Hence,

$$(\gamma - \frac{\delta}{\sqrt{n+1}})\mu^k \leq \mu^+ \leq (\gamma + \frac{\delta}{\sqrt{n+1}})\mu^k$$

and then using (20) we obtain

$$(1 - \frac{5\beta}{4\sqrt{n+1}})\mu^k \leq \mu^+ \leq (1 - \frac{3\beta}{4\sqrt{n+1}})\mu^k$$

and ii) has been proved. Therefore,

$$\left\|X^+s^+ - \mu^+ e\right\| \leq \delta\mu^k \leq \frac{\beta}{4}\mu^k \leq \beta\mu^+$$

which proves iii). Note that $\|X^+s^+ - \mu^+ e\| < \beta\mu^+$ implies $s^+ > 0$, since $x^+ > 0$.

The proof of iv) follows ii) of the preceding lemma and the choices of α and η.

Since we assume that the problem has a nontrivial solution and we start at an initial point such that $\|y\| \geq 1$ in the neighborhood of the central, (19) implies

$$\left\|r_h^+ - \gamma r_h\right\| \leq \frac{\delta\mu^k}{\|y^k\|} \leq \delta\mu^k.$$

It follows

$$\left\|r_h^+\right\| \leq \gamma\left\|r_h\right\| + \delta\mu^k.$$

Therefore,

$$\begin{aligned}\left\|r_h^+\right\| + \sqrt{n+1}\mu^+ &\leq \gamma\left\|r_h\right\| + \delta\mu^k + (\gamma + \tfrac{\delta}{\sqrt{n+1}})\sqrt{n+1}\mu^k \\ &\leq \gamma\left\|r_h\right\| + (\gamma + \delta(\tfrac{1}{\sqrt{n+1}} + \tfrac{1}{\sqrt{n+1}}))\sqrt{n+1}\mu^k \\ &\leq (\gamma + \delta\tfrac{2}{\sqrt{n+1}})(\left\|r_h^k\right\| + \sqrt{n+1}\mu^k) \\ &\leq (1 - \tfrac{\beta}{\sqrt{n+1}} + \tfrac{\beta}{4}\tfrac{2}{\sqrt{n+1}})(\left\|r_h^k\right\| + \sqrt{n+1}\mu^k) \\ &\leq (1 - \tfrac{\beta}{2\sqrt{n+1}})(\left\|r_h^k\right\| + \sqrt{n+1}\mu^k)\end{aligned}$$

Q.E.D.

Based on this theorem, we see that if $(HMCP)$ has a non-trivial solution in y and if the initial point is chosen such that $\|y\| \geq 1$ in the neighborhood of the central path, the algorithm generates a sequence (y^k, x^k) converges to a maximal solution.

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