Complexity Analysis of the Analytic Center Cutting Plane Method That Uses Multiple Cuts [1]

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Abstract

We analyze the complexity of the analytic center cutting plane or column generation algorithm for solving general convex problems defined by a separation oracle. The oracle is called at the analytic center of a polytope, which contains a solution set and is given by the intersection of the linear inequalities previously generated from the oracle. If the center is not in the solution set, separating hyperplanes will be placed through the center to shrink the containing polytope. While the complexity result has been recently established for the algorithm when one cutting plane is placed in each iteration, the result remains open when multiple cuts are added. Moreover, adding multiple cuts actually is a key to practical effectiveness in solving many problems and it presents theoretical difficulties in analyzing cutting plane methods. In this paper, we show that the analytic center cutting plane algorithm, with multiple cuts added in each iteration, still is a fully polynomial approximation algorithm.

Keywords: Convex feasibility problem, analytic center, potential reduction, column generation, cutting planes.

AMS subject classification: 90C25, 90C26, 90C60.

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1 Introduction

The problem studied is to find a point in a convex set Γ, where we assume $\Gamma \subset \mathbb{R}^m$ is contained in the cube $\Omega^0 = \{y \in \mathbb{R}^m : 0 \leq y \leq e\} = [0,1]^m$, where e denotes the vector of all ones. Through out this paper we consider nontrivial cases where $m \geq 2$. We assume that Γ has a nonempty interior and it contains a full dimensional closed ball with ϵ ($< \frac{1}{2}$) radius. The set Γ is defined implicitly by a separating oracle, which for every $\hat{y} \in \mathbb{R}^m$ either answers $\hat{y} \in \Gamma$ or generates a separating hyperplane $a \in \mathbb{R}^m$ such that $\{y \in \mathbb{R}^m : a^T y \leq a^T \hat{y}\} \supset \Gamma$. Without loss of generality, we assume that a is normalized so that $\|a\| = 1$.

The analytic center cutting plane or column generation algorithm has been very effective for solving this problem (see [2], [3] and references therein). The algorithm starts with Ω^0 as an initial polytope that contains Γ. Then, in each iteration the oracle is called at the (approximate) analytic center of the containing polytope given by the intersection of the linear inequalities previously generated from the oracle. If the center in not in Γ yet, separating hyperplanes will be placed through the center to trim down the containing polytope further.

Recently, several complexity results have been established for this algorithm when one plane is placed in each iteration, see Nesterov [6], Goffin et al. [3], and Altman and Kiwiel [1]. However, the complexity result remains open when multiple cuts are added in each iteration. Intriguingly, adding multiple cuts is a key to practical effectiveness of the algorithm in solving many problems (e.g., Bahn et al. [2]) and it presents theoretical difficulties in analyzing all cutting plane methods (including the ellipsoid method). In this paper, we show that the analytic center cutting plane algorithm, with multiple cuts (less than or equal to m) added in each iteration, still is a fully polynomial approximation algorithm.

The paper is organized as follows. We derive a key potential reduction theorem and its corollary in the next section. Then, in Section 3 we bound the total number of cutting planes needed to generate a solution in Γ. Finally, we bound the total number of Newton's steps before the algorithm terminates, which is the main computational cost of the algorithm, in Section 4. We show that the total number of Newton's steps is as the same order as the total number of cutting planes, and both of them are polynomials in m and $1/\epsilon$.

2 Max-Potential Reduction

Let

$$\Omega = \{y \in \mathbb{R}^m : \quad c - A^T y \geq 0\},$$

defined by n $(> m)$ linear inequalities, be bounded and have a nonempty interior. Let its analytic center be $\bar{y}$. Define

$$\bar{s} = c - A^T \bar{y}$$

and the max-potential of Ω

$$P(\Omega) = \sum_{j=1}^{n} \log \bar{s}_j := \max_{y \in \Omega} \sum_{j=1}^{n} \log(c - A^T y)_j.$$

Then, there is a unique $\bar{x} > 0$ with $A\bar{x} = 0$ such that

$$\bar{X}\bar{s} - e = 0, \tag{2.1}$$

where $\bar{X} = \mathrm{diag}(\bar{x})$. Let us place η (≥ 1) cuts at $\bar{y}$, that is, add η new inequalities $a_{n+i}^T y \leq a_{n+i}^T \bar{y}$, $i = 1, ..., \eta$, to Ω, and consider the new set

$$\Omega^+ = \{y : A^T y \leq c, \quad a_{n+i}^T y \leq a_{n+i}^T \bar{y}, \ i = 1, ..., \eta\},$$

where we assume

$$\|a_{n+i}\| = 1, \quad i = 1, ..., \eta.$$

Define

$$r_{n+i} = \sqrt{a_{n+i}^T (A\bar{S}^{-2}A^T)^{-1} a_{n+i}}, \quad i = 1, ..., \eta.$$

Then, the following result, which was first stated in Ye [9], is an extension of Theorem 2 of Ye [8].

Theorem 2.1 *(Ye [9]) Let $\bar{y}^+$ be the analytic center of Ω^+ and*

$$\bar{s}^+ = c - A^T \bar{y}^+ \in \mathbb{R}^n, \quad \bar{s}_{n+i}^+ = a_{n+i}^T \bar{y} - a_{n+i}^T \bar{y}^+, \ i = 1, ..., \eta.$$

Then,

$$P(\Omega^+) = \sum_{j=1}^{n+\eta} \log \bar{s}_j^+ \leq P(\Omega) + \sum_{i=1}^{\eta} \log(r_{n+i}) + \frac{\eta}{2} \log(\frac{n}{n-1}) + (\eta + 1) \log(\eta + 1) - \frac{\eta(\eta + 2)}{\eta + 1}.$$

Proof. Note that we have, for $i = 1, ..., \eta$,

$$\begin{aligned}
\bar{s}^+_{n+i} &= a^T_{n+i}(\bar{y} - \bar{y}^+) \\
&= a^T_{n+i}(A\bar{S}^{-2}A^T)^{-1}(A\bar{S}^{-2}A^T)(\bar{y} - \bar{y}^+) \\
&= a^T_{n+i}(A\bar{S}^{-2}A^T)^{-1}A\bar{S}^{-2}(A^T\bar{y} - A^T\bar{y}^+) \\
&= a^T_{n+i}(A\bar{S}^{-2}A^T)^{-1}A\bar{S}^{-2}(-c + A^T\bar{y} + c - A^T\bar{y}^+) \\
&= a^T_{n+i}(A\bar{S}^{-2}A^T)^{-1}A\bar{S}^{-2}(\bar{s}^+ - \bar{s}) \\
&= a^T_{n+i}(A\bar{S}^{-2}A^T)^{-1}A\bar{S}^{-1}(\bar{S}^{-1}\bar{s}^+ - e) \\
&\leq \|a^T_{n+i}(A\bar{S}^{-2}A^T)^{-1}A\bar{S}^{-1}\|\|\bar{S}^{-1}\bar{s}^+ - e\| \\
&= r_{n+i}\|\bar{S}^{-1}\bar{s}^+ - e\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\exp P(\Omega^+)}{(\prod_i^\eta r_{n+i})\exp P(\Omega)} &= \prod_{i=1}^{\eta} \frac{\bar{s}^+_{n+i}}{r_{n+i}} \prod_{j=1}^{n} \frac{\bar{s}^+_j}{\bar{s}_j} \\
&\leq \|\bar{S}^{-1}\bar{s}^+ - e\|^\eta \prod_{j=1}^{n} \frac{\bar{s}^+_j}{\bar{s}_j}. \qquad (2.2)
\end{aligned}$$

Note also that we have

$$e^T\bar{S}^{-1}\bar{s}^+ = n.$$

This can be seen as follows:

$$\begin{aligned}
e^T\bar{S}^{-1}\bar{s}^+ &= e^T\bar{X}\bar{s}^+ \\
&= e^T\bar{X}(c - A^T\bar{y}^+) \\
&= e^T\bar{X}c \\
&= e^T\bar{X}(c - A^T\bar{y}) \\
&= e^T\bar{X}\bar{s} \\
&= e^Te = n.
\end{aligned}$$

Let $\alpha = \bar{S}^{-1}\bar{s}^+ \in \mathbb{R}^n$. Then, to evaluate the quantity of (2.2), we face a maximum problem

$$\begin{aligned}
\max \quad & f(\alpha) = \|\alpha - e\|^\eta \prod_{j=1}^{n} \alpha_j \\
\text{s.t.} \quad & e^T\alpha = n, \quad \alpha > 0.
\end{aligned}$$

The maximizer is achieved, without loss of generality, at $\alpha_1 = \beta > 1$ and

$$\alpha_2 = ... = \alpha_n = \frac{n-\beta}{n-1} > 0.$$

Hence

$$\begin{aligned} f(\alpha) &= \left(\frac{n}{n-1}\right)^{\eta/2} (\beta-1)^{\eta}\beta\left(\frac{n-\beta}{n-1}\right)^{n-1} \\ &\le \left(\frac{n}{n-1}\right)^{\eta/2} (\eta+1)^{\eta+1}\left(\frac{\beta-1}{\eta+1}\right)^{\eta}\frac{\beta}{\eta+1}\left(\frac{n-\beta}{n-1}\right)^{n-1} \\ &\le \left(\frac{n}{n-1}\right)^{\eta/2} (\eta+1)^{\eta+1}\left(\frac{n-\eta/(\eta+1)}{n+\eta}\right)^{n+\eta} \\ &\le \left(\frac{n}{n-1}\right)^{\eta/2} (\eta+1)^{\eta+1}\exp(-\eta(\eta+2)/(\eta+1)). \end{aligned}$$

This gives the desired result.

$\square$

The above theorem also holds when cuts are placed at an approximate analytic center. Let a (primal) vector $x \geq 0$, with $Ax = 0$, represent the Lagrangean multiplier for Ω, where Ω is usually called the dual system. Then, we say (x, y, s) is a δ-approximate center pair if $s = c - A^T y > 0$, $x > 0$, $Ax = 0$, $c^T x = n$, and

$$\|X(c - A^T y) - e\| = \|Xs - e\| \leq \delta < 1.$$

A primal homogeneous potential function is defined as

$$\phi(x) = n\log(c^T x) - \sum_{j=1}^{n} \log x_j$$

for all $x > 0$ and $Ax = 0$. Again, let $\bar{y}$ be the analytic center of Ω and $\bar{s} = c - A^T\bar{y}$. We also consider a primal-dual potential function

$$\psi(x, \bar{s}) = n\log(\bar{s}^T x) - \sum_{j=1}^{n}\log(x_j\bar{s}_j) \geq n\log n.$$

Note that

$$\psi(x, \bar{s}) = \phi(x) - \sum_{j=1}^{n}\log(\bar{s}_j) = \phi(x) - P(\Omega).$$

We have the following results whose proofs can be found in Goffin et al. [3] and Ye [8].

Lemma 2.1 *Let (x, y, s) be a δ-approximate center pair and let $(\bar{y}, \bar{s})$ be the analytic center of Ω. Then,*

(i)

$$0 \leq -\sum_{j=1}^{n} \log x_j - P(\Omega) \leq \frac{\delta^2}{2(1-\delta)}$$

and

$$0 \leq P(\Omega) - \sum_{j=1}^{n} \log s_j \leq \frac{\delta^2}{2(1-\delta)}.$$

(ii)

$$\|S^{-1}\bar{s} - e\| \leq \frac{\delta}{1-\delta}$$

and

$$\|\bar{S}^{-1}s - e\| \leq \frac{\delta}{1-2\delta}.$$

□

Let $(\hat{y}, \hat{s})$ be a δ-approximate center and let us place η (≥ 1) cuts at $\hat{y}$, that is, add η new inequalities $a_{n+i}^T y \leq a_{n+i}^T \hat{y}$, $i = 1, ..., \eta$, to Ω, and consider the new set

$$\Omega^+ = \{y : A^T y \leq c, \quad a_{n+i}^T y \leq a_{n+i}^T \hat{y}, \ i = 1, ..., \eta\}.$$

Again, we assume

$$\|a_{n+i}\| = 1, \quad i = 1, ..., \eta.$$

Then we still have the following result.

Corollary 2.1 *Let $\bar{y}^+$ be the analytic centers of Ω^+ and*

$$\bar{s}^+ = c - A^T \bar{y}^+ \in \mathbb{R}^n, \quad \bar{s}_{n+i}^+ = a_{n+i}^T \hat{y} - a_{n+i}^T \bar{y}^+, \ i = 1, ..., \eta.$$

Then,

$$P(\Omega^+) = \sum_{j=1}^{n+\eta} \log \bar{s}_j^+ \leq P(\Omega) + \sum_{i=1}^{\eta} \log(r_{n+i}) + \eta \log\left((\frac{n}{n-1})^{1/2} (\eta+1)^{\frac{\eta+1}{\eta}} \exp(\frac{-\eta-2}{\eta+1}) + \frac{\delta}{1-2\delta} \right).$$

Proof. Let $(\bar{y}, \bar{s})$ be the analytic centers of Ω. Note that we have, for $i = 1, ..., \eta$,

$$\begin{aligned}
\bar{s}^+_{n+i} &= a^T_{n+i}(\hat{y} - y^+) \\
&= a^T_{n+i}(A\bar{S}^{-2}A^T)^{-1}(A\bar{S}^{-2}A^T)(\hat{y} - \bar{y}^+) \\
&= a^T_{n+i}(A\bar{S}^{-2}A^T)^{-1}A\bar{S}^{-2}(A^T\hat{y} - A^T\bar{y}^+) \\
&= a^T_{n+i}(A\bar{S}^{-2}A^T)^{-1}A\bar{S}^{-2}(-c + A^T\hat{y} + c - A^T\bar{y}^+) \\
&= a^T_{n+i}(A\bar{S}^{-2}A^T)^{-1}A\bar{S}^{-2}(\bar{s}^+ - \hat{s}) \\
&= a^T_{n+i}(A\bar{S}^{-2}A^T)^{-1}A\bar{S}^{-1}(\bar{S}^{-1}\bar{s}^+ - \bar{S}^{-1}\hat{s}) \\
&\le \|a^T_{n+i}(A\bar{S}^{-2}A^T)^{-1}A\bar{S}^{-1}\|\|\bar{S}^{-1}\bar{s}^+ - \bar{S}^{-1}\hat{s}\| \\
&= r_{n+i}\|\bar{S}^{-1}\bar{s}^+ - e + e - \bar{S}^{-1}\hat{s}\| \\
&\le r_{n+i}(\|\bar{S}^{-1}\bar{s}^+ - e\| + \|e - \bar{S}^{-1}\hat{s}\|) \\
&\le r_{n+i}(\|\bar{S}^{-1}\bar{s}^+ - e\| + \frac{\delta}{1-2\delta}) \quad \text{(from (ii) of Lemma 2.1).}
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\exp P(\Omega^+)}{(\prod_i^\eta r_{n+i})\exp P(\Omega)} &= \prod_{i=1}^{\eta}\frac{\bar{s}^+_{n+i}}{r_{n+i}}\prod_{j=1}^{n}\frac{\bar{s}^+_j}{\bar{s}_j} \\
&\le (\|\bar{S}^{-1}\bar{s}^+ - e\| + \frac{\delta}{1-2\delta})^\eta \prod_{j=1}^{n}\frac{\bar{s}^+_j}{\bar{s}_j}.
\end{aligned}$$

Again let $\alpha = \bar{S}^{-1}\bar{s}^+ > 0$. Note that $e^T\alpha = n$ and, thereby, $\prod_{j=1}^n \alpha_j \le 1$. Then,

$$\begin{aligned}
\frac{\exp P(\Omega^+)}{(\prod_i^\eta r_{n+i})\exp P(\Omega)} &\le (\|\alpha - e\| + \frac{\delta}{1-2\delta})^\eta \prod_{j=1}^{n}\alpha_j \\
&= \left(\|\alpha - e\|(\prod_{j=1}^{n}\alpha_j)^{1/\eta} + \frac{\delta}{1-2\delta}(\prod_{j=1}^{n}\alpha_j)^{1/\eta}\right)^\eta \\
&\le \left(\|\alpha - e\|(\prod_{j=1}^{n}\alpha_j)^{1/\eta} + \frac{\delta}{1-2\delta}\right)^\eta.
\end{aligned}$$

From Theorem 2.1, we have

$$\|\alpha - e\|^\eta(\prod_{j=1}^{n}\alpha_j) \le \left(\frac{n}{n-1}\right)^{\eta/2}(\eta+1)^{\eta+1}\exp(-\eta(\eta+2)/(\eta+1)),$$

which gives the desired result.

$\square$

We now describe the algorithm. Recall that $\Omega^0 = \{y \in \mathbb{R}^m : 0 \le y \le e\}$ and there exists an oracle which for every $\hat{y} \in \mathbb{R}^m$ either returns $\hat{y} \in \Gamma$ or generates separating hyperplanes, $\{y : a_i^T y \le a_i^T \hat{y}\} \supset \Gamma$, with $\|a_i\| = 1$.

The analytic center cutting plane algorithm:

- **Step 0:** Let

$$A^0 = (I, -I) \in \mathbb{R}^{m\times 2m}, \quad c^0 = \begin{pmatrix} e \\ 0 \end{pmatrix} \in \mathbb{R}^{2m},$$

$$y^0 = \frac{1}{2}e \in \mathbb{R}^m, \quad s^0 = c^0 - (A^0)^T y^0 = \frac{1}{2}e \in \mathbb{R}^{2m}, \quad x^0 = 2e \in \mathbb{R}^{2m},$$

  and

$$\delta = 1/4, \quad k = 0 \quad \text{and} \quad n(0) = 0.$$

- **Step 1:** Let (x^k, y^k, s^k) be a δ-approximate analytic center pair of $\Omega^k = \{y \in \mathbb{R}^m : c^k - (A^k)^T y \ge 0\}$, i.e., $s^k = c^k - (A^k)^T y^k > 0$, $x^k > 0$, $A^k x^k = 0$, $(c^k)^T x^k = 2m + n(k)$, and $\|X^k s^k - e\| \le \delta$. Then, Query the oracle to see if $y^k \in \Gamma$ or not.

  If yes, stop; otherwise generate hyperplanes, $i = 1, ..., \eta(k)$, $\{y : a_{n(k)+i}^T y \le a_{n(k)+i}^T y^k\} \supset \Gamma$ with $\|a_{n(k)+i}\| = 1$, and let

$$\Omega^{k+1} = \{y \in \mathbb{R}^m : c^{k+1} - (A^{k+1})^T y \ge 0\}$$

  where

$$A^{k+1} = \left(A^k, a_{n(k)+i}, ..., a_{n(k)+\eta(k)}\right)$$

  and

$$c^{k+1} = \begin{pmatrix} c^k \\ a_{n(k)+1}^T y^k \\ \cdots \\ a_{n(k)+\eta(k)}^T y^k \end{pmatrix} \in \mathbb{R}^{2m+n(k)+\eta(k)}.$$

- **Step 2:** Compute a δ-approximate center pair for Ω^{k+1} and set $n(k+1) = n(k) + \eta(k)$, $k := k+1$, and return to Step 1.

$\square$

Here, $\eta(k)$ denotes the number of cuts added at iteration k and $n(k)$ represents the number of total cuts at the beginning of iteration k, where $k = 0, 1, \ldots$. We will elaborate how to perform Step 2 later.

From Theorem 2.1, its Corollary 2.1, and noting that $\delta = 1/4$, $\eta(k) \geq 1$ and $n \geq 4$, clearly the following relations hold, provided that termination has not occurred: for all $k \geq 0$,

$$\Gamma \subset \Omega^k, \tag{2.3}$$

and

$$\begin{aligned} & P(\Omega^{k+1}) \\ \leq \;& P(\Omega^k) + \sum_{i=1}^{\eta(k)} \log(r_{n(k)+i}) + \eta(k) \log\left((\frac{4}{3})^{1/2} (\eta(k)+1)^{\frac{\eta(k)+1}{\eta(k)}} \exp(\frac{-\eta(k)-2}{\eta(k)+1}) + \frac{\delta}{1-2\delta} \right) \\ \leq \;& P(\Omega^k) + \sum_{i=1}^{\eta(k)} \log(r_{n(k)+i}) + \eta(k) \log(\eta(k)+1) \end{aligned} \tag{2.4}$$

where

$$r_{n(k)+i} = \sqrt{a_{n(k)+i}^T (A^k (\bar{S}^k)^{-2} (A^k)^T)^{-1} a_{n(k)+i}},$$

and $(\bar{y}^k, \bar{s}^k)$ denotes the exact analytic center of Ω^k. We remark that the exact analytic center is used only in our analysis and it is not needed in computation.

3 Bound on the total number of cuts

First, we have the following lemma to bound $P(\Omega^k)$.

Lemma 3.1 *(Goffin et al. [3]) For all $k \geq 0$,*

$$P(\Omega^k) \geq (2m + n(k)) \log \epsilon.$$

Proof. From (2.3), $\Gamma \subset \Omega^k$. Thus Ω^k contains a full dimensional ball with ϵ radius. Let the center of this ball be $\bar{y}$. Then $c^k - (A^k)^T \bar{y} \geq \epsilon e$; thus

$$P(\Omega^k) = \sum_{j=1}^{2m+n(k)} \log(c^k - (A^k)^T y^k)_j \geq \sum_{j=1}^{2m+n(k)} \log(c^k - (A^k)^T \bar{y})_j \geq \sum_{j=1}^{2m+n(k)} \log \epsilon.$$

□

Lemma 3.2 *Let $s = c^k - (A^k)^T y \geq 0$ for any $y \in \Omega^k$. Then*

1. $0 \leq s_j \leq 1, \quad j = 1, \ldots, 2m$
2. $0 \leq s_j \leq \sqrt{m}, \quad j = 2m+1, \ldots, 2m+n(k)$.

Proof. The proof is almost identical to the one in [3]; we include it here for completeness.

For $j = 1, \ldots, m$, $s_j = 1 - y_j$; since $0 \leq y_j \leq 1$, $0 \leq s_j \leq 1$.

For $j = m+1, \ldots, 2m$, $s_j = y_{j-m}$; since $0 \leq y_{j-m} \leq 1$, $0 \leq s_j \leq 1$.

For $j = 2m+1, \ldots, 2m+n(k)$,

$$s_j = a_{j-2m}^T y^i - a_{j-2m}^T y \leq \|a_{j-2m}\| \; \|y^i - y\| = \|y^i - y\| \leq \sqrt{m}$$

for some $y^i \in \Omega^0$ $(0 \leq i < k)$. Note that the last inequality is due to the fact that, for all pairs $y^i, \; y \in \Omega^0$,

$$-e \leq y^i - y \leq e.$$

□

Lemma 3.3 *Let $s = c^k - (A^k)^T y > 0$, and let $B^0 = 8I$ and*

$$B^{k+1} = B^k + \frac{1}{m} \sum_{i=1}^{\eta(k)} a_{n(k)+i} a_{n(k)+i}^T.$$

Then

$$A^k S^{-2} (A^k)^T \succeq B^k;$$

that is,

$$A^k S^{-2} (A^k)^T - B^k$$

is positive semi-definite.

Proof. Let $Y = \operatorname{diag}(y)$. Then

$$\begin{aligned}
A^k S^{-2} (A^k)^T &= Y^{-2} + (I - Y)^{-2} + \sum_{j=1}^{n(k)} \frac{a_j a_j^T}{(s_{2m+j})^2} \\
&\succeq Y^{-2} + (I - Y)^{-2} + \frac{1}{m} \sum_{j=1}^{n(k)} a_j a_j^T \quad \text{(by Lemma 3.2)} \\
&\succeq 8I + \frac{1}{m} \sum_{j=1}^{n(k)} a_j a_j^T \quad (\text{as } 0 < y < e) \\
&= B^k.
\end{aligned}$$

$\square$

The above lemma leads to the following lemma.

Lemma 3.4 *Let $\bar{y}^k$ be the analytic center of Ω^k, $\bar{s}^k = c^k - (A^k)^T\bar{y}^k$ and, for $i = 1, ..., \eta(k)$, $(\omega_{n(k)+i})^2 = a^T_{n(k)+i}(B^k)^{-1}a_{n(k)+i}$. Then, for $i = 1, ..., \eta(k)$,*

$$(\omega_{n(k)+i})^2 \geq a^T_{n(k)+i}(A^k(\bar{S}^k)^{-2}(A^k)^T)^{-1}a_{n(k)+i} = (r_{n(k)+i})^2. \tag{3.1}$$

$\square$

This lemma implies that upper bounds on the series of $(\omega_.)^2$ will lead to same bounds on the series $(r_.)^2$. The following lemma is the key to establish our main result.

Lemma 3.5 *Let $1 \leq \eta(k) \leq \eta \leq m$ for all k. Then,*

$$\sum_{j=1}^{n(k+1)} (\omega_j)^2 \leq \frac{18m^2}{15}\log(1 + \frac{n(k+1)}{8m^2}).$$

Proof. Note that

$$\det B^{k+1} = \det(B^k + \frac{1}{m}\sum_{i=1}^{\eta(k)} a_{n(k)+i}a^T_{n(k)+i}) = (1 + \frac{\omega^2}{m})\det(B^k + \frac{1}{m}\sum_{i=2}^{\eta(k)} a_{n(k)+i}a^T_{n(k)+i}),$$

where

$$\omega^2 = a^T_{n(k)+1}(B^k + \frac{1}{m}\sum_{i=2}^{\eta(k)} a_{n(k)+i}a^T_{n(k)+i})^{-1}a_{n(k)+1}.$$

Clearly,

$$\omega^2 \leq a^T_{n(k)+1}(B^k)^{-1}a_{n(k)+1} = (\omega_{n(k)+1})^2.$$

On the other hand, consider the matrix

$$I + \frac{1}{m}\sum_{i=2}^{\eta(k)}(B^k)^{-1/2}a_{n(k)+i}a^T_{n(k)+i}(B^k)^{-1/2}.$$

We claim that its largest eigenvalue,

$$\lambda \leq 9/8.$$

This is because that for any $y \in \mathbb{R}^m$ and $\|y\| = 1$, we have

$$
\begin{aligned}
& y^T \left(I + \frac{1}{m} \sum_{i=2}^{\eta(k)} (B^k)^{-1/2} a_{n(k)+i} a_{n(k)+i}^T (B^k)^{-1/2} \right) y \\
= \; & \|y\|^2 + \frac{1}{m} \sum_{i=2}^{\eta(k)} (y^T (B^k)^{-1/2} a_{n(k)+i})^2 \\
\leq \; & \|y\|^2 + \frac{1}{m} \sum_{i=2}^{\eta(k)} \|y\|^2 \|(B^k)^{-1/2} a_{n(k)+i})\|^2 \\
= \; & 1 + \frac{1}{m} \sum_{i=2}^{\eta(k)} a_{n(k)+i}^T (B^k)^{-1} a_{n(k)+i} \\
\leq \; & 1 + \frac{1}{m} \sum_{i=2}^{\eta(k)} \|a_{n(k)+i}\|^2/8 \quad (B^k \succeq 8I) \\
\leq \; & 1 + \frac{\eta(k) - 1}{8m} \leq 9/8.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\omega^2 &= a_{n(k)+1}^T (B^k + \frac{1}{m} \sum_{i=2}^{\eta(k)} a_{n(k)+i} a_{n(k)+i}^T)^{-1} a_{n(k)+1} \\
&= a_{n(k)+1}^T (B^k)^{-1/2} \left(I + \frac{1}{m} \sum_{i=2}^{\eta(k)} (B^k)^{-1/2} a_{n(k)+i} a_{n(k)+i}^T (B^k)^{-1/2} \right)^{-1} (B^k)^{-1/2} a_{n(k)+1} \\
&\geq a_{n(k)+1}^T (B^k)^{-1/2} (\frac{8}{9} I) (B^k)^{-1/2} a_{n(k)+1} \\
&= \frac{8}{9} (\omega_{n(k)+1})^2.
\end{aligned}
$$

Continue this process for $i = 2, ..., \eta(k)$, we have

$$
\log \det B^{k+1} \geq \log \det B^k + \sum_{i=1}^{\eta(k)} \log(1 + \frac{8(\omega_{n(k)+i})^2}{9m}).
$$

But, for $i = 1, ..., \eta(k)$,

$$
\frac{8(\omega_{n(k)+i})^2}{9m} \leq \frac{8}{9m} \frac{1}{8} \|a_{n(k)+i}\|^2 = \frac{1}{9m} \leq \frac{1}{9};
$$

hence

$$
\log(1 + \frac{8(\omega_{n(k)+i})^2}{9m}) \quad \geq \quad \frac{8(\omega_{n(k)+i})^2}{9m} - \frac{(\frac{8(\omega_{n(k)+i})^2}{9m})^2}{2(1 - \frac{8(\omega_{n(k)+i})^2}{9m})}
$$

$$
\begin{aligned}
&= \frac{8(\omega_{n(k)+i})^2}{9m}\left(1 - \frac{\frac{8(\omega_{n(k)+i})^2}{9m}}{2(1 - \frac{8(\omega_{n(k)+i})^2}{9m})}\right) \\
&\geq \frac{15(\omega_{n(k)+i})^2}{18m}.
\end{aligned}
$$

Thus we have

$$
\log\det B^{k+1} \geq \log\det B^k + \sum_{i=1}^{\eta(k)} \frac{15(\omega_{n(k)+i})^2}{18m}.
$$

Using induction on k, we have

$$
\log\det B^{k+1} \geq \log\det B^0 + \sum_{j=1}^{n(k+1)} \frac{15(\omega_j)^2}{18m} = m\log 8 + \sum_{j=1}^{n(k+1)} \frac{15(\omega_j)^2}{18m}.
$$

But

$$
\frac{1}{m}\log\det B^{k+1} \leq \log\frac{\text{trace } B^{k+1}}{m} = \log(8 + \frac{n(k+1)}{m^2}).
$$

Thus

$$
\sum_{j=1}^{n(k+1)} \frac{15(\omega_j)^2}{18m} \leq m\log(8 + \frac{n(k+1)}{m^2}) - m\log 8
$$

or

$$
\sum_{j=1}^{n(k+1)} (\omega_j)^2 \leq \frac{18m^2}{15}\log(1 + \frac{n(k+1)}{8m^2}).
$$

□

Now we present our main result.

Theorem 3.1 *Let the number of cuts generated in each step of the analytic center cutting plane algorithm be between* 1 *and* η ($\leq m$)*. Then, the algorithm stops with a solution in* Γ *as soon as* $n(k+1)$ *satisfies:*

$$
\frac{\epsilon^2}{(\eta+1)^2} \geq \frac{\frac{m}{2} + \frac{18m^2}{15}\log(1 + \frac{n(k+1)}{8m^2})}{2m + n(k+1)}.
$$

Proof. If the algorithm does not terminate, from relation (2.4) and Lemma 3.1

$$
\begin{aligned}
(2m + n(k+1))\log\epsilon &\leq P(\Omega^{k+1}) \\
&\leq P(\Omega^k) + \frac{1}{2}\sum_{i=1}^{\eta(k)} \log(r_{n(k)+i})^2 + \eta(k)\log(\eta(k)+1)
\end{aligned}
$$

$$
\begin{aligned}
&\leq P(\Omega^k) + \frac{1}{2}\sum_{i=1}^{\eta(k)} \log(r_{n(k)+i})^2 + \eta(k)\log(\eta+1) \\
&\leq P(\Omega^0) + \frac{1}{2}\sum_{j=1}^{n(k+1)} \log(r_j)^2 + n(k+1)\log(\eta+1) \\
&= 2m\log\frac{1}{2} + \frac{1}{2}\sum_{j=1}^{n(k+1)} \log(r_j)^2 + n(k+1)\log(\eta+1).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\log\epsilon - \log(\eta+1) &\leq \frac{1}{2(2m+n(k+1))}\left[2m\log\frac{1}{4} + \sum_{j=1}^{n(k+1)} \log(r_j)^2\right] \\
&\leq \frac{1}{2}\log\frac{2m\frac{1}{4} + \sum_{j=1}^{n(k+1)}(r_j)^2}{2m+n(k+1)} \quad \text{(from the concavity of } \log) \\
&\leq \frac{1}{2}\log\frac{\frac{m}{2} + \sum_{j=1}^{n(k+1)}(\omega_j)^2}{2m+n(k+1)} \quad \text{(from Lemma 3.4)} \\
&\leq \frac{1}{2}\log\frac{\frac{m}{2} + \frac{18m^2}{15}\log(1+\frac{n(k+1)}{8m^2})}{2m+n(k+1)} \quad \text{(from Lemma 3.5)}
\end{aligned}
$$

or

$$
\frac{\epsilon^2}{(\eta+1)^2} \leq \frac{\frac{m}{2} + \frac{18m^2}{15}\log(1+\frac{n(k+1)}{8m^2})}{2m+n(k+1)}. \tag{3.2}
$$

This proves the theorem.

□

Theorem 3.1 implies that the complexity of the analytic center cutting plane algorithm, counted by the total number of cuts generated from the oracle, is $O^*(\frac{\eta^2m^2}{\epsilon^2})$; the notation O^* means that lower order terms are ignored.

4 Bound on the total number of Newton's steps

In each iteration of the cutting plane algorithm, we need to compute an approximate analytic center pair of Ω^{k+1}. This can be done by using an LP interior-point algorithm, which is based on Newton's method. The number of Newton's steps is bounded by $O(\sqrt{2m+n(k+1)}\log(1/\epsilon))$. Let k^* be the first k such that $n(k^*+1)$ satisfies the inequality in Theorem 3.1. Then, the complexity of the analytic center cutting plane algorithm, counted by the total number of Newton's steps, is bounded by

$$
O^*((k^*+1)\frac{m\eta}{\epsilon}\log\frac{1}{\epsilon}) \leq O^*(\frac{m^3\eta^3}{\epsilon^3}\log\frac{1}{\epsilon}).
$$

However, in this section we show that a (primal) potential reduction algorithm will lead to a much lower bound for the total number of Newton's steps. For

$$\Omega^k = \{y \in \mathbb{R}^m : \quad c^k - (A^k)^T y \geq 0\},$$

let the primal homogeneous potential function be

$$\phi^k(x) := (2m + n(k)) \log((c^k)^T x) - \sum_{j=1}^{2m+n(k)} \log x_j$$

for all $x > 0$ and $A^k x = 0$. Let $\bar{y}^k$ denote the analytic center of Ω^k and $\bar{s}^k = c^k - (A^k)^T \bar{y}^k$. Note again that the primal-dual potential function

$$\psi^k(x, \bar{s}^k) = (2m + n(k)) \log((c^k)^T x) - \sum_{j=1}^{2m+n(k)} \log(x_j \bar{s}_j^k)$$

$$= \phi^k(x) - \sum_{j=1}^{2m+n(k)} \log(\bar{s}_j^k) = \phi(x) - P(\Omega^k) \geq (2m + n(k)) \log(2m + n(k)).$$

At Step 2 of the algorithm, starting from a $\hat{x} > 0$ with $A^{k+1}\hat{x} = 0$, we apply the (primal) potential reduction algorithm to reduce $\phi^{k+1}(x)$ (e.g., Ye [10]). Each step of the algorithm uses a Newton's step to reduce $\phi^{k+1}(x)$ by a *constant*, as long as a (y, s), a δ-approximate analytic center for Ω^{k+1} satisfying

$$\|\frac{1}{\mu} Xs - e\| \leq \delta, \quad \text{where} \quad \mu = \frac{(c^{k+1})^T x}{2m + n(k+1)} = \frac{\bar{s}^T x}{2m + n(k+1)} \text{ and } s = c^{k+1} - (A^{k+1})^T y,$$

is not yet generated. On the other hand, if (x, y, s) satisfy this inequality, then we let $(y^{k+1}, s^{k+1}) = (y, s)$ and $x^{k+1} = x/\mu$. Note that $(c^{k+1})^T x^{k+1} = 2m + n(k+1)$. Hence, $(x^{k+1}, y^{k+1}, s^{k+1})$ is a δ-approximate center pair for Ω^{k+1}. Thus, the total number of Newton's steps to complete iteration k, denoted by $\mathcal{N}(k)$, is bounded by

$$\gamma \mathcal{N}(k) \leq \phi^{k+1}(\hat{x}) - P(\Omega^{k+1}) - (2m + n(k+1)) \log(2m + n(k+1)), \tag{4.1}$$

for a *constant* $0 < \gamma < 1$.

Given (x^k, y^k, s^k), the δ-approximate analytic center pair of Ω^k, i.e., $s^k = c^k - (A^k)^T y^k > 0$, $x^k > 0$, $A^k x^k = 0$, $(c^k)^T x^k = 2m + n(k)$, and

$$\|X^k s^k - e\| \leq \delta,$$

we now compute a δ-approximate analytic center pair of Ω^{k+1}, where

$$\Omega^{k+1} = \{y \in \mathbb{R}^m : c^{k+1} - (A^{k+1})^T y \geq 0\},$$

$$A^{k+1} = \left(A^k, a_{n(k)+i}, \ldots, a_{n(k)+\eta(k)}\right),$$

and

$$c^{k+1} = \begin{pmatrix} c^k \\ a^T_{n(k)+1} y^k \\ \cdots \\ a^T_{n(k)+\eta(k)} y^k \end{pmatrix} \in \mathbb{R}^{2m+n(k+1)}.$$

Using the technique of Mitchell and Todd [5], we first construct a $\hat{x} \in \mathbb{R}^{2m+n(k+1)}$ from x^k such that $A^{k+1}\hat{x} = 0$ with $\hat{x} > 0$. Let

$$a^k = \sum_{i=1}^{\eta(k)} a_{n(k)+i}$$

and

$$r^k = \sqrt{(a^k)^T (A^k X^k (S^k)^{-1} (A^k)^T)^{-1} a^k}, \tag{4.2}$$

$$\Delta x = -(\beta / r^k) X^k (S^k)^{-1} (A^k)^T (A^k X^k (S^k)^{-1} (A^k)^T)^{-1} a^k. \tag{4.3}$$

Then we assign

$$\hat{x} = \begin{pmatrix} x^k + \Delta x \\ \beta / r^k \\ \cdots \\ \beta / r^k \end{pmatrix} \in \mathbb{R}^{2m+n(k+1)},$$

where $\beta = 1/2$. It can be readily verified that

$$A^{k+1}\hat{x} = A^k x^k - (\beta/r^k) a^k + (\beta/r^k) a^k = 0,$$

$$\begin{aligned}
(c^{k+1})^T \hat{x} &= (c^k)^T x^k + (c^k)^T \Delta x + (\beta/r^k)(y^k)^T a^k \\
&= (c^k)^T x^k + (c^k - (A^k)^T y^k + (A^k)^T y^k)^T \Delta x + (\beta/r^k)(y^k)^T a^k \\
&= (c^k)^T x^k + (s^k)^T \Delta x + (y^k)^T A^k \Delta x + (\beta/r^k)(y^k)^T a^k \\
&= (c^k)^T x^k + (y^k)^T A^k \Delta x + (\beta/r^k)(y^k)^T a^k \quad (\text{since } A^k x^k = 0) \\
&= (c^k)^T x^k - (\beta/r^k)(y^k)^T a^k + (\beta/r^k)(y^k)^T a^k \\
&= (c^k)^T x^k = 2m + n(k),
\end{aligned}$$

and

$$x^k + \Delta x = x^k - (\beta/r^k)X^k(S^k)^{-1}(A^k)^T(A^kX^k(S^k)^{-1}(A^k)^T)^{-1}a^k = X^k(e - p^k),$$

where

$$p^k = (\beta/r^k)(S^k)^{-1}(A^k)^T(A^kX^k(S^k)^{-1}(A^k)^T)^{-1}a^k.$$

Note that we have

$$\|p^k\| \le \frac{\beta}{\sqrt{1-\delta}}. \tag{4.4}$$

Thus, we have $\hat{x} > 0$ as $\delta = 1/4$ and $\beta = 1/2$.

Consider the primal potential function for Ω^{k+1}

$$\phi^{k+1}(x) = (2m + n(k+1))\log((c^{k+1})^T x) - \sum_{j=1}^{2m+n(k+1)} \log x_j$$

for all $x > 0$ and $A^{k+1}x = 0$. Then

$$\begin{aligned}
&\phi^{k+1}(\hat{x})\\
&= (2m + n(k+1))\log((c^{k+1})^T\hat{x}) - \sum_{j=1}^{2m+n(k+1)} \log \hat{x}_j\\
&= (2m + n(k+1))\log(2m + n(k)) - \sum_{j=2m+n(k)+1}^{2m+n(k+1)} \log \hat{x}_j - \sum_{j=1}^{2m+n(k)} \log \hat{x}_j\\
&= (2m + n(k+1))\log(2m + n(k)) - \eta(k)\log(\beta/r^k) - \sum_{j=1}^{2m+n(k)} \log \hat{x}_j\\
&= (2m + n(k+1))\log(2m + n(k)) - \eta(k)\log(\beta/r^k) - \sum_{j=1}^{2m+n(k)} \log x_j^k - \sum_{j=1}^{n(k)} \log(1 - p_j^k)\\
&\le (2m + n(k+1))\log(2m + n(k)) - \eta(k)\log(\beta/r^k) + P(\Omega^k) + \frac{\delta^2}{2(1-\delta)} - \sum_{j=1}^{n(k)} \log(1 - p_j^k)\\
&\quad \text{(from (i) of Lemma 2.1)}\\
&\le (2m + n(k+1))\log(2m + n(k)) - \eta(k)\log(\beta/r^k) + P(\Omega^k) + \frac{\delta^2}{2(1-\delta)}\\
&\quad + \frac{\beta^2}{2(1-\delta-\beta\sqrt{1-\delta})} \quad (\text{as } \|p^k\| \le \beta/\sqrt{1-\delta}\ )\\
&\le (2m + n(k+1))\log(2m + n(k)) + \eta(k)\log r^k - \eta(k)\log\beta + P(\Omega^k) + 1.
\end{aligned}$$

Thus, from relation (4.1) the number of Newton's steps of the potential reduction algorithm at iteration k, starting from $\hat{x}$, is bounded by

$\gamma\mathcal{N}(k)$

$$
\begin{aligned}
&\le \phi^{k+1}(\hat{x}) - P(\Omega^{k+1}) - (2m+n(k+1))\log(2m+n(k+1))\\
&\le (2m+n(k+1))\log(2m+n(k)) - (2m+n(k+1))\log(2m+n(k+1))\\
&\quad +1+\eta(k)\log r^k - \eta(k)\log\beta + P(\Omega^k) - P(\Omega^{k+1})\\
&\le (2m+n(k+1))\log(2m+n(k+1)-1) - (2m+n(k+1))\log(2m+n(k+1))\\
&\quad +1+\eta(k)\log r^k - \eta(k)\log\beta + P(\Omega^k) - P(\Omega^{k+1}) \quad (\text{since } n(k+1)\ge n(k)+1\ )\\
&= (2m+n(k+1))\log(1-\frac{1}{2m+n(k+1)})+1\\
&\quad +\eta(k)\log r^k - \eta(k)\log\beta + P(\Omega^k) - P(\Omega^{k+1})\\
&\le \eta(k)\log r^k - \eta(k)\log\beta + P(\Omega^k) - P(\Omega^{k+1}).
\end{aligned}
$$

Therefore, the total number of Newton's steps till the algorithm terminates

$$
\begin{aligned}
\gamma\sum_{j=0}^{k^*}\mathcal{N}(j) &\le \sum_{j=0}^{k^*}\Big(\eta(j)\log(r^j) - \eta(j)\log\beta\Big) + P(\Omega^0) - P(\Omega^{k^*+1})\\
&= \frac{1}{2}\sum_{j=0}^{k^*}(\eta(j)\log(r^j)^2) + P(\Omega^0) - P(\Omega^{k^*+1}) - n(k^*+1)\log\beta\\
&= \frac{1}{2}\sum_{j=0}^{k^*}(\eta(j)\log(r^j)^2) + 2m\log\frac{1}{2} - P(\Omega^{k^*+1}) - n(k^*+1)\log\beta.
\end{aligned}
$$

Let

$$(\omega^k)^2 = (a^k)^T(B^k)^{-1}a^k,$$

where B^k is defined in Lemma 3.3 of the last section and

$$a^k = \sum_{i=1}^{\eta(k)} a_{n(k)+i}.$$

Then, from Lemma 3.3

$$
\begin{aligned}
(\omega^k)^2 &\ge (a^k)^T(A^k(S^k)^{-2}(A^k)^T)^{-1}a^k\\
&\ge (1-\delta)(a^k)^T(A^kX^k(S^k)^{-1}(A^k)^T)^{-1}a^k\\
&= (1-\delta)(r^k)^2, \qquad (4.5)
\end{aligned}
$$

since

$$A^kX^k(S^k)^{-1}(A^k)^T \succeq (1-\delta)A^k(S^k)^{-2}(A^k)^T.$$

Thus, the total number of Newton's steps till the algorithm terminates

$$\gamma\sum_{j=0}^{k^*}\mathcal{N}(j)$$

$$
\begin{aligned}
&\leq \frac{1}{2}\sum_{j=0}^{k^*}(\eta(j)\log(\omega^j)^2) - \sum_{j=0}^{k^*}(\eta(j)\log(\sqrt{1-\delta})) + 2m\log\frac{1}{2} - P(\Omega^{k^*+1}) - n(k^*+1)\log\beta \\
&= \frac{1}{2}\left(\sum_{j=0}^{k^*}(\eta(j)\log(\omega^j)^2) + 2m\log\frac{1}{4}\right) - P(\Omega^{k^*+1}) - n(k^*+1)\log(\beta\sqrt{1-\delta}) \\
&\leq \frac{2m+n(k^*+1)}{2}\log\frac{\frac{2m}{4}+\sum_{j=0}^{k^*}\eta(j)(\omega^j)^2}{2m+n(k^*+1)} - P(\Omega^{k^*+1}) - n(k^*+1)\log(\beta\sqrt{1-\delta}) \\
&\quad \text{(from the concavity of } \log) \\
&\leq \frac{2m+n(k^*+1)}{2}\log\frac{\frac{m}{2}+\eta\sum_{j=0}^{k^*}(\omega^j)^2}{2m+n(k^*+1)} - P(\Omega^{k^*+1}) - n(k^*+1)\log(\beta\sqrt{1-\delta}). \qquad (4.6)
\end{aligned}
$$

Moreover, we have

Lemma 4.1 *Let $1 \leq \eta(k) \leq \eta \leq m$ for all k. Then,*

$$\sum_{j=0}^{k}(\omega^j)^2 \leq \frac{14}{13}\eta m^2\log(1+\frac{n(k+1)}{8m^2}).$$

Proof. Note that

$$\det B^{k+1} = \det(B^k + \frac{1}{m}\sum_{i=1}^{\eta(k)} a_{n(k)+i}a_{n(k)+i}^T) \geq (1+\frac{(\omega^k)^2}{\eta(k)m})\det(B^k),$$

where we used the fact that

$$B^{k+1} \succeq B^k + \frac{1}{\eta(k)m}a^k(a^k)^T$$

since

$$\sum_{i=1}^{\eta(k)} a_{n(k)+i}a_{n(k)+i}^T \succeq \frac{1}{\eta(k)}a^k(a^k)^T.$$

Furthermore,

$$\frac{(\omega^k)^2}{\eta(k)m} \leq \frac{1}{8\eta(k)m}\|a^k\|^2 \leq \frac{\eta(k)}{8\eta(k)m}(\sum_{i=1}^{\eta(x)}\|a_{n(k)+i}\|^2) = \frac{\eta(k)}{8m} \leq \frac{1}{8};$$

hence

$$
\begin{aligned}
\log(1+\frac{(\omega^k)^2}{\eta(k)m}) &\geq \frac{(\omega^k)^2}{\eta(k)m} - \frac{(\frac{(\omega^k)^2}{\eta(k)m})^2}{2(1-\frac{(\omega^k)^2}{\eta(k)m})} \\
&= \frac{(\omega^k)^2}{\eta(k)m}(1-\frac{\frac{(\omega^k)^2}{\eta(k)m}}{2(1-\frac{(\omega^k)^2}{\eta(k)m})}) \\
&\geq \frac{13(\omega^k)^2}{14\eta(k)m}.
\end{aligned}
$$

Hence, we have

$$\log\det B^{k+1} \geq \log\det B^k + \frac{13(\omega^k)^2}{14\eta(k)m}.$$

Using induction on k, we have

$$\log\det B^{k+1} \geq \log\det B^0 + \sum_{j=0}^{k} \frac{13(\omega^j)^2}{14\eta(j)m} = m\log 8 + \sum_{j=0}^{k} \frac{13(\omega^j)^2}{14\eta(j)m}.$$

But again

$$\frac{1}{m}\log\det B^{k+1} \leq \log\frac{\text{trace } B^{k+1}}{m} = \log(8 + \frac{n(k+1)}{m^2}).$$

Thus

$$\sum_{j=0}^{k} \frac{13(\omega^j)^2}{14\eta(j)m} \leq m\log(8 + \frac{n(k+1)}{m^2}) - m\log 8$$

or

$$\sum_{j=0}^{k} (\omega^j)^2 \leq \frac{14}{13}\eta m^2 \log(1 + \frac{n(k+1)}{8m^2}).$$

$\square$

From relation (4.6) and the above lemma, the total number of Newton's steps

$$\begin{aligned}
&\gamma\sum_{j=0}^{k^*}\mathcal{N}(j)\\
\leq\ & \frac{2m+n(k^*+1)}{2}\log\frac{\frac{2m}{4} + \frac{14\eta^2m^2}{13}\log(1+\frac{n(k^*+1)}{8m^2})}{2m+n(k^*+1)}\\
&-P(\Omega^{k^*+1}) - n(k^*+1)\log(\beta\sqrt{1-\delta})\\
\leq\ & \frac{2m+n(k^*+1)}{2}\log\frac{\frac{\eta^2m}{2} + \frac{18\eta^2m^2}{15}\log(1+\frac{n(k^*+1)}{8m^2})}{2m+n(k^*+1)}\\
&-P(\Omega^{k^*+1}) - n(k^*+1)\log(\beta\sqrt{1-\delta})\\
\leq\ & \frac{2m+n(k^*+1)}{2}\log\frac{\frac{\eta^2m}{2} + \frac{18\eta^2m^2}{15}\log(1+\frac{n(k^*+1)}{8m^2})}{2m+n(k^*+1)}\\
&-(2m+n(k^*+1))\log\epsilon - n(k^*+1)\log(\beta\sqrt{1-\delta}) \quad \text{(from Lemma 3.1)}\\
=\ & \frac{2m+n(k^*+1)}{2}\log\frac{\frac{\eta^2m}{2} + \frac{18\eta^2m^2}{15}\log(1+\frac{n(k^*+1)}{8m^2})}{\epsilon^2(2m+n(k^*+1))} - n(k^*+1)\log(\beta\sqrt{1-\delta})\\
\leq\ & -n(k^*+1)\log(\beta\sqrt{1-\delta})\\
=\ & n(k^*+1)\log\frac{1}{\beta\sqrt{1-\delta}}.
\end{aligned}$$

The last inequality holds since k^* satisfies the condition of Theorem 3.1, i.e.,

$$\frac{\frac{\eta^2 m}{2} + \frac{18\eta^2 m^2}{15}\log(1+\frac{n(k^*+1)}{8m^2})}{\epsilon^2(2m+n(k^*+1))} \le \frac{\frac{(\eta+1)^2 m}{2} + \frac{18(\eta+1)^2 m^2}{15}\log(1+\frac{n(k^*+1)}{8m^2})}{\epsilon^2(2m+n(k^*+1))} \le 1.$$

Therefore, we have

Theorem 4.1 *Let the number of cuts generated in each iteration of the analytic center cutting plane algorithm be between 1 and η ($\le m$). Then, the total number of Newton's steps of the algorithm is bounded by $n(k^*+1) \le O^*(\frac{\eta^2 m^2}{\epsilon^2})$, i.e., the same bound for the total number of cuts needed to terminate the algorithm.*

□

Obviously, the total number of iterations is bounded by the total number of cutting planes. If the number of cuts in each iteration is about η, then the total number of iterations is bounded by $O^*(\frac{\eta m^2}{\epsilon^2})$, and, therefore, the average number of Newton's steps per iteration is bounded by η. We have recently learnt that Luo, in a subsequent paper [4], proved a complexity bound $O^*(\frac{\eta m^2}{\epsilon^2})$ for the total number of cuts in a similar analytic center cutting plane algorithm. His result has improved our bound by a factor η.

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