Interior-Point Algorithms: 1996 Annual Progress Report on NSF Grant DMI-9522507

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March 3, 1997

1 Introduction

This document is the first year progress report on an optimization project funded by NSF Grant DMI-9522507. The aim of the proposal is to develop and analyze interior-point algorithms and their variants (e.g., [2]) for solving large-scale linear, quadratic, and nonlinear optimization problems that arise in production management and resource allocation. It is a continuation of the PI's previous projects funded by NSF, and it is motivated by the demand to resolve important remaining theoretical and practical issues in the field. In the new project the PI will do research to strengthen existing results and to explore new techniques for improving linear and nonlinear programming algorithms. He will also do research on implementation and application to solve real-world problems.

In general, results in the area of developing efficient algorithms for solving large-scale optimization problems will be of great importance in improving the efficiency of manufacturing systems, communication networks, aircraft routing, multi-commodity-flow operations, and resources planning. Strengthening research in this area will definitely contribute to the national interest in industrial competitiveness and scientific knowledge.

More specific, my research topics include the following.

1. Complexity issues: Continue to develop the complexity theory for linear programming (LP) with data given by real numbers, analyze "condition" numbers that determine the degree of difficulty of the LP problem, and "precondition" the problem and study **strongly** polynomial-time algorithms. The latter probably is **the** most challenging open question in linear programming.

2. Algorithm developments: Extend LP interior-point algorithms to solving nonlinear optimization problems arising from engineering design and control. In particular, develop efficient initialization and termination techniques to generate initial points, obtain optimal basic solutions and detect infeasibility. This research has both theoretical and practical values.

3. Decomposition and column generation: Develop interior-point algorithms, coupled with various decomposition and column generation schemes, for solving large-scale, semi-infinite dimensional, and combinatorial optimization problems. This type of problems have wide industrial applications.

4. Implementation and Application: Continue to implement the state of the art interior-point algorithm for both linear and nonlinear programming, and to develop public-domain computer programs based on these implementations. Continue to apply the algorithm to solving real world problems.

5. Education: Continue to educate undergraduate and graduate students at the University of Iowa via their participating in the project. Three Ph. D. students graduated during the past and they are now faculties or post-docs in various US or overseas institutions. The PI is writing a text book on this topic and directing a computational optimization laboratory (COL) in his department. Currently, the lab has one post-doc associate, and one visiting Ph. D. students fully supported by the Brazil government.

Progresses have been made in each of these areas in 1996, and the results are better than what I expected:

- Six new publications have been resulted from the project in *SIAM J. on Optimization* and *Mathematical Programming*. Several working papers were submitted for possible publication. These papers are listed in Section 5, where [J*] for journal publication and [W*] for working paper.

- A computational optimization laboratory (COL) has been partially supported by the grant. Currently we have one post-doctor associate, and one visiting Ph.D. student who is fully supported by the Brazil government.

- Two implementation programs are being developed and enhanced: one solves linear programs and the other solves nonlinear optimization problems.

- We start a network design project for the Rockwell International at Cedar Rapids, where some of our computer programs will be used.

2 Project Progress

I briefly summarize several results obtained in 1996 and discuss ongoing research activities.

2.1 Progress in Complexity Issues

Consider a linear programming (LP) problem in the standard form:

$$\begin{array}{lll} (LP) & \text{minimize} & c^T x \\ & \text{subject to} & Ax = b, \; x \geq 0, \end{array}$$

where $c \in R^n$, $A \in R^{m \times n}$ and $b \in R^m$ are given, $x \in R^n$, and ^{T} denotes transpose.

Based on our previous, Vavasis and I [W1] recently developed a simplified proof concerning the paper [3] "A Primal-Dual Interior Point Method Whose Running Time Depends Only on the Constraint Matrix". In particular, we prove that Case II, one of the three cases in the method, can never occur. This result simplifies the proof of the main theorem and improves certain bounds established in the paper.

I [J2] study whether or not certain kinds of partial knowledge could help in solving LP problems and how they impact the complexity. If such knowledge is helpful and available, then the owner of a problem instance might be able to use it to solve the instance more efficiently. In particular, if a linear system, $Ax = b$ and $x \geq 0$, has an interior feasible point, then I show that finding a feasible point to this system can be done in $O(n^{2.5}c(A))$ iterations by the layered interior-point method, and each iteration solves a least-squares problem, where n is the dimension of vector x and $c(A)$ is the condition number of matrix A defined in [3]. This complexity bound is reduced by a factor n from that when this property does not exists. I also present a result for solving the problem using a little strong knowledge.

2.2 Progress in Algorithm Developments

There are four major results in algorithm developments.

First, Andersen and I [J4] presented an extension of a homogeneous self-dual algorithm [4] to solving the monotone complementarity problem (MCP). The algorithm does not need to use any "big-M" parameter or two-phase method, and it generates either a solution converging towards feasibility and complementarity simultaneously or a certificate proving infeasibility. Moreover, if the MCP is polynomially solvable with an interior feasible starting point, then it can be polynomially solved without using or knowing such information at all. To our knowledge, this is the first interior-point and infeasible-starting algorithm for solving the MCP that possesses these desired features. We [W3][W5] also did a computational study of the homogeneous algorithm for large-scale convex optimization, where computational results are presented.

Second, Nesterov, Todd and I [W4] developed homogeneous methods for solving conic nonlinear programs, These methods try to find a recession direction of the feasible set of a self-dual homogeneous primal-dual problem. The methods under consideration generate an ϵ-solution for an ϵ-perturbation of an initial strictly (primal and dual) feasible problem in $O(\sqrt{\nu} \ln \frac{\nu}{\epsilon \rho_f})$ iterations, where ν is the parameter of a self-concordant barrier for the cone, ϵ is a relative accuracy and ρ_f is a feasibility measure. We also discussed the behavior of path-following methods as applied to infeasible problems. We prove that strict infeasibility (primal or dual) can be detected in $O(\sqrt{\nu} \ln \frac{\nu}{\rho_.})$ iterations, where $\rho_.$ is a primal or dual infeasibility measure.

Third, in [J3] the problem of minimizing a sum of Euclidean norms is studied. This problem is convex but not everywhere differentiable. By transforming the problem into a standard convex programming problem in conic form, we show that an ϵ-optimal solution can be computed efficiently using interior-point algorithms. As applications to this problem, polynomial time algorithms are derived for the Euclidean single facility location problem, the Euclidean multi-facility location problem, and the shortest network under a given tree topology. In particular, by solving the Newton equation in linear time using *Gaussian elimination on leaves of a tree*, we present an algorithm which computes an ϵ-optimal solution to the shortest network under a given full Steiner topology interconnecting N regular points, in $O(N\sqrt{N}(\log(\bar{c}/\epsilon) + \log N))$ arithmetic operations where $\bar{c}$ is the largest pair wise distance among the given points. Previous best known result on this problem is a graphical algorithm which requires $O(N^2)$ arithmetic operations under certain conditions.

Finally, my colleagues Kortanek, Xu and I [J5] developed some theoretical and computational results for solving the posynomial geometric programming (GP) problem using an infeasible primal-dual interior-point algorithm. We applied the Newton method for a perturbed KKT system arising from the dual GP problem and develop effective techniques in choosing iterate direction and step length. Our computational results indicate that the algorithm, coupled with these new techniques, leads to an efficient and stable implementation for solving GP programs. Competitive numerical results are presented for 40 or so of the most challenging posynomial geometric programming problems in the field, where for a significant number of these we see similar numerical performance to linear programming.

2.3 Progress in Decomposition and Column Generation

Methods, which allow for column generation—the addition of columns or constraints to the system only when needed—permit a great deal of flexibility for solving optimization problems in which the number of columns is either very large or not explicitly known. Thus, we need to develop interior-point algorithms that allow decomposition or column generation and are implementable with practical efficiency. This will advance the practical solvability for a large class of optimization problems, such as semi-infinite programming, convex nonlinear programming, and combinatorial optimization.

Several approaches for column generation or decomposition were proposed and developed. One method, the analytic center cutting plane method was developed and analyzed by several people, including myself, to solve general convex problems defined by a separation oracle. The oracle is called at the analytic center of a polytope, which contains a solution set and is given by the intersection of the linear inequalities previously generated from the oracle. If the center is not in the solution set, separating hyperplanes will be placed through the center to shrink the containing polytope. Its complexity or convergence result has been established when one cutting plane is placed in each step. Thus, the question remains open when multiple cuts are added. Moreover, adding multiple cuts actually is a key to practical effectiveness and it presents theoretical difficulties in analyzing other cutting plane methods. In paper [J6], I overcome this difficulty and show that the analytic center cutting plane algorithm, with multiple cuts added in each step, still is a fully polynomial approximation algorithm.

2.4 Progress in Global Optimization

More recently, I [J1] present a potential reduction algorithm to approximate a Karush-Kuhn-Tucker (KKT) point of general quadratic programming. We show that the algorithm is a fully polynomial-time approximation scheme, and its running-time dependency on accuracy $\epsilon \in (0,1)$ is $O((\frac{1}{\epsilon})\log(\frac{1}{\epsilon})\log(\log(\frac{1}{\epsilon})))$, compared to the previously best-known result $O((\frac{1}{\epsilon})^2)$. Furthermore, the limit of the KKT point satisfies the second-order necessary optimality condition of being a local minimizer.

Moreover, Fu, Luo and I [W2] consider the problem of approximating the global minimum of a general quadratic program (QP) with n variables subject to m ellipsoidal constraints. For $m = 1$, we rigorously show that an ϵ-minimizer, where error $\epsilon \in (0,1)$, can be obtained in polynomial time, meaning that the number of arithmetic operations is a polynomial in n, m, and $\log(1/\epsilon)$. For $m \geq 2$, we present a polynomial-time $(1-\frac{1}{m^2})$-approximation algorithm as well as a semidefinite programming relaxation for this problem. In addition, we present approximation algorithms for solving QP under the box constraints and the assignment polytope constraints.

2.5 Progress in Implementation and Application

Two implementation programs are being developed and enhanced: one solves linear programs and the other solves nonlinear optimization problems. We have also started a network design project for the Rockwell International at Cedar Rapids, where some of our computer programs will be used.

3 Education and Human Resources

Beneficiaries, educationally, of the project are the Ph.D. students at the University of Iowa. We have supported several Ph.D. students at UI from this NSF grant. Some results mentioned above are part of their Ph.D. theses.

Another outgrowth of our research has been a new seminar, meeting once a week, among graduate students and faculty members from several departments. I have also brought some of the new research topics and results to an optimization and mathematical programming course at the graduate level. I am writing a text book "Interior-Point Algorithms: Theory and Analysis."

4 Publications and Technical Reports

4.1 New Journal Papers

[J1] "On the complexity of approximating a KKT point of quadratic programming," to appear in *Mathematical Programming.*
[J2] " How partial knowledge helps to solve linear programs," *Journal of Complexity* **12** (1996) 480-491.
[J3] "Efficient algorithms for minimizing a sum of Euclidean norms with applications," (G. Xue and Y. Ye), to appear in *SIAM J. Optimization.*
[J4] "On a homogeneous algorithm for the monotone complementarity problem," (E. Andersen and Y. Ye), to appear in *Mathematical Programming.*
[J5] "An infeasible interior-point algorithm for solving primal and dual geometric programs," (K. O. Kortanek, X. Xu, and Y. Ye), *Mathematical Programming* **76** (1997) 155-182.
[J6] "Complexity analysis of the analytic center cutting plane method that uses multiple cuts," to appear in *Mathematical Programming.*

4.2 New Working Papers

[W1] "A simplification to 'A Primal-Dual Interior Point Method Whose Running Time Depends Only on the Constraint Matrix'," (S. Vavasis and Y. Ye), Working Paper, Department of Management Sciences, The University of Iowa (December, 1996).

[W2] "Approximation algorithms for quadratic programming," (M. Fu, Z.-Q. Luo, and Y. Ye), Working Paper, Department of Management Sciences, The University of Iowa (July, 1996).
[W3] "A computational study of the homogeneous algorithm for large-scale convex optimization," (E. Andersen and Y. Ye), Working Paper, Department of Management Sciences, The University of Iowa (May, 1996).
[W4] "Infeasible-start primal-dual methods and infeasibility detectors for nonlinear programming problems," (Yu. Nesterov, M.J. Todd, and Y. Ye), Working Paper, Department of Management Sciences, The University of Iowa (April, 1996).
[W5] "On a Homogeneous Algorithm for a Monotone Complementarity Problem with Nonlinear Equality Constraints," (E. Andersen and Y. Ye), Working Paper, Department of Management Sciences, The University of Iowa (Jan., 1996).

References

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[2] N. Karmarkar, "A new polynomial-time algorithm for linear programming," *Combinatorica* 4 (1984) 373-395.

[3] S. Vavasis and Y. Ye, "A primal-dual interior-point method whose running time depends only on the constraint matrix," *Mathematical Programming* 74 (1996) 79-120..

[4] Y. Ye, M. J. Todd, and S. Mizuno, "An $O(\sqrt{n}L)$-iteration homogeneous and self-dual linear programming algorithm," *Mathematics of Operations Research* 19 (1994) 53-67.