On Solving the P_0 Matrix Linear Complementarity Problem

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September 1997

Abstract

In this short note we show that interior-point algorithms can solve the P_0 matrix linear complementarity problem if its solution set is nonempty and bounded. These algorithms generate an iterative sequence converging to the solution set.

Key words. linear complementarity problem, P_0 matrix, interior-point algorithm.

AMS subject classifications. 65H10, 90C30, 90C33.

*This author's work was supported by the Australian Research Council.

†This author is supported in part by NSF Grants DMI-9522507 and DMS-9703490, and by the Australian Research Council.

We are concerned with the linear complementarity problem (LCP)

$$x^T y = 0, \quad y = Mx + q, \quad \text{and} \quad (x, y) \geq 0,$$

where M is an $n \times n$ matrix and q is an n dimensional vector. We say that the LCP is feasible if there is pair (x, y) such that

$$y = Mx + q \quad \text{and} \quad (x, y) \geq 0;$$

the LCP has an interior feasible point if there is pair (x, y) such that

$$y = Mx + q \quad \text{and} \quad (x, y) > 0.$$

A matrix $M \in R^{n\times n}$ is called a

(i) P_0 matrix if

$$\max_{i:x_i\neq 0} x_i(Mx)_i \geq 0, \quad \text{for all} \quad x \in R^n, \ x \neq 0;$$

(ii) P matrix if

$$\max_{i:x_i\neq 0} x_i(Mx)_i > 0, \quad \text{for all} \quad x \in R^n, \ x \neq 0;$$

(iii) monotone matrix if

$$x^T Mx \geq 0, \quad \text{for all} \quad x \in R^n.$$

Obviously, every P matrix is a P_0 matrix and every monotone matrix is also a P_0 matrix (Cottle et al. [2]).

Recently, there are tremendous research activities on solving the P_0-matrix LCP, see Chen et al. [1] and references therein. Among these, Kojima et al. [4] have showed that if the complementarity level set,

$$S_{++}(z) = \{(x, y) : \ x^T y \leq z, \ y = Mx + q, \ (x, y) > 0\},$$

is bounded for any fixed $z \geq 0$, then several interior-point algorithms will generate a sequence converging to the solution set of the LCP.

Pardalos et al. [5] have further showed that if the potential level set,

$$S_{++,\phi}(z) = \{(x, y) : \ \phi(x, y) \leq z, \ y = Mx + q, \ (x, y) > 0\},$$

is bounded for any fixed z, then the potential reduction algorithm will solve the LCP as well, where

$$\phi(x, y) = (2n) \log(x^T y) - \sum_{i=1}^{n} \log(x_i y_i).$$

Note that the potential level set is contained in the complementarity level set, since $n \log(x^T y) \leq \phi(x, y) - n \log n$.

Facchinei and Kanzow have applied a regularization approach for solving the P_0-function nonlinear complementarity problem. In the P_0-matrix LCP case, first they define a P matrix $M(\epsilon) = M + \epsilon I$ where I is the identity matrix and $\epsilon > 0$. Let $x(\epsilon)$ be the unique solution of the $(M(\epsilon), q)$ LCP. Then, they have showed that $x(\epsilon)$ tends to an solution of the original P_0-matrix LCP as ϵ tends to 0, if the solution set of the original P_0-matrix LCP is nonempty and bounded.

In this short note we show that interior-point algorithms can solve the P_0 matrix linear complementarity problem as long as it has a nonempty and bounded solution set. In other words, they will generate an iterative sequence (x^k, y^k), such that $y^k = Mx^k + q$, $(x^k, y^k) > 0$, and

$$(x^k)^T y^k \to 0.$$

Our result is based on two results about the P_0-matrix LCP. The first result is due to Chen et al. [1] and Ravindran and Gowda [6], which states

Theorem 1 *If M is a P_0 matrix and its solution set is nonempty and bounded, then it has an interior feasible point.*

Note that the converse is not true, that is, the existence of an interior feasible point does not imply that the solution set is bounded. For instance,

Example 1

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It is not difficult to verify that M is a P_0-matrix and that this problem has an interior point $(x, y) = (1,1,1,1,1,1)^T$. However, the solution set of the LCP contains the unbounded ray $(x_1, 0, 0, 0, 0, 1)^T$ for all $x_1 \geq 0$.

The first result indicates that we can always find an initial interior feasible point (in polynomial time) to start those interior point algorithms, described by Kojima et al. [4] and Pardalos et al. [5], to solve the P_0-matrix LCP.

Thus, the remaining issue is whether the complementarity or potential level set is bounded or not. In what follows we show the second result that the level set is indeed bounded as long as the solution set is bounded.

We first define some notations. Let S_+ be the set of all feasible points, S_0 be the solution set, and $S_+(z)$ be the level set:

$$S_+(z) = \{(x, y) \in S_+ \ : \ x^T y \leq z, \ z \geq 0\}.$$

We now prove the following theorem:

Theorem 2 *If S_0 is nonempty and bounded, then $S_+(z)$ is nonempty and bounded for any positive number z.*

Proof: Since $S_0 \subseteq S_+(z)$, $S_+(z) \neq \emptyset$. We now prove it is bounded. Suppose on the contrary that there is a sequence $\{x^k, y^k\} \subseteq S_+(z)$ and $\|x^k\| \to \infty$.

Let $(x^*, y^*) \in S_0$, $d_x^k = x^k - x^*$ and $d_y^k = y^k - y^*$, and $\lambda_k = \|d_x^k\|$. Then

$$d_y^k = M d_x^k.$$

Moreover, since the solution set is bounded and $\|x^k\| \to \infty$, $\lambda_k = \|d_x^k\| \to \infty$. Note that

$$0 \leq x^k = x^* + d_x^k = x^* + \lambda_k \frac{d_x^k}{\|d_x^k\|} \tag{1}$$

and

$$0 \leq y^k = y^* + d_y^k = y^* + \lambda_k M \frac{d_x^k}{\|d_x^k\|}. \tag{2}$$

Let $(\bar{d}_x, \bar{d}_y)$ be an accumulate point of $\{d_x^k/\|d_x^k\|, d_y^k/\|d_x^k\|\}$. Since $\lambda_k \to \infty$, (1) and (2) imply

$$\|\bar{d}_x\| = 1, \quad \bar{d}_x \geq 0, \quad \text{and} \quad \bar{d}_y = M\bar{d}_x \geq 0.$$

Furthermore, from

$$\begin{aligned}
(d_x^k)^T M d_x^k &= (x^k - x^*)^T (y^k - y^*) \\
&= (x^k)^T y^k + (x^*)^T y^* - (x^k)^T y^* - (x^*)^T y^k \\
&= (x^k)^T y^k - (x^k)^T y^* - (x^*)^T y^k,
\end{aligned}$$

and $\{x^k, y^k\} \subseteq S_+(z)$, we have

$$z \geq (x^k)^T y^k = (x^k)^T y^* + (x^*)^T y^k + (d_x^k)^T M d_x^k \geq 0.$$

This, together with (1) and (2), gives

$$\begin{aligned}
z &\geq (x^* + d_x^k)^T y^* + (x^*)^T (y^* + d_y^k) + (d_x^k)^T M d_x^k \\
&= \lambda_k \left((y^*)^T \frac{d_x^k}{\|d_x^k\|} + (x^*)^T M \frac{d_x^k}{\|d_x^k\|} \right) + \lambda_k^2 \frac{(d_x^k)^T}{\|d_x^k\|} M \frac{d_x^k}{\|d_x^k\|} \\
&\geq 0.
\end{aligned}$$

Dividing both sides by λ_k and from $\lambda_k \to \infty$ and z being bounded, we have

$$(y^*)^T \bar{d}_x + (x^*)^T M \bar{d}_x + \lim_{k \to \infty} \left(\lambda_k (\bar{d}_x)^T M \bar{d}_x \right) = 0;$$

Again, since $\lambda_k \to \infty$ and $(y^*)^T \bar{d}_x + (x^*)^T M \bar{d}_x$ being bounded, we must have

$$(\bar{d}_x)^T M \bar{d}_x = 0.$$

Hence, we have

$$(y^*)^T \bar{d}_x + (x^*)^T M \bar{d}_x = 0.$$

Let $\tilde{x}^k = x^* + \lambda_k \bar{d}_x$ and $\tilde{y}^k = y^* + \lambda_k \bar{d}_y$. Then $(\tilde{x}^k, \tilde{y}^k) \in S_+$. Moreover,

$$(\tilde{x}^k)^T \tilde{y}^k = \lambda_k (x^*)^T M \bar{d}_x + \lambda_k (y^*)^T \bar{d}_x + \lambda_k^2 \bar{d}_x^T \bar{d}_y = 0,$$

i.e., $(\tilde{x}^k, \tilde{y}^k) \in S_0$. However, $\tilde{x}_k = x^* + \lambda_k \bar{d}_x \to \infty$. This contradicts the boundedness of the solution set S_0. ∎

Based on Theorem 1 and Theorem 2, we conclude that

Corollary 1 *The interior point algorithms described in Kojima et al. [4] and Pardalos et al. [5] solve the P_0-matrix LCP if its solution set is nonempty and bounded. That is, they generate an iterative sequence (x^k, y^k), such that $y^k = Mx^k + q$, $(x^k, y^k) > 0$, and*

$$(x^k)^T y^k \to 0.$$

The remaining question is whether there exists a converging algorithm to solve the P_0-matrix LCP even if the solution set is unbounded. Facchinei and Kanzow [3] showed that it is impossible to remove the boundedness assumption of the solution set for the regularization method to solve the P_0-matrix LCP. The following example is given by them [3]:

Example 2

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The solution set of the LCP is given by

$$\mathcal{X} = \{(x_1, 1, 0, 0), x_1 \geq 0\} \cup \{(0, x_2, x_2 - 1, 0), x_2 \geq 1\}.$$

Note that this problem has no interior feasible point.

However, interior point methods and general smoothing methods might still be able to solve the P_0-matrix problem as long as it has an interior feasible point. Consider Example 1 again and its central path point:

$$(x, y) > 0, \quad y = Mx + q, \quad x_i y_i = \mu, \quad i = 1, 2, 3$$

for any positive μ. We can calculate this point:

$$x_1 = \frac{4\mu}{\sqrt{1 + 8\mu} + 1}, \quad x_2 = \frac{\sqrt{1 + 8\mu} + 1}{4}, \quad x_3 = \frac{4\mu}{\sqrt{1 + 8\mu} + 1}.$$

We see this path is bounded for any fixed $\mu \geq 0$ and it converges to a strictly complementarity solution $(0, 1/2, 0, 1/2, 0, 1/2)$ as $\mu \to 0$.

On the other hand, the unique solution of the regularized $(M(\epsilon), q)$ LCP is

$$x_1(\epsilon) = 0, \quad x_2(\epsilon) = 0, \quad x_3(\epsilon) = 0$$

for any $\epsilon > 0$. Hence regularization methods can only generate a trivial solution $(0, 0, 0, q)$.

References

[1] B. Chen, X. Chen, and C. Kanzow, "A penalized Fischer-Burmeister NCP-function: theoretical investigation and numerical results", Department of Management and Systems, Washington State University, Pullman, (1997).

[2] R. W. Cottle, J.-S. Pang and R. E. Stone, *The Linear Complementarity Problem*, Computer Science and Scientific Computing, Academic Press Inc, San Diego, CA92101, (1990).

[3] F. Facchinei and C. Kanzow, "Beyond monotonicity in regularization methods for nonlinear complementarity problems", Dipartmento di Informatica e Sistemistica, Universita di Roma, Roma, Italy, (1997).

[4] M. Kojima, N. Megiddo, T. Noma, and A. Yoshise, *A unified approach to interior point algorithms for linear complementarity problems*, LNCS 538, Springer-Verlag, (1991).

[5] P. M. Pardalos, Y. Ye, C-G. Han, and J. A. Kaliski, "Solution of P_0-matrix linear complementarity problems using a potential reduction algorithm", *SIAM J. Matrix Anal. Appl. 14*, (1993), 1048-1060.

[6] G. Ravindran and M. S. Gowda, "Regularization of P_0-functions in box variational inequality problems", Department of Mathematics & Statistics, University of Maryland Baltimore County, Baltimore, (1997).