Approximating global quadratic optimization with convex quadratic constraints

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Abstract

We consider the problem of approximating the global maximum of a quadratic program (QP) subject to convex non-homogeneous quadratic constraints. We prove an approximation quality bound that is related to a condition number of the convex feasible set; and it is the currently best for approximating certain problems, such as quadratic optimization over the assignment polytope, according to the best of our knowledge.

Key words. Quadratic programming, global optimizer, approximation algorithm

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1 Introduction

Consider the quadratic optimization (QP) problem with convex non-homogeneous quadratic inequality constraints

$$\text{(QP)} \qquad \begin{array}{rll} \bar{q}(C) := & \text{Maximize} & q(x) := x^T C x \\ & \text{Subject to} & \|A_i x + b_i\|^2 \leq 1, \ i = 1, \ldots, m \end{array}$$

where symmetric matrices $C \in \Re^{n \times n}$, $A_i \in \Re^{r_i \times n}$ $(r_i \leq n)$, and $b_i \in \Re^{r_i}$ are given. Note that if $r_i = 1$, then it becomes a linear constraint with (nonsymmetric) upper and lower bounds. Denote by $\bar{q} = \bar{q}(C)$ the maximum value of the problem.

We assume that the (QP) problem is feasible, bounded and it has an nonempty interior; and thereby, assume that $x = 0$ is a "center" of the convex feasible region, which implies that $1 - \|b_i\|^2 > 0$, $i = 1, \ldots, m$. For simplicity, we also assume that C is positive semidefinite. Thus, the global minimum value of $q(x)$ in the feasible region, $\underline{q} = -\bar{q}(-C)$, is 0.

Normally, there may be a linear term in the objective function, or/and C is not positive semidefinite, or/and $x = 0$ is not in the interior of the feasible region. We will address later how to solve QP problems in more general forms.

Recently, there were several results on approximating this quadratic maximum. Bellare and Rogaway [3] established several negative result on approximating this problem; Goemans and Williamson [8] (also see Frieze and Jerrum [4]) proved an approximation result for the Maxcut problem; Nesterov [11] generalized their result to approximating the Maxcut problem with a more general objective matrix; Ye [18] extended their result to solving the continuous (QP) with simple bound constraints; Ye [19] and Nesterov [12] proved constant approximation quality for problem (QP) such that A_i is diagonal and $b_i = 0$; Nesterov [13] and Nemirovski et al. [10] established a quality bound when A_i is positive semidefinite and $b_i = 0$; Fu et al. [5] constructed a quality bound for approximating (QP) without those assumptions. Their results can be summarized as follows.

- Bellare and Rogaway [3] show that if the feasible regions is a polytope, then there is a constant δ such that the problem has no polynomial algorithm to yield a solution x with
$$q(x) \geq 2^{-\log^{\delta} n} \bar{q},$$
unless NP$\subset\tilde{\text{P}}$; there is a constant $\epsilon \leq 1/3$ such that it has no polynomial algorithm to yield a solution x with
$$q(x) \geq (1 - \epsilon)\bar{q},$$
unless P=NP.

- If $i = 1$, A_1 is positive definite and $b_i = 0$, then Vavasis [17] and Ye [20] develop a procedure to obtain a feasible x with
$$q(x) \geq (1 - \epsilon)\bar{q},$$
whose running time is polynomial in n and $|\ln \epsilon|$ for any $1 > \epsilon > 0$.

- If all A_i are diagonal (or mutually commute) and $b_i = 0$; then Nesterov [12] and Ye [19] are able to generate a feasible (randomized) solution $\hat{x}$ such that

$$\mathrm{E}(q(\hat{x})) \geq \frac{2}{\pi}\bar{q}.$$

- If all A_i are positive semidefinite and $b_i = 0$, then Nesterov [13] and Nemirovskii et al. [10] produce a feasible (randomized) solution $\hat{x}$ such that, with constant probability,

$$q(\hat{x}) \geq \frac{1}{2\ln(2m\min(m,n))}\bar{q}.$$

- If all A_i are positive semidefinite and the feasible region is bounded and has nonempty interior, Fu et al. [5] (also see Bellare and Rogaway [3], Vavasis [17] and Ye [20]) compute a feasible solution x such that

$$q(x) \geq \frac{1}{O(m^2)}\bar{q}.$$

In this paper, we approximate the problem where $b_i \neq 0$, that is, the problem containing convex non-homogeneous quadratic constraints.

2 Positive Semi-Definite Relaxation

Let

$$Q := \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \succeq 0 \quad \text{and} \quad Q_i := \begin{pmatrix} \|b_i\|^2 & b^T A_i \\ A_i^T b & A_i^T A_i \end{pmatrix} \succeq 0.$$

(Here, $X \succeq Z$ means that $X - Z$ is positive semi-definite.) Then, the problem is equivalent to

$$\text{(QP)} \qquad \begin{array}{rll} \bar{q} := & \text{Maximize} & (1;x)^T Q (1;x) \\ & \text{Subject to} & (1;x)^T Q_i (1;x) \leq 1, \ i = 1, \ldots, m, \end{array}$$

where column vector $(1;x) \in \Re^{n+1}$ with 1 at its top position. Our approximation algorithm for (QP) is to solve the positive semi-definite programming (SDP) relaxation problem

$$\text{(SDP)} \qquad \begin{array}{rll} \bar{p}(Q) := & \text{Maximize} & \langle Q, X \rangle \\ & \text{Subject to} & \langle I_1, X \rangle = 1, \\ & & \langle Q_i, X \rangle \leq 1, \ i = 1, \ldots, m, \\ & & X \succeq 0, \end{array} \tag{1}$$

where I_1 is the diagonal matrix with 1 in its first diagonal position and 0 everywhere else. Here, unknown $X \in \Re^{(n+1)\times(n+1)}$ is a symmetric matrix. Furthermore, $\langle \cdot, \cdot \rangle$ is the matrix inner product $\langle Q, X \rangle = \mathrm{trace}(Q^T X)$. Since the original QP problem is feasible, so is the SDP relaxation.

The dual of (SDP) is

$$\begin{array}{rll} \bar{p}(Q) = & \text{Minimize} & y_0 + e^T y \\ & \text{Subject to} & \sum_{i=0}^m y_i Q_i \succeq Q, \ y \geq 0 \in \Re^m, \end{array} \tag{2}$$

where $Q_0 = I_1$. Note that the primal is feasible and the dual has an interior so that there is no duality gap between the primal and dual. Denote by $\bar{X}(Q)$ and $(\bar{y}_0(Q), \bar{y}(Q))$ an optimal solution pair for the primal (1) and dual (2).

The positive semi-definite relaxation was first proposed by Lovász and Shrijver [9], also see recent papers by Alizadeh [1], Anstreicher and Wolkowicz[2], Fujie and Kojima [6] and Polijak, Rendl and Wolkowicz [16]. This relaxation problem can be solved in polynomial time, e.g., see Nesterov and Nemirovski [14].

We have the following relations between (QP) and (SDP).

Proposition 1 *Let $\bar{q} = \bar{q}(C)$, $\underline{q} = -\bar{q}(-C)$, $\bar{p} = \bar{p}(Q)$, and $\underline{p} = -\bar{p}(-Q)$. Then,*

$$\underline{p} \leq \underline{q} \leq \bar{q} \leq \bar{p}.$$

■

3 Rank Reduction

We now generate a feasible solution for (QP), similar to Nemirovski et al. [10]. Let $\bar{X} = \bar{X}(Q) \succeq 0$ be an optimizer of (SDP), and let the lower triangle factorization matrix of $\bar{X}$ be $\bar{L} = (\bar{l}_1; \ldots; \bar{l}_n) \in \Re^{n \times n}$, i.e., $\bar{l}_j$ is the jth row of $\bar{L}$, such that $\bar{X} = \bar{L}\bar{L}^T$. Then, we have

$$\langle \bar{L}^T Q \bar{L}, I \rangle = \langle Q, \bar{L}\bar{L}^T \rangle = \langle Q, \bar{X} \rangle = \bar{p},$$

and for $i = 1, ..., m$,

$$\langle \bar{L}^T Q_i \bar{L}, I \rangle = \langle Q_i, \bar{L}\bar{L}^T \rangle = \langle Q_i, \bar{X} \rangle \leq 1,$$

Also, since $\bar{X}_{11} = \langle I_1, X \rangle = 1$, $\bar{l}_1$ has 1 in its first position and 0 everywhere else.

We generate a random vector $\hat{u} \in \Re^{n+1}$ with independent components taking the value 1 or -1 with equal probability, then assign

$$\hat{v} = \bar{L}\hat{u} \in \Re^{n+1}. \tag{3}$$

Note that with probability one

$$\hat{v}_1^2 = (\bar{l}_1 \hat{u})^2 = (\hat{u}_1)^2 = 1,$$

$$\hat{v}^T Q \hat{v} = \langle \bar{L}^T Q \bar{L}, \hat{u}\hat{u}^T \rangle \leq \langle \bar{p} I, \hat{u}\hat{u}^T \rangle = n\bar{p},$$

and, for $i = 1, ..., m$,

$$\hat{v}^T Q_i \hat{v} = \langle \bar{L}^T Q_i \bar{L}, \hat{u}\hat{u}^T \rangle \leq \langle I, \hat{u}\hat{u}^T \rangle = n.$$

Now, we generate a vector solution $\hat{x} \in \Re^n$ from

$$\hat{x} = (\hat{v}_2; ..., \hat{v}_{n+1})/\hat{v}_1. \tag{4}$$

Then, the expected value

$$\mathrm{E}(\hat{x}^T C\hat{x}) = \mathrm{E}(\hat{v}^T Q\hat{v}) = \mathrm{E}\left(\langle \bar{L}^T Q\bar{L}, \hat{u}\hat{u}^T\rangle\right) = \langle \bar{L}^T Q\bar{L}, \mathrm{E}(\hat{u}\hat{u}^T)\rangle = \langle \bar{L}^T Q\bar{L}, I\rangle = \bar{p},$$

and

$$\mathrm{E}(\|A_i\hat{x} + b_i\|^2) = \mathrm{E}(\hat{v}^T Q_i\hat{v}) = \mathrm{E}\left(\langle \bar{L}^T Q_i\bar{L}, \hat{u}\hat{u}^T\rangle\right) = \langle \bar{L}^T Q_i\bar{L}, \mathrm{E}(\hat{u}\hat{u}^T)\rangle = \langle \bar{L}^T Q_i\bar{L}, I\rangle \leq 1.$$

To summarize, we have

Proposition 2 *With probability one*

$$\begin{gathered}\hat{v}_1^2 = 1, \\ q(\hat{x}) = \hat{v}^T Q\hat{v} \leq n\bar{p}, \\ \|A_i\hat{x} + b_i\|^2 = \hat{v}^T Q_i\hat{v} \leq n, \;\; i = 1, ..., m.\end{gathered}$$

Their expected values

$$\begin{gathered}\mathrm{E}(q(\hat{x})) = \mathrm{E}(\hat{x}^T C\hat{x}) = \mathrm{E}(\hat{v}^T Q\hat{v}) = \bar{p} \\ \mathrm{E}(\|A_i\hat{x} + b_i\|^2) = \mathrm{E}(\hat{v}^T Q_i\hat{v}) \leq 1, \;\; i = 1, ..., m.\end{gathered}$$

■

Although $\hat{x}$ is expected to meet the constraints of (QP), we have no guarantee that it will do so for sure. If not, we need to shrink it from

$$\tilde{x} = \hat{\alpha}\hat{x}, \tag{5}$$

where

$$\hat{\alpha} = \min_i\{\hat{\alpha}_i\};$$

and $\hat{\alpha}_i = 1$ if $\|A_i\hat{x} + b_i\|^2 \leq 1$, else $\hat{\alpha}_i < 1$ is the positive root of the quadratic equation of α:

$$q_i(\alpha) := \|\alpha(A_i\hat{x}) + b_i\|^2 = 1.$$

Next, we will prove that $\tilde{x}$ is feasible for (QP).

4 Approximation Analysis

We now analyzing the solution $\tilde{x}$ resulted from the rank reduction procedure. We first present a technical lemma.

Lemma 1 *Let $\beta_i \geq 0$ and*

$$\beta_i := \|A_i \hat{x} + b_i\|.$$

Then, for any $0 \leq \alpha \leq 1$,

$$\|A_i(\alpha \hat{x}) + b_i\|^2 \leq \alpha(\beta_i^2 - \|b_i\|^2) + \|b_i\|^2 \leq \max\{\beta_i^2, \|b_i\|^2\}.$$

If $\beta_i > 1$, then the root of the quadratic equation of $q_i(\alpha) = 1$

$$\hat{\alpha}_i \geq \frac{1 - \|b_i\|}{\sqrt{\beta_i^2 - \|b_i\|^2}} \geq \frac{1 - \|b_i\|}{\beta_i}.$$

Proof. Note that $q_i(\alpha) = \|A_i(\alpha\hat{x}) + b_i\|^2$ is a convex function of α and $q_i(0) = \|b_i\|^2$ and $q_i(1) = \beta_i^2$. But $\alpha(\beta_i^2 - \|b_i\|^2) + \|b_i\|^2$ is the linear function crossing $\|b_i\|^2$ and β_i^2 at $\alpha = 0$ and $\alpha = 1$, respectively. Thus, for $\alpha \in [0, 1]$,

$$q_i(\alpha) \leq \alpha(\beta_i^2 - \|b_i\|^2) + \|b_i\|^2.$$

Furthermore, consider the case $\beta_i^2 > 1$. If $b_i^T A_i \hat{x} \leq 0$, then

$$q_i(\alpha) \leq \alpha^2(\beta_i^2 - \|b_i\|^2) + \|b_i\|^2,$$

Thus, the root

$$\hat{\alpha}_i \geq \frac{\sqrt{1 - \|b_i\|^2}}{\sqrt{\beta_i^2 - \|b_i\|^2}}.$$

On the other hand, if $b_i^T A_i \hat{x} \geq 0$, then

$$\hat{x}^T A_i^T A_i \hat{x} \leq \beta_i^2 - \|b_i\|^2,$$

$$q_i(\alpha) \leq (\alpha\|A_i\hat{x}\| + \|b_i\|)2.$$

Thus, the root

$$\hat{\alpha}_i \geq \frac{1 - \|b_i\|}{\|A_i\hat{x}\|} \geq \frac{1 - \|b_i\|}{\sqrt{\beta_i^2 - \|b_i\|^2}}.$$

Noting the second bound is less than the first, we have proved the desired result. ■

We are ready to prove the following theorem.

Theorem 1 *With probability one, $\tilde{x}$ generated in the above rank reduction procedure is a feasible solution for (QP).*

Proof. We need to prove that for $i = 1, ..., m$,

$$\|A_i \tilde{x} + b_i\|^2 \leq 1.$$

Note that $0 \le \hat{\alpha} \le 1$. Thus, for any i, if $\beta_i^2 = \|A_i\hat{x} + b_i\|^2 \le 1$, we must have from the above lemma

$$\|A_i\tilde{x} + b_i\|^2 \le \max\{\beta_i^2, \|b_i\|^2\} \le 1.$$

Otherwise, since $\hat{\alpha} \le \hat{\alpha}_i$,

$$\begin{aligned} \|A_i\tilde{x} + b_i\|^2 &= \|A_i(\hat{\alpha})\hat{x} + b_i\|^2 \\ &\le \max\{\|A_i(\hat{\alpha}_i)\hat{x} + b_i\|^2, \|b_i\|^2\} \\ &= \max\{1, \|b_i\|^2\} \\ &= 1. \end{aligned}$$

■

Now we analyze the objective function value

$$q(\tilde{x}) = \tilde{x}^T C\tilde{x} = \hat{\alpha}^2\hat{x}^T C\hat{x} = \hat{\alpha}^2\hat{v}^T Q\hat{v}.$$

Note that $\hat{\alpha}$ is also a random number here. Recall

$$\hat{\alpha} = \min_i\{\hat{\alpha}_i\} \ge \min_{\{i,\ \|A_i\hat{x}+b_i\|^2>1\}} \left(\frac{1 - \|b_i\|}{\sqrt{\|A_i\hat{x} + b_i\|^2 - \|b_i\|^2}}\right).$$

Since

$$\beta_i^2 = \|A_i\hat{x} + b_i\|^2 = \hat{v}^T Q_i\hat{v},$$

the following lemma is a straightforward extension of the result proved by Nemirovski et al. [10].

Lemma 2 *For any $\beta \ge 1$, The probability*

$$\Pr\left(\max_i\{\|A_i\hat{x} + b_i\|^2\} > \beta^2\right) < 2m\bar{r}\exp(-\beta^2/2),$$

where $\bar{r} = \max_i\{r_i\}$, the row dimension of A_i.

■

Hence, we have, in particular,

$$\Pr\left(\max_i\{\beta_i^2 = \|A_i\hat{x} + b_i\|^2\} > 2\ln(4mn\bar{r})\right) < 2m\bar{r}\exp(-\ln(4mn\bar{r})) = \frac{1}{2n}.$$

Now we prove the following theorem.

Theorem 2 *The expected value*

$$\mathrm{E}(q(\tilde{x})) = \mathrm{E}(\tilde{x}^T C\tilde{x}) \ge \frac{(1 - \max_i\{\|b_i\|\})^2}{4\ln(4mn\bar{r})} \cdot \bar{p} \ge \frac{(1 - \max_i\{\|b_i\|\})^2}{4\ln(4mn\bar{r})} \cdot \bar{q},$$

Proof. For simplicity, let

$$\underline{\alpha} = \frac{1 - \max_i\{\|b_i\|\}}{\sqrt{2\ln(4mn\bar{r})}}.$$

Note that

$$\Pr(\hat{\alpha} < \underline{\alpha}) \le \Pr\left(\max_i\{\beta_i^2 = \|A_i\hat{x} + b_i\|^2\} > 2\ln(4mn\bar{r})\right) < \frac{1}{2n}.$$

Recall that

$$\mathrm{E}(\hat{v}^T Q \hat{v}) = \bar{p},$$

and with probability one

$$\hat{v}^T Q \hat{v} \le n\bar{p}.$$

$$\begin{array}{rl} & \mathrm{E}(\tilde{x}^T C \tilde{x}) \\ = & \mathrm{E}(\hat{\alpha}^2 \hat{v}^T Q \hat{v}) \\ = & \mathrm{E}\left(\hat{\alpha}^2 \hat{v}^T Q \hat{v} | \ \hat{\alpha} \ge \underline{\alpha}\right) + \mathrm{E}\left(\hat{\alpha}^2 \hat{v}^T Q \hat{v} | \ \hat{\alpha} < \underline{\alpha}\right) \\ \ge & \mathrm{E}\left(\hat{\alpha}^2 \hat{v}^T Q \hat{v} | \ \hat{\alpha} \ge \underline{\alpha}\right) \quad (\text{since } \hat{\alpha}^2 \hat{v}^T Q \hat{v} \ge 0) \\ \ge & \underline{\alpha}^2 \mathrm{E}\left(\hat{v}^T Q \hat{v} | \ \hat{\alpha} \ge \underline{\alpha}\right) \quad (\text{since } \hat{\alpha} \ge \underline{\alpha}) \\ = & \underline{\alpha}^2 \mathrm{E}(\hat{v}^T Q \hat{v}) - \underline{\alpha}^2 \mathrm{E}\left(\hat{v}^T Q \hat{v} | \ \hat{\alpha} < \underline{\alpha}\right) \\ \ge & \underline{\alpha}^2 \bar{p} - \underline{\alpha}^2 (n\bar{p}) \Pr(\hat{\alpha} < \underline{\alpha}) \quad (\text{since } \hat{\alpha}^2 \hat{v}^T Q \hat{v} \le n\bar{p}) \\ \ge & \underline{\alpha}^2 \bar{p} - \underline{\alpha}^2 \bar{p}/2 \quad (\text{since } \Pr(\hat{\alpha} < \underline{\alpha}) \le 1/2n) \\ = & \dfrac{\underline{\alpha}^2}{2} \cdot \bar{p}. \end{array}$$

■

The quality of the approximation bound is dependent on the the value $(1 - \max_i\{\|b_i\|\})$, which is related to the condition number of the convex feasible set measuring how "round" the interior of the set is. This condition number also dictates the computational complexity of linear optimization over this convex set.

If we relax the feasibility by a constant fraction ϵ, then we have

Corollary 1 *In the rank reduction procedure, let $\hat{\alpha}_i$ be the root of*

$$q_i(\alpha) = \|\alpha(A_i\hat{x}) + b_i\|^2 = 1 + \epsilon.$$

Then, with probability one

$$\|A_i\tilde{x} + b_i\|^2 \le 1 + \epsilon, \ i = 1, ..., m,$$

and

$$\mathrm{E}(\tilde{x}^T C \tilde{x}) \ge \frac{(\epsilon + 1 - \max_i\{\|b_i\|\})^2}{4\ln(4mn\bar{r})} \cdot \bar{q} \ge \frac{\epsilon^2}{4\ln(4mn\bar{r})} \cdot \bar{q}.$$

■

5 Linear inequality constraints

Consider the problem

$$\text{(LQP)} \qquad \begin{array}{rll} \bar{q} := & \text{Maximize} & y^T C y \\ & \text{Subject to} & -l \le A^T y \le u \end{array}$$

where $A \in \Re^{n\times m}$ and $y = 0$ is the analytic center of the polytope $\{y : \ -l \le A^T y \le u\}$ (hence the given bounds $l, s > 0$). Then, the problem can be written as

$$\begin{array}{ll} \text{Maximize} & y^T C y \\ \text{Subject to} & (a_i^T y + l_i)(a_i^T y - u_i) \le 0, \ \ i = 1, ..., m, \end{array}$$

where a_i is the ith column of A.

The quadratic constraints can be further written as

$$(a_i^T y + \frac{l_i - u_i}{2})^2 \le \frac{(l_i - u_i)^2}{4}$$

or

$$(\frac{2}{l_i + u_i} a_i^T y + \frac{l_i - u_i}{l_i + u_i})^2 \le 1.$$

Thus, they are in the quadratic constraint form of (QP). Furthermore, in the worst case,

$$\frac{1}{m} \le \frac{l_i}{u_i} \le m.$$

Therefore, from Theorem 2 and noting $\bar{r} = 1$, we have

Corollary 2 *The approximation algorithm presented in this paper will generate a feasible point $\tilde{x}$ for the problem (LQP) and*

$$\mathrm{E}(\tilde{x}^T C \tilde{x}) \ge \frac{1}{O(m^2)\ln(4mn)} \cdot \bar{q}.$$

■

Note that this bound is worse than the bound of $\frac{1}{O(m^2)}$ of Bellare and Rogaway [3], Fu et al. [5], Vavasis [17], and Ye [20]. However, if $\frac{l_i}{u_i}$ is close to 1, then the SDP relaxation and the rank reduction could yield a better approximation.

Consider the assignment polytope

$$\sum_{j=1}^{n} x_{ij} = 1, \ \ i = 1, ..., n,$$

$$\sum_{i=1}^{n} x_{ij} = 1, \; j = 1, ..., n,$$
$$x_{ij} \geq 0, \; i = 1, ..., n, \; j = 1, ..., n.$$

Let us transform x_{ij} to

$$y_{ij} = x_{ij} - \frac{1}{n}.$$

Then, the system becomes

$$\sum_{j=1}^{n} y_{ij} = 0, \; i = 1, ..., n,$$
$$\sum_{i=1}^{n} y_{ij} = 0, \; j = 1, ..., n,$$
$$y_{ij} \geq -\frac{1}{n}, \; i = 1, ..., n, \; j = 1, ..., n.$$

Adding seemly redundant upper bounds and eliminating $2n$ variables from $2n$ homogeneous equations, the system can be written as

$$-\frac{1}{n} \leq a_{ij}^T y \leq 1 - \frac{1}{n}, \; i = 1, ..., n, \; j = 1, ..., n,$$

for given vectors $a_{ij} \in \Re^{n^2-2n}$ and unknown $y \in \Re^{n^2-2n}$. Note that $y = 0$ is the analytic center of the polytope and

$$1 - \frac{l_i - u_i}{l_i + u_i} = 1 - (1 - \frac{2}{n}) = \frac{2}{n}.$$

Let the objective be maximizing $y^T C y$ and C be positive semidefinite, and the global maximum value be $\bar{q}$. Then, we have

Corollary 3 *The approximation algorithm presented in this paper will generate a feasible point $\tilde{y}$ for the problem (LQP) over the assignment polytope, and*

$$\mathrm{E}(\tilde{y}^T C \tilde{y}) \geq \frac{1}{n^2 \ln(4n^4)} \cdot \bar{q}.$$

■

This bound is better than the bound of $\frac{1}{O(n^3)}$ of Fu et al. [5].

6 Linear term in the objective function

If C is not positive semidefinite and there is a linear term in the objective function of (QP):

$$\begin{array}{ll} \text{Maximize} & q(x) := x^T C x + 2c^T x \\ \\ \text{Subject to} & \|A_i x + b_i\|^2 \leq 1, \; i = 1, \ldots, m, \end{array}$$

we could still solve the SDP relaxation with

$$Q := \begin{pmatrix} 0 & c^T \\ c & C \end{pmatrix} \quad \text{and} \quad Q_i := \begin{pmatrix} \|b_i\|^2 & b^T A_i \\ A_i^T b & A_i^T A_i \end{pmatrix} \succeq 0.$$

Let $\bar{X}(Q)$ be an optimizer of the (SDP) relaxation. Then

$$\bar{p} = \langle Q, \bar{X}(Q) \rangle \geq \bar{q},$$

and

$$\underline{p} = -\langle -Q, \bar{X}(-Q) \rangle = -\underline{y}_0(-Q) - e^T \underline{y}(-Q) \leq \underline{q}.$$

From $\bar{L}\bar{L}^T = \bar{X} = \bar{X}(Q)$, we use the same rank reduction procedure to generate $\hat{u}$, $\hat{v}$, and $\hat{x}$, then we consider both $\hat{x}$ and $-\hat{x}$. We have

Proposition 3 *With probability one*

$$\hat{v}_1^2 = 1,$$

$$q(\hat{x}) = \hat{x}^T C \hat{x} + c^T \hat{x} = \hat{v}^T Q \hat{v},$$

$$q(-\hat{x}) = \hat{x}^T C \hat{x} - c^T \hat{x} = \hat{v}^T \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix} Q \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix} \hat{v},$$

$$q_i(\hat{x}) = \|A_i \hat{x} + b_i\|^2 = \hat{v}^T Q_i \hat{v},$$

$$q_i(-\hat{x}) = \|A_i \hat{x} - b_i\|^2 = \|A_i \hat{x} + b_i\|^2 - 4b_i^T (A_i \hat{x} + b_i) + 4\|b_i\|^2 =$$
$$\hat{v}^T \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix} Q_i \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix} \hat{v} \leq (\|A_i \hat{x} + b_i\| + 2\|b_i\|)^2, \; i = 1, ..., m.$$

Their expected values

$$\mathrm{E}(\max\{q(\hat{x}), q(-\hat{x})\}) \geq \mathrm{E}(q(\hat{x})) = \mathrm{E}(\hat{x}^T C \hat{x}) = \mathrm{E}(\hat{v}^T Q \hat{v}) = \bar{p},$$

$$\mathrm{E}(\|A_i \hat{x} + b_i\|^2) = \mathrm{E}(\hat{v}^T Q_i \hat{v}) \leq 1, \; i = 1, ..., m,$$

$$\mathrm{E}(\|A_i \hat{x} - b_i\|^2) \leq (\sqrt{\mathrm{E}(\|A_i \hat{x} + b_i\|^2)} + 2\|b_i\|)^2 \leq 9, \; i = 1, ..., m.$$

■

Let

$$\hat{\alpha}^+ = \min_i \{\hat{\alpha}_i^+\}$$

where for $i = 1, ..., m$,

$$\hat{\alpha}_i^+ = \begin{cases} 1 & \text{if } \|A_i \hat{x} + b_i\|^2 \leq 1, \\ \text{the positive root of } \|\alpha(A_i \hat{x}) + b_i\|^2 = 1, & \text{otherwise;} \end{cases}$$

and

$$\hat{\alpha}^- = \min_i \{\hat{\alpha}_i^-\}$$

where for $i = 1, ..., m$,

$$\hat{\alpha}_i^- = \begin{cases} 1 & \text{if } \|A_i\hat{x} - b_i\|^2 \leq 1, \\ \text{the positive root of } \|\alpha(A_i\hat{x}) - b_i\|^2 = 1, & \text{otherwise.} \end{cases}$$

Then, we select

$$\tilde{x} = \begin{cases} \hat{\alpha}^+ \cdot \hat{x}, & \text{if } q(\hat{\alpha}^+ \cdot \hat{x}) \geq q(\hat{\alpha}^- \cdot (-\hat{x})), \\ \hat{\alpha}^- \cdot (-\hat{x}), & \text{otherwise,} \end{cases} \tag{6}$$

That is, we shrink both $\hat{x}$ and $-\hat{x}$ to meet the constraints and choose the better one among the two.

Again, one can verify that $\tilde{x}$ is feasible for (QP) from Theorem 1. Let

$$\hat{\alpha} = \min\{\hat{\alpha}^+, \hat{\alpha}^-\}.$$

Then, since $q(\hat{x})$ and $q(-\hat{x})$ share the same quadratic term, the linear term of $\max\{q(\hat{x}), q(-\hat{x})\}$ is always non-negative, which implies that

$$q(\tilde{x}) = \max\{q(\hat{\alpha}^+ \cdot \hat{x}), q(\hat{\alpha}^- \cdot (-\hat{x}))\} \geq \max\{q(\hat{\alpha} \cdot \hat{x}), q(\hat{\alpha} \cdot (-\hat{x}))\} \geq \hat{\alpha}^2 \max\{q(\hat{x}), q(-\hat{x})\} \geq \hat{\alpha}^2 q(\hat{x}).$$

Moreover, if $\hat{\alpha}$ is fixed

$$\mathrm{E}(q(\tilde{x})) \geq \hat{\alpha}^2 \mathrm{E}(\max\{q(\hat{x}), q(-\hat{x})\}) \geq \hat{\alpha}^2 \mathrm{E}(q(\hat{x})) \geq \hat{\alpha}^2 \bar{p} \geq \hat{\alpha}^2 \bar{q}.$$

The remaining question is how small $\hat{\alpha}$ could be. We have established a bound for $\hat{\alpha}^+$ early such that

$$\hat{\alpha}^+ \geq \min_i \{\hat{\alpha}_i^+\} \geq \min_{\{i,\ \|A_i\hat{x}+b_i\|^2>1\}} \left(\frac{1 - \|b_i\|}{\sqrt{\|A_i\hat{x} + b_i\|^2 - \|b_i\|^2}} \right).$$

Similarly, we can bound

$$\hat{\alpha}^- \geq \min_i \{\hat{\alpha}_i^-\} \geq \min_{\{i,\ \|A_i\hat{x}-b_i\|^2>1\}} \left(\frac{1 - \|b_i\|}{\sqrt{\|A_i\hat{x} - b_i\|^2 - \|b_i\|^2}} \right).$$

Recalling

$$\|A_i\hat{x} - b_i\|^2 \leq (\|A_i\hat{x} + b_i\| + 2\|b_i\|)^2$$

and using Lemma 2, we must have

Lemma 3 *For any $\beta \geq 1$, The probability*

$$\Pr\left(\max_i\{\|A_i\hat{x} - b_i\|^2\} > (\beta + 2\|b_i\|)^2\right) < 2m\bar{r}\exp(-\beta^2/2),$$

where $\bar{r} = \max_i\{r_i\}$, the row dimension of A_i.

■

Hence, we have, in particular,

$$\Pr\left(\max_i\{\|A_i\hat{x}-b_i\|^2\} > (\sqrt{2\ln(4mn\bar{r})}+2\|b_i\|)^2\right) < 2m\bar{r}\exp(-\ln(4mn\bar{r})) = \frac{1}{2n}.$$

Since $\|b_i\| < 1$ for all i, we can now prove the following theorem

Theorem 3 *Let us fix*

$$\hat{\alpha} = \underline{\alpha} := \frac{1-\max_i\{\|b_i\|\}}{\sqrt{2\ln(4mn\bar{r})}+2}$$

in the above rank reduction procedure. Then,

$$\mathrm{E}(q(\tilde{x})) \geq \frac{(1-\max_i\{\|b_i\|\})^2}{(\sqrt{2\ln(4mn\bar{r})}+2)^2}\cdot\bar{p} \geq \frac{(1-\max_i\{\|b_i\|\})^2}{(\sqrt{2\ln(4mn\bar{r})}+2)^2}\cdot\bar{q},$$

and, with probability greater than $(1-2/n)$, $\tilde{x}$ is a feasible point for (QP).

■

Thus, as $n\to\infty$, $\tilde{x}$ is surely a feasible point for (QP) and its expected objective value is bounded below by a fraction of $\bar{q}$ from the theorem.

If $\hat{\alpha}$ is assigned from the random vector $\hat{x}$, we can prove a quality bound using a measure similar to the one in Nesterov [12] and Ye [19]. First, we note again that

$$\Pr(\hat{\alpha} < \underline{\alpha}) \leq \frac{1}{2n}.$$

Let $(\underline{y}_0, \underline{y}) = -(\bar{y}_0(-Q), \bar{y}(-Q)) \in \Re^{m+1}$. Then $\underline{y} \leq 0$, $\underline{p} = \underline{y}_0 + e^T\underline{y}$,

$$\underline{S} := Q - \sum_{i=0}^{m}\underline{y}_i Q_i \succeq 0,$$

and

$$\langle \underline{S}, \bar{X}\rangle = \bar{p} - \sum_{i=0}^{m}\underline{y}_i\langle Q_i, \bar{X}\rangle = \bar{p} - \tilde{p} \leq \bar{p} - \underline{p},$$

since $\langle Q_0, \bar{X}\rangle = 1$ and $\langle Q_i, \bar{X}\rangle \leq 1$, $i = 1, ..., m$, so that

$$\tilde{p} := \underline{y}_0 + \sum_{i=1}^{m}\underline{y}_i\langle Q_i, \bar{X}\rangle \geq \underline{p}.$$

Therefore, with probability one

$$\hat{v}^T\underline{S}\hat{v} \leq n(\bar{p} - \tilde{p}),$$

and

$$\mathrm{E}(\hat{v}^T \underline{S}\hat{v}) = \langle \underline{S}, \bar{X}\rangle = \bar{p} - \tilde{p}$$

and

$$\mathrm{E}\left(\hat{v}^T\left(\sum_{i=0}^m \underline{y}_i Q_i\right)\hat{v}\right) = \sum_{i=0}^m \underline{y}_i \langle Q_i, \bar{X}\rangle = \tilde{p}.$$

Furthermore, if $\underline{y}_0 > 0$, then

$$\underline{S} + \underline{y}_0 Q_0 = \underline{S} + \underline{y}_0 I_1 \succeq 0,$$

and with probability one

$$\hat{v}^T(\underline{S} + \underline{y}_0 Q_0)\hat{v} \le n(\bar{p} - \tilde{p} + \underline{y}_0),$$

and

$$\mathrm{E}(\hat{v}^T(\underline{S} + \underline{y}_0 Q_0)\hat{v}) = \langle \underline{S} + \underline{y}_0 Q_0), \bar{X}\rangle = \bar{p} - \tilde{p} + \underline{y}_0$$

and

$$\mathrm{E}\left(\hat{v}^T\left(\sum_{i=1}^m \underline{y}_i Q_i\right)\hat{v}\right) = \sum_{i=1}^m \underline{y}_i \langle Q_i, \bar{X}\rangle = \tilde{p} - \underline{y}_0.$$

We first consider the case $\underline{y}_0 \le 0$. Then,

$$\begin{aligned}
&\mathrm{E}(q(\tilde{x}))\\
&\ge \mathrm{E}\left(\hat{\alpha}^2 q(\hat{x})\right)\\
&= \mathrm{E}\left(\hat{\alpha}^2 \hat{v}^T Q \hat{v}\right)\\
&= \mathrm{E}\left(\hat{\alpha}^2\hat{v}^T\left(\sum_{i=0}^m \underline{y}_i Q_i + Q - \sum_{i=0}^m \underline{y}_i Q_i\right)\hat{v}\right)\\
&= \mathrm{E}\left(\hat{\alpha}^2\hat{v}^T\underline{S}\hat{v}\right) + \mathrm{E}\left(\hat{\alpha}^2\hat{v}^T\left(\sum_{i=0}^m \underline{y}_i Q_i\right)\hat{v}\right)\\
&\ge \mathrm{E}\left(\hat{\alpha}^2\hat{v}^T\underline{S}\hat{v}\right) + \mathrm{E}\left(\hat{v}^T\left(\sum_{i=0}^m \underline{y}_i Q_i\right)\hat{v}\right)\\
&\qquad (\text{since } \hat{\alpha}^2 \le 1,\ \underline{y}_i \le 0,\ \hat{v}^T Q_i \hat{v} \ge 0,\ i = 0, ..., m)\\
&= \mathrm{E}\left(\hat{\alpha}^2\hat{v}^T\underline{S}\hat{v}\right) + \tilde{p}\\
&\ge \mathrm{E}\left(\hat{\alpha}^2\hat{v}^T\underline{S}\hat{v} |\ \hat{\alpha} \ge \underline{\alpha}\right) + \tilde{p} \quad (\text{since } \hat{v}^T\underline{S}\hat{v} \ge 0)\\
&\ge \underline{\alpha}^2\mathrm{E}\left(\hat{v}^T\underline{S}\hat{v} |\ \hat{\alpha} \ge \underline{\alpha}\right) + \tilde{p} \quad (\text{since } \hat{\alpha} \ge \underline{\alpha})\\
&= \underline{\alpha}^2\mathrm{E}\left(\hat{v}^T\underline{S}\hat{v}\right) - \underline{\alpha}^2\mathrm{E}\left(\hat{v}^T\underline{S}\hat{v} |\ \hat{\alpha} < \underline{\alpha}\right) + \tilde{p}\\
&= \underline{\alpha}^2(\bar{p} - \tilde{p}) - \underline{\alpha}^2\mathrm{E}\left(\hat{v}^T\underline{S}\hat{v} |\ \hat{\alpha} < \underline{\alpha}\right) + \tilde{p}\\
&\ge \underline{\alpha}^2(\bar{p} - \tilde{p}) - \underline{\alpha}^2 n(\bar{p} - \tilde{p})\Pr(\hat{\alpha} < \underline{\alpha}) + \tilde{p} \quad (\text{since } \hat{v}^T\underline{S}\hat{v} \le n(\bar{p} - \tilde{p}))\\
&\ge \underline{\alpha}^2(\bar{p} - \tilde{p}) - \underline{\alpha}^2(\bar{p} - \tilde{p})/2 + \tilde{p} \quad (\text{since } \Pr(\hat{\alpha} < \underline{\alpha}) \le 1/2n)\\
&= \frac{\underline{\alpha}^2}{2}(\bar{p} - \tilde{p}) + \tilde{p}.
\end{aligned}$$

Secondly, consider the case $\underline{y}_0 > 0$. Then,

$$
\begin{aligned}
&\mathrm{E}(q(\tilde{x}))\\
&\geq \mathrm{E}\left(\hat{\alpha}^2 q(\hat{x})\right)\\
&= \mathrm{E}\left(\hat{\alpha}^2 \hat{v}^T Q \hat{v}\right)\\
&= \mathrm{E}\left(\hat{\alpha}^2 \hat{v}^T \left(\sum_{i=0}^m \underline{y}_i Q_i + Q - \sum_{i=0}^m \underline{y}_i Q_i\right)\hat{v}\right)\\
&= \mathrm{E}\left(\hat{\alpha}^2 \hat{v}^T (\underline{S} + \underline{y}_0 Q_0)\hat{v}\right) + \mathrm{E}\left(\hat{\alpha}^2 \hat{v}^T \left(\sum_{i=1}^m \underline{y}_i Q_i\right)\hat{v}\right)\\
&\geq \mathrm{E}\left(\hat{\alpha}^2 \hat{v}^T (\underline{S} + \underline{y}_0 Q_0)\hat{v}\right) + \mathrm{E}\left(\hat{v}^T \left(\sum_{i=1}^m \underline{y}_i Q_i\right)\hat{v}\right)\\
&\quad (\text{since } \hat{\alpha}^2 \leq 1,\ \underline{y}_i \leq 0,\ \hat{v}^T Q_i \hat{v} \geq 0,\ i = 1, ..., m)\\
&= \mathrm{E}\left(\hat{\alpha}^2 \hat{v}^T (\underline{S} + \underline{y}_0 Q_0)\hat{v}\right) + (\tilde{p} - \underline{y}_0)\\
&\geq \mathrm{E}\left(\hat{\alpha}^2 \hat{v}^T (\underline{S} + \underline{y}_0 Q_0)\hat{v} |\ \hat{\alpha} \geq \underline{\alpha}\right) + (\tilde{p} - \underline{y}_0) \quad (\text{since } \hat{v}^T (\underline{S} + \underline{y}_0 Q_0)\hat{v} \geq 0)\\
&\geq \underline{\alpha}^2 \mathrm{E}\left(\hat{v}^T (\underline{S} + \underline{y}_0 Q_0)\hat{v} |\ \hat{\alpha} \geq \underline{\alpha}\right) + (\tilde{p} - \underline{y}_0) \quad (\text{since } \hat{\alpha} \geq \underline{\alpha})\\
&= \underline{\alpha}^2 \mathrm{E}\left(\hat{v}^T (\underline{S} + \underline{y}_0 Q_0)\hat{v}\right) - \underline{\alpha}^2 \mathrm{E}\left(\hat{v}^T (\underline{S} + \underline{y}_0 Q_0)\hat{v} |\ \hat{\alpha} < \underline{\alpha}\right) + (\tilde{p} - \underline{y}_0)\\
&= \underline{\alpha}^2 (\bar{p} - \tilde{p} + \underline{y}_0) - \underline{\alpha}^2 \mathrm{E}\left(\hat{v}^T (\underline{S} + \underline{y}_0 Q_0)\hat{v} |\ \hat{\alpha} < \underline{\alpha}\right) + (\tilde{p} - \underline{y}_0)\\
&\geq \underline{\alpha}^2 (\bar{p} - \tilde{p} + \underline{y}_0) - \underline{\alpha}^2 n(\bar{p} - \tilde{p} + \underline{y}_0)\Pr(\hat{\alpha} < \underline{\alpha}) + (\tilde{p} - \underline{y}_0)\\
&\quad (\text{since } \hat{v}^T \underline{S} \hat{v} \leq n(\bar{p} - \tilde{p} + \underline{y}_0))\\
&\geq \underline{\alpha}^2 (\bar{p} - \tilde{p} + \underline{y}_0) - \underline{\alpha}^2 (\bar{p} - \tilde{p} + \underline{y}_0)/2 + (\tilde{p} - \underline{y}_0) \quad (\text{since } \Pr(\hat{\alpha} < \underline{\alpha}) \leq 1/2n)\\
&= \frac{\underline{\alpha}^2}{2}(\bar{p} - \tilde{p} + \underline{y}_0) + (\tilde{p} - \underline{y}_0)\\
&\geq \frac{\underline{\alpha}^2}{2}(\bar{p} - \tilde{p}) + \tilde{p}. \quad (\text{since } \underline{y}_0 > 0,\ \frac{\underline{\alpha}^2}{2} < 1)
\end{aligned}
$$

Both inequalities imply that

$$\mathrm{E}(q(\tilde{x})) - \tilde{p} \geq \frac{\underline{\alpha}^2}{2}(\bar{p} - \tilde{p}),$$

which further implies that

$$\mathrm{E}(q(\tilde{x})) - \underline{p} \geq \frac{\underline{\alpha}^2}{2}(\bar{p} - \underline{p}),$$

since $0 \geq \tilde{p} \geq \underline{p}$. This leads to

Theorem 4 *The expected value*

$$\mathrm{E}(q(\tilde{x})) - \underline{p} \geq \frac{(1 - \max_i\{\|b_i\|\})^2}{2(\sqrt{2\ln(4mn\bar{r})} + 2)^2} \cdot (\bar{p} - \underline{p}) \geq \frac{(1 - \max_i\{\|b_i\|\})^2}{2(\sqrt{2\ln(4mn\bar{r})} + 2)^2} \cdot (\bar{q} - \underline{p}).$$

■

We remark that if $C \succeq 0$ and $c = 0$, then $Q \succeq 0$ and $\underline{p} = 0$, and the theorem establishes the same bound as in Theorem 2.

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